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A CHARACTERIZATION OF TOTALLY GEODESIC SUBMANIFOLDS IN S^N AND CP^N BY AN INEQUALITY (*)

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Chapter 0. Introduction.

This work is mainly devoted to the study of differential geometry of submanifolds in space forms, although most of the arguments and results in Chapters I and II are intrinsic.

In 1952, Chern and Kuiper [2] introduced the notions of nullity and relative nullity of submanifolds in Euclidean space; they showed that the nullity and relative nullity distributions are of C^∞, involutive and totally geodesic on the open sets where they are constant. Later in 1959, Hartman and Nirenberg [9](**) proved that a complete hypersurface in $(n+1)$-dimensional Euclidean space with the Gauss map of rank at most 1 is cylindrical, i. e., a Riemannian product of $(n-1)$-dimensional Euclidean space and a plane curve. This theorem gave the first global determination of flat hypersurfaces in Euclidean space and led to other, so called, cylinder theorems, under the assumption of constancy of relative nullity and some restrictions on the sectional curvatures, by O'Neill [15] or Hartman [8]. However, it should be remarked here that without the restrictions of sectional curvature, it has not yet been known whether or not the above cylinder theorems have further generalizations, even under the assumption of constancy of relative nullity. For these theorems, completeness of the leaves of the relative nullity distribution is crucial, as easily seen in their proofs. Thus a natural question arose as to whether the leaves of the minimum relative nullity distribution are complete or not under more general situations.

Meanwhile, Maltz [11] stated that the leaves of the minimum nullity distribution in the sense of [2] are complete if the considered manifold is complete. As a further extension of the notion, A. Gray [7] recently studied the nullity distributions of curvature-like tensors and showed completeness of the leaves of the minimum nullity distribution under the conditions that the curvature-like tensors

(*) This paper is a portion of the author's doctoral dissertation at Brown University under the direction of Professor K. Nomizu ; during its preparation, he held National Science Foundation Assistantships.

(**) The author has the complex version of this theorem, see [1].

are recurrent and the considered manifold is complete. In this case, also, the problem about completeness of the leaves of the minimum nullity distribution remains open for more general situations.

It is the purpose of this work to establish the completeness in the above two cases and to give some applications to submanifolds in space forms.

In Chapter I, we shall define a certain curvature-like tensor D relative to a real number k on a Riemannian manifold M, and shall call it the relative curvature tensor with constant k. Especially, if $k=0$, then D will be the curvature tensor of M as in [2] and [11]. Also if M is a submanifold of a space form $\widetilde{M}$ with constant sectional curvature c, then the difference of the curvature operators on M and $\widetilde{M}$ considered as a tensor on M will give us a typical example of D with constant c.

With D, we shall prove our first main result, THEOREM 1.4.1, describing completeness of the leaves induced from the minimum nullity distribution of D, when M is complete. As an application of THEOREM 1.4.1. we shall prove THEOREM 1.6.2, which, incidentally, gives a partial answer to Gray's conjecture in [7].

Now let M^n be a complete Riemannian submanifold of a space form $\widetilde{M}(c)$ of constant sectional curvature c. Then applying to the second fundamental form quite similar arguments to those in the proof of THEOREM 1.4.1, we shall get THEOREM 1.8.1, which tells us completeness of the leaves of the minimum relative nullity distribution.

In Chapter II, we shall treat the complex analogues of the results in Chapter I under a slightly different definition of complex relative curvature tensors on a Kählerian manifold. Our main results in this chapter are described as THEOREMS 2.2.1 and 2.3.1.

Chapters III and IV will be devoted to an application of the results obtained in Chapters I and II to submanifolds in space forms.

We shall genralize a theorem by Nomizu [13] on a characterization of totally geodesic Kählerian hypersurfaces in the complex projective space of constant holomorphic sectional curvature 1. Here his proof, although elegant, heavily depends on algebraic geometrical results such as Chow's theorem and is applicable only to compact imbedded hypersurfaces. In Chapters III and IV, we shall develop a more differential geometric method, which therefore will be applicable not only to the complex case but also to the real case under more general situations. Our main results in Chapters III and IV will be stated in THEOREMS 3.2.1, 3.2.2, 4.2.1, 4.2.2, and 4.2.3. The author would like to express his sincere gratitude to his advisor, Professor K. Nomizu, for his help and encouragement during the preparation of this work. The author also thanks those people who gave encouragement during his stay at Tôhoku University.

Chapter 1. Completeness of the leaves of the nullity and relative nullity distributions.

This chapter will be devoted to the Riemannian case. However, some notations and definitions to be introduced here will be used, throughout this paper, in the Kählerian case which will be treated later as well. We shall follow Kobayashi-Nomizu [10] for most of the fundamental notations, to which we shall refer as K-N in the following chapters.

1.1. Curvature-like tensor fields and their indexes of nullity. Let M be a Riemannian manifold with the metric tensor g and let ∇, TM and TM_x be the Riemannian connection, the tangent bundle and the tangent space at x, respectively.

DEFINITION 1.1.1. *A tensor field T of type $(1,3)$ is curvature-like if it has the following properties:*

(1.1.1) *$T(X,Y)$ is skew symmetric endomorphism of TM;*
(1.1.2) $T(X,Y)=-T(Y,X)$;
(1.1.3) *The first Bianchi identity, i.e.,* $\underset{X,Y,Z}{\mathfrak{S}} T(X,Y)Z=0$;
(1.1.4) *The second Bianchi identity, i.e.,* $\underset{X,Y,Z}{\mathfrak{S}} (\nabla_X T)(Y,Z)=0$;
(1.1.5) $g(T(X,Y)Z,W)=g(T(Z,W)X,Y)$.

Here X,Y,Z and W are in TM and $\underset{X,Y,Z}{\mathfrak{S}}$ means the cyclic sum over X,Y and Z.

DEFINITION 1.1.2. *For any point x in M, the subspace defined by $TN(x) = \{X\in TM_x : T(X,Y)=0$ for all Y in $TM_x\}$ is called the nullity space of T at x, and its dimension is defined to be the nullity of T at x, say $\mu(x)$.*

Under Definition 1.1.2., $\mu(x)$ becomes an integer-valued function and is upper semi-continuous. Thus the subset of M, say G, where $\mu(x)$ assumes the minimum is open.

DEFINITION 1.1.3. *The minimum of $\mu(x)$ in M is called the index of nullity of T and is denoted by $\mu=\mu(T,M)$.*

1.2. The nullity distributions of the curvature-like tensors. In Section 1.1, we defined the index of nullity μ and an open set G where $\mu(x)=\mu$, i.e., the nullity is minimum.

DEFINITION 1.2.1. *Let TN be the distribution on G which assigns $TN(x)$ to x. TN is called the nullity distribution of T on M.*

The following propositions are known, for example, see A. Gray [6] or K-N [10].

PROPOSITION 1.2.1. *The nullity distrition TN is of* C^∞ *and involutive.*

By Proposition 1.2.1, we can speak of the integral manifolds of TN on G. The leaves of this foliation will be called the leaves of nullity.

PROPOSITION 1.2.2. *The leaves of nullity are totally geodesic submanifolds of M.*

1.3. Relative curvature tensor field with constant K as an example of curvature-like tensors. In Section 1.1, we introduced the notion of curvature-like tensor. The most typical example is the curvature tensor field of M itself, from which the notion comes. Another interesting example will be given in the following definition, which we shall mainly investigate in Chapter I and Chapter III.

DEFINITION 1.3.1. *Let R be the curvature tensor of M. Define a new tensor field D of type* $(1,3)$ *by*

$$D(X,Y) = R(X,Y) - kX \wedge Y, \tag{1.3.1}$$

where $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ and k is a real number. Call D the relative curvature tensor of M with constant k. Especially, R is the relative curvature tensor of M with constant 0.

It is just a matter of verification to check the following proposition.

PROPOSITION 1.3.1. *D is a curvature-like tensor on M.*

REMARK 1.3.1. Let $\widetilde{M}$ be a Riemannian space form of constant sectional curvature c with $\widetilde{R}(X,Y)=cX\wedge Y$ as its curvature tensor. Let M be a Riemannian submanifold of $\widetilde{M}$ with R as its curvature tensor. Then $D(X,Y)=R(X,Y) - \widetilde{R}(X,Y) = R(X,Y) - cX\wedge Y$ for X and Y in TM is a natural and significant example of the relative curvature tensor of M with constant c.

1.4. Completeness of the leaves of nullity for the relative curvature tensors. Now we shall state our first main theorem. Let DN be the nullity distribution of D which assigns $DN(x)$, the nullity space at x, to x. Also, we denote by μ the index of nullity for D.

THEOREM 1. 4. 1.(*) *Under the notations mentioned in the previous sections, if M is complete, then the leaves of nullity for the relative curvature tensors are complete.*

In the case $k=0$, i. e., $D=R$, R. Maltz stated completeness of the leaves of nullity, see [11]. A. Gray [7] has a similar theorem when D is recurrent, i. e., there exists a 1-form ω on M such that $\nabla_x D=\omega(X)D$ for all X in TM.

The fundamental idea in our proof is similar to that of Maltz [11]. However, in our case, the following lemmas, which are not mentioned in [11], are essential.

We begin our proof with some preparations.

Let L be a leaf in G and p be a point in L. Consider a unit speed geodesic, say $\bar{\gamma}(t)$, in L. Since L is totally geodesic in M, $\bar{\gamma}(t)$ can be regarded as a geodesic of M and can be extended infinitely in M.

Define $p_*=\bar{\gamma}(s_*)$ to be the point in M such that for any s, $0\leqq s<s_*$, $\bar{\gamma}([0,s])$ is contained in L but p_* itself is not. If p_* is in G, then by continuity of D and by the argument similar to that in the proof of Lemma 2, p. 86, K-N [10], p_* is actually in L. Thus we can assume that p_* is not in G. Under this assumption, our aim is to show a contradiction.

REMARK 1. 4. 1. We may assume that there exists a point in L and a geodesic starting at that point in L such that there is a point in the geodesic which corresponds to the above p_* and satisfies the above conditions characterizing p_*. Because, if not, then every geodesic in L is extendable infinitely in L, i. e., L is complete.

Let $B(p_*,\varepsilon)$ be an ε-ball with p_* as its center such that for any x in $B(p_*,\varepsilon)$, $\mathrm{Exp}_x: TM_x\to M$ gives a diffeomorphism of the 2ε-ball in TM_x with its image that is contained in a normal convex neighborhood of x. It is possible to choose such an ε by well known arguments in Riemannian geometry.

Notice that the exponential maps at points in $B(p_*,\varepsilon)$ restricted to the leaves are nothing but the exponential maps with respect to the leaves on the neighborhood, since leaves are totally geodesic.

For convenience, let us take another fixed point, say q, in $\bar{\gamma}(t)\cap L\cap B(p_*,\varepsilon)$ and reparametrize $\bar{\gamma}$ to get a new unit speed geodesic $\gamma(t)$ such that $\gamma(0)=q$ and $\gamma(t_*)=p_*$ for some t_*.

LEMMA 1. 4. 1. *There exists a Frobenius coordinate system on a neigh-*

(*) After the author completed this work, his attention was called to the following paper : The K-Nullity Spaces of the Curvature Operator, by Yeaton H. Clifton and Robert Maltz, Mich. Math. Jour. Vol. 17(1970), pp. 85–89. Its main result is the same as Theorem 1. 4. 1. However, the author would like to point out that their definition of $[X_\alpha, X_\beta]$ on p. 88 is not clear, and that there is also the same lack of details in [11] by Maltz.

borhood U of q, say $(U; x^1, \cdots, x^n; \zeta)$, such that $\zeta(q)=(0, \cdots, 0)$ in R^n, $\partial/\partial x^1, \cdots, \partial/\partial x^n$ are orthonormal at q and finally, its first μ coordinates are those of slices by the leaves of nullity, where μ is the index of nullity, i.e., the dimension of the leaves.

PROOF. A routine construction by Gram-Schmidt orthonormalization process which starts with a Frobenius coordinate system around q. Q. E. D.

From now on, we assume that U which is constructed in Lemma 1. 4. 1. is contained in $G \cap B(p_*, \varepsilon)$, for convenience.

Let Σ be the transversal slice determined by $x^1 = \cdots = x^\mu = 0$, where $(U, x^1, \cdots, x^n, \zeta)$ is the local coordinate system in Lemma 1. 4. 1.

Let $E_1, \cdots, E_\mu$ be μ orthonormal vector fields in DN on Σ such that $E_1(q) = \dot{\gamma}(q)$, i. e., the velocity vector of γ at q.

Denote by ϕ the restriction of ζ to Σ. Then ϕ gives a diffeomorphism of Σ with a neighborhood W of the origin $\overbrace{(0, \cdots, 0)}^{n-\mu}$ in $R^{n-\mu}$. Define a C^∞-mapping $F: R^\mu \times W \to M$ by

$$F((t^1, \cdots, t^\mu), (x)) = \mathrm{Exp}_{\phi^{-1}(x)}\left(\sum_{i=1}^{\mu} t^i E_i(\phi^{-1}(x))\right), \tag{1. 4. 1}$$

where $(x)=(x^{\mu+1}, \cdots, x^n)$ is a point in $R^{n-\mu}$ such thnt $\zeta \circ \phi^{-1}(x) = (0, \cdots, 0, x^{\mu+1}, \cdots, x^n)$ in the coordinate system constructed in Lemma 1. 4. 1.

REMARK 1. 4. 2. We can prove that F is of C^∞ as in Nomizu [11].

The following lemma is most crucial in the proof of Theorem 1. 4. 1.

LEMMA 1. 4. 2. *Let H be the subset $\{(\overbrace{t^1, 0, \cdots, 0}^{\mu}, \overbrace{0, \cdots, 0}^{n-\mu})$ in $R^\mu \times W$: $0 \leqq t^1 < t_*\}$ of $R^\mu \times W$. Then F is regular on H except possibly at finitely many points.*

PROOF. Let $N_1, \cdots, N_n$ be the natural rectangular vector fields in $R^\mu \times W$ as a subset of $R^\mu \times R^{n-\mu} = R^n$.

First of all, we recall that the exponential map of M at q restricted to the tangent subspace to L at q is nothing but the exponential map of L at q, since L is totally geodesic.

By the definition of the mapping F and the fact mentioned above, the first μ natural vectors $N_1(h), \cdots, N_\mu(h)$ at h in H are mapped upon linearly independent vectors $F_*(h)(N_1(h)), \cdots, F_*(h)(N_\mu(h))$ which are tangent to L on the fixed geodesic γ.

Now in order to prove regularity of F on H, it is sufficient to show that the normal components to L of $F_*(h)(N_{\mu+1}(h)), \cdots, F_*(h)(N_n(h))$ are linearly independent at h in H except possibly at finitely many points.

Let $H_\alpha, \alpha=\mu+1, \cdots, n$, be the subset of $R^\mu \times W$ given by $\{(\overbrace{t^1, 0, \cdots, 0,}^{\mu} \overbrace{0, \cdots, x^\alpha, 0, \cdots, 0}^{n-\mu})$ in $R^\mu \times W\}$, where x^α occurs in the α-th component in $R^\mu \times W \subset R^n$. Let $V_\alpha, \alpha=\mu+1, \cdots, n$, be the restriction of F to H_α. Then for each $\alpha, V_\alpha(t, x)$ defines a geodesic variation along the geodesic $\gamma(t)=V_\alpha(t, 0)$. By well known arguments in variation theory, such variations induce Jacobi fields along γ. Now let us denote by $X_\alpha(t)$ the associated Jacobi field for each α. A well known theorem tells us, see [10],

$$\nabla^2_{\dot\gamma(t)} X_\alpha(t) + R(X_\alpha, \dot\gamma(t))\dot\gamma(t) = 0, \quad \alpha=\mu+1, \cdots, n. \tag{1.4.2}$$

Notice that if $t=0$, then $X_\alpha(0)=$ velocity vector of the α-th coordinate curve, i.e., $\partial/\partial x^\alpha(q)$. Thus from this, we know that one of the initial conditions of (1.4.2) for X_α is $X_\alpha(0)=\partial/\partial x^\alpha(q)$.

Since $\dot\gamma(t)$ is in $DN(\gamma(t))$ for $0 \leqq t < t_*$,

$$D(X_\alpha, \dot\gamma)\dot\gamma = R(X_\alpha, \dot\gamma)\dot\gamma - k(X_\alpha \wedge \dot\gamma)\dot\gamma = 0 .$$

Thus the equation (1.4.2) can be rewritten as

$$\nabla^2_{\dot\gamma(t)} X_\alpha + k(X_\alpha \wedge \dot\gamma)\dot\gamma = 0 , \tag{1.4.3}$$

i.e.,

$$\nabla^2_{\dot\gamma} X_\alpha + k\{g(\dot\gamma, \dot\gamma)X_\alpha - g(X_\alpha, \dot\gamma)\dot\gamma\} = 0 . \tag{1.4.4}$$

For convenience, let us introduce a parallel orthonormal adapted frame field on $\gamma(t)$, say $e_1(t), \cdots, e_\mu(t), e_{\mu+1}(t), \cdots, e_n(t)$, defined as follows: The first μ vectors e_i's are given by displacing E_i's parallelly along $\gamma(t)$ and e_α's are given by displacing $\partial/\partial x^\alpha(q)$ parallelly along $\gamma(t)$. Then, of course, $e_1(t)=\dot\gamma(t)$. Also note that the parallel transformation sends an adapted frame to an adapted frame in this case, because L is totally geodesic.

By the frame obtained as above, let $X_\alpha(t)$ be expressed as follows:

(1. 4. 5) $$X_\alpha(t)=\sum_{i=1}^{n} x_\alpha^i(t)e_i(t).$$

Then (1. 4. 4) is

(1. 4. 6) $$\nabla^2_{\dot\gamma(t)}\left\{\sum_{i=1}^{n} x_\alpha^i(t)e_i(t)\right\}+k\left\{\sum_{i=1}^{n} x_\alpha^i(t)e_i(t)-x_\alpha^1(t)e_1(t)\right\}=0.$$

So by performing covariant differentiation,

(1. 4. 7) $$\sum_{i=1}^{n} e_1^2(t)(x_\alpha^i(t))\cdot e_i(t)+k\left\{\sum_{i=1}^{n} x_\alpha^i(t)e_i(t)-x_\alpha^1(t)e_1(t)\right\}=0.$$

Thus we have the following system of differential equations for each $\alpha=\mu+1,\cdots,n$:

(1. 4. 8) $$(x_\alpha^1)''(t)=0$$

(1. 4. 9) $$(x_\alpha^i)''(t)+kx_\alpha^i(t)=0 \quad \text{for} \quad 2\leqq i\leqq n.$$

Since we are only interested in linear independence of the normal components of X_α's to L, it is sufficient to take care of (1. 4. 9) for $\mu+1\leqq i\leqq n$.

By elementary theory for linear differential equations, (1. 4. 9) has the following general solutions for the cases $k>0$, $k=0$ and $k<0$, respectively:

For $k>0$

(1. 4. 10) $$x_\alpha^\beta(t)=a_\alpha^\beta\cos\sqrt{k}\,t+b_\alpha^\beta\sin\sqrt{k}\,t,\ \mu+1\leqq\alpha,\beta\leqq n,$$

where a_α^β's and b_α^β's are independent of t, but depend on the initial conditions.

Now one of our initial conditions is $x_\alpha^\beta(0)=\delta_\alpha^\beta$, where δ_α^β is Kronecker's delta.

Hence (1. 4. 10) has more simplified form as follows:

(1. 4. 11) $$x_\alpha^\beta(t)=\delta_\alpha^\beta\cos\sqrt{k}\,t+b_\alpha^\beta\sin\sqrt{k}\,t,\quad \mu+1\leqq\alpha,\beta\leqq n.$$

Similarly, for $k=0$,

(1. 4. 12) $$x_\alpha^\beta(t)=\delta_\alpha^\beta+b_\alpha^\beta t,\quad \mu+1\leqq\alpha,\beta\leqq n.$$

For $k<0$,

(1. 4. 13) $$x_\alpha^\beta(t)=\delta_\alpha^\beta\sinh\sqrt{-k}\,t+b_\alpha^\beta\cosh\sqrt{-k}\,t,\quad \mu+1\leqq\alpha,\beta\leqq n.$$

Note that these solutions have only finitely many zeroes on $[0, t_*]$.

From now on to the end of this proof, we shall study the case $k>0$. The cases $k=0$ and $k<0$ can be proved in the same fashion.

Since X_α^β's, $\mu+1\leqq\alpha,\beta\leqq n$, are the normal components of X_α's, $\mu+1\leqq\alpha,\beta\leqq n$, to L, it is sufficient to show that the following $(n-\mu)\times(n-\mu)$ matrix, say $B(t)$, has non-zero determinent on H except possibly at finitely many points:

$$(1.4.14)\quad B(t)=\begin{pmatrix} \cos\sqrt{k}\,t+b_{\mu+1}^{\mu+1}\sin\sqrt{k}\,t & \cdots\cdots & b_n^{\mu+1}\sin\sqrt{k}\,t \\ \vdots & & \vdots \\ b_{\mu+1}^n\ \sin\sqrt{k}\,t & \cdots\cdots & \cos\sqrt{k}\,t+b_n^n\sin\sqrt{k}\,t \end{pmatrix}.$$

As we see easily, the determinant of $B(t)$ is a real-valued analytic function of t. Since the determinant of $B(0)=1\neq 0$, we have only finitely many zeroes of $B(t)$ on $[0, t_*]$. This fact shows that the normal components of X_α's, $\mu+1\leqq\alpha\leqq n$, to L are linearly independent except at finitely many points possibly.

To conclude the proof of Lemma 1.4.2, we recall that we showed that $F_*N_1,\cdots,F_*N_\mu$ are linearly independent on H and are tangent to L. By the definition of X_α's, we know that $X_\alpha(\gamma(t))=F_*(t)(N_\alpha(t))$, $\mu+1\leqq\alpha\leqq n$; so together with the above fact, we have shown that $F_*(h)(N_1(h)),\cdots,F_*(h)(N_n(h))$ are linearly independent on H except possibly at finitely many points. Q.E.D.

Coming back to the proof of Theorem (1.4.1), in Lemma (1.4.2), we showed that only finitely many singular points possibly exist on H. Let h_* be the greatest of the first coordinate in $R^\mu\times W$ among such singular points in H. Then there exists an open neighborhood N of the set $\{(t,0,\cdots,0)$ in H: $h_*<t<t_*\}$, say H', where the rank of F_* is n constantly by Lemma 1.4.2 and lower semi-continuity of the rank of F_*, namely, F restricted to N gives an immersion from N to M.

By the inverse function theorem, at any point x in H', we have a neighborhood N_x where F becomes a diffeomorphism.

Since $R^\mu\times W$ has the canonical coordinate frame $N_1,\cdots,N_n$ which are induced from those in $R^\mu\times R^{n-\mu}=R^n$, we can introduce a frame field, say $\partial/\partial x^1,\cdots,\partial/\partial x^n$, on $F(N_x)$ by $(F*|N_x)(N_1)=\partial/\partial x_1,\cdots,(F_*|N_x)(N_n)=\partial/\partial x^n$ such that the first μ vector fields $\partial/\partial x^1,\cdots,\partial/\partial x^\mu$ are tangent to leaves in $N_x\cap G$.

Note here that by taking N_x small enough, we can assume $N_x\cap G=N_x$, so from now on, we shall always consider such N_x at each x.

Let X be any parallel vector field along $\gamma(t)$. We extend X to a vector field on a neighborhood of the set $\gamma([h_*,t_*])$ and denote it by the same letter X for convenience.

Our next aim is to show $\nabla_{\dot\gamma(t)}\{D(X_\alpha,X_\beta)X\}=0$, i.e. $D(X_\alpha,X_\beta)X$ is parallel along $\gamma((h_*,t_*))$ for $\alpha,\beta=\mu+1,\cdots,n$.

Since X_α, X_β, X and $D(X_\alpha, X_\beta)X$ are all defined on $\gamma([h_*, t_*])$ and since $D(X_\alpha, X_\beta)X$ is a tensor field along γ, it does not depend on a local extension of the considered vector fields.

Expecially, for each $N_x, X_\alpha(\gamma(t)) = F_*(N_\alpha | N_x) = \partial/\partial x^\alpha(\gamma(t))$ on $H \cap N_x$ for $\alpha = \mu+1, \cdots, n$, so to prove that $D(X_\alpha, X_\beta)X$ is parallel along $\gamma((h_*, t_*))$, it suffices to show that $D(\partial/\partial x_\alpha, \partial/\partial x_\beta)X$ is parallel along γ on each $F(N_x)$.

By the second Bianchi identity (1.1.4), we have

$$\underset{\partial/\partial x_1, \partial/\partial x_\alpha, \partial/x_\beta}{\mathfrak{S}} (\nabla_{\partial/\partial x^1} D)(\partial/\partial x_\alpha, \partial/\partial x_\beta) = 0\,, \tag{14.15}$$

i. e.,

$$\begin{aligned}
&\nabla_{\partial/\partial x^1}(D(\partial/\partial x^\alpha, \partial/\partial x^\beta)X) - D(\nabla_{\partial/\partial x^1}\partial/\partial x^\alpha, \partial/\partial x^\beta)X - D(\partial/\partial x^\alpha, \nabla_{\partial/\partial x^1}\partial/\partial x^\beta)X \\
&\quad - D(\partial/\partial x^\alpha, \partial/\partial x^\beta)\nabla_{\partial/\partial x^1}X + \nabla_{\partial/\partial x^\alpha}(D(\partial/\partial x^\beta, \partial/\partial x^1)X) \\
&\quad - D(\nabla_{\partial/\partial x^\alpha}\partial/\partial x^\beta, \partial/\partial x^1)X - D(\partial/\partial x^\beta, \nabla_{\partial/\partial x^\alpha}\partial/\partial x^1)X \\
&\quad - D(\partial/\partial x^\beta, \partial/\partial x^1)\nabla_{\partial/\partial x^\alpha}X + \nabla_{\partial/\partial x^\beta}(D(\partial/\partial x^1, \partial/\partial x^\alpha)X) \\
&\quad - D(\nabla_{\partial/\partial x^\beta}\partial/\partial x^1, \partial/\partial x^\alpha)X - D(\partial/\partial x^1, \nabla_{\partial/\partial x^\beta}\partial/\partial x^\alpha)X \\
&\quad - D(\partial/\partial x^1, \partial/\partial x^\alpha)\nabla_{\partial/\partial x^\beta}X = 0\,.
\end{aligned}$$

Since $\partial/\partial x^1$ is in DN in $F(N_x)$, we have $D(\partial/\partial x^i, \partial/\partial x^1) = 0$ for $0 \leqq i \leqq n$. Also the choice of X gives that $\nabla_{\partial/\partial x^1}X = 0$ on $\gamma(t)$. Finally, by the facts that $D(X, Y) = -D(Y, X)$ and that $\nabla_{\partial/\partial x^i}\partial/\partial x^j - \nabla_{\partial/\partial x^j}\partial/\partial x^i = [\partial/\partial x^i, \partial/\partial x^j] = F_*([N_i, N_j]) = 0$ for $1 \leqq i, j \leqq n$, on $F(N_x)$, we can reduce the above equation to

$$\nabla_{\partial/\partial x^1}(D(\partial/\partial x^\alpha, \partial/\partial x^\beta)X) = 0 \text{ on } \gamma((h_*, t_*)) \cap F(N_x)\,,$$

i. e.,

$$\nabla_{\dot\gamma(t)}(D(X_\alpha, X_\beta)X) = 0 \text{ for } \mu+1 \leqq \alpha, \beta \leqq n\,, \tag{1.4.16}$$

i. e., $D(X_\alpha, X_\beta)X$ is parallel on $\gamma((h_*, t_*))$ for $\mu+1 \leqq \alpha, \beta \leqq n$.

Let t be a fixed point chosen from (h_*, t_*) and let Y be any vector field parallel along γ.

Now let $Y(\gamma(t_*))$ be in $DN(t_*)$. Then we claim that $D(V, W)Y(\bar t) = 0$ for all V and W in $TM_{\gamma(\bar t)}$, i. e., $Y(\bar t)$ is in $DN(\bar t)$. This last assertion follows from (1.1.5) directly if we know $D(V, W)Y(t) = 0$ for any V and W in TM.

Let $V = V' + \sum_{\alpha=\mu+1}^{n} V^\alpha X_\alpha$ and $W = W' + \sum_{\beta=\mu+1}^{n} W^\beta X_\beta$, where V' and W' are DN

components of V and W, respectively. Then

$$D(V,W)Y(\bar{t}) = D(V',W')Y(\bar{t}) + D\left(V', \sum W^\beta X_\beta\right)Y(\bar{t}) + D\left(\sum V^\alpha X_\alpha, W\right)Y(\bar{t})$$

$$+ D\left(\sum V^\alpha X_\alpha, \sum W^\beta X_\beta\right)Y(\bar{t}).$$

Here the first three terms vanish. On the other hand, if one of $D(X_\alpha, X_\beta)Y(t) \neq 0$, then by (1.4.16) and continuity of D, we would have $D(X_\alpha(t_*), X_\beta(t_*))Y(t_*) \neq 0$. Thus we have a contradiction that $Y(t_*)$ is in $DN(t_*)$. Thus we have shown that $D(V,W)Y(t)=0$ for any V and W in TM.

This fact tells us that every $Y(t_*)$ in $DN(t_*)$ is the image of a vector $Y(t)$ in $DN(t)$ by the parallel displacement along γ.

Since the parallel displacement is an isomorphism between the two tangent spaces, the dimension of $DN(t_*)$ is less than or equal to the dimension of $DN(t)$. However, the dimension of $DN(t)=\mu$, the index of nullty of D, is supposed to be the minimum among the nullity $\mu(x)$ corresponding to every x in M. Consequently, the dimension of $DN(t_*)$ must be equal to that of $DN(t)$. Thus $\gamma(t_*)=p_*$ is in G by its definition. Then by the argument described just below the statement of Theorem 1.4.1, we can show that p_* is in L and can extend γ in an eighborhood of p_*.

It is a contradiction to the choice of γ and p_*. Q.E.D.

1.5. Riemannian submanifolds in Riemannian manifolds. Let $\widetilde{M}^N$ be an N-dimensional Riemannian manifold with $\tilde{g}$, $\widetilde{\nabla}$ and $\widetilde{R}$ as its Riemannian metric, Riemannian connection and Riemannian curvature, respectively. Let M^n be an n-dimensional Riemannian submanifold of $\widetilde{M}^N$ isometrically immersed by f with g, ∇ and R as its Riemannian metric, Riemannian connection and Riemannian curvature, respectively.

Denote by $\xi_1, \cdots, \xi_{N-n}$ a normal frame to M^n orthonormal to each other in $\widetilde{M}^N$. Then we have the following basic formulas, see K-N [10]:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{N-n} h_i(X,Y)\xi_i = \nabla_X Y + \alpha(X,Y); \tag{1.5.1}$$

$$\widetilde{\nabla}_X \xi_i = -A_i X + \widehat{\nabla}_X \xi_i, \qquad 1 \leqq i \leqq N-n; \tag{1.5.2}$$

where X and Y are vector fields tangent to M, $\widehat{\nabla}$ is the normal connection, and α is the second fundamental form.

It is just a simple verification to see that h_i's are symmetric bilinear forms and A_i's are tensor fields of type (1, 1). Moreover, we have that $h_i(X, Y) = g(A_iX, Y)$, $1 \leqq i \leqq N-n$.

Finally, under the notations introduced here, we have Gauss equation:

$$(1.5.3)\quad R(X, Y)Z = \text{tangential component of } \widetilde{R}(X, Y)Z + \sum_{i=1}^{N-n} (A_iX \wedge A_iY)Z,$$

where X, Y and Z are vector fields tangent to M and $A_iX \wedge A_iY$, for $1 \leqq i \leqq N-n$, means the linear operator of TM defined by $(A_iX \wedge A_iY)(Z) = g(A_iY, Z)A_iX - g(A_iX, Z)A_iY$ for Z in TM.

1.6. Some direct results of Theorem 1.4.1. Let L be a leaf of nullity of D in Section 1.4. Consider L as a Riemannian submanifold of M with the induced Riemannian structure, say g_L, ∇_L and R_L, respectively, by those of M. Then we have

THEOREM 1.6.1. *L has constant sectional curvature k. Especially, if $k > 0$, then L has the standard sphere of radius $\frac{1}{\sqrt{k}}$ as its universal covering manifold. Conequently, L is compact.*

PROOF. The first half is trivial by Gauss equation (1.5.3.) in Section 1.5.

For the rest of the proof, since L is complete by Theorem 1.4.1 and is of constant sectional curvature k, it is well known that L has a complete universal covering manifold of constant curvature k. Especially, if $k > 0$, then such universal coverings are isometric to the standard μ-dimensional sphere with radius $\frac{1}{\sqrt{k}}$ and L is compact. Q. E. D.

T. Frankel proved the following in [5]:

LEMMA 1.6.1 (T. Frankel). *Let M^n be a complete connected manifold with positive sectional curvature and let V^l and W^m be compact totally geodesic submanifolds of M with dimensions l and m, respectively. Then if $l + m \geqq n$, then V^l and W^m have non-empty intersection.*

Thus Lemma 1 6.1 leads us to

THEOREM 1.6.2. *Let M^n be a complete connected Riemannian manifold with positive sectional curvature. If the index of nullity μ of the relative curvature with constant k satisfies $2\mu \geqq n$, then M^n has constant sectional* *curvature k and also has the n-dimensional standard sphere $S^n(k)$ of radius $\frac{1}{\sqrt{k}}$ as its universal covering manifold.*

PROOF. Let $2\mu \geqq n$, but $\mu \neq n$. Then there exist two leaves different from each other, but Lemma 1.6.1 tells us that any such leaves must intersect each other. This is a contradiction.

So only one possible case is that $\mu = n$. Since we know that L is complete and M is connected, so $L = M$. Using the same argument as in the proof of Theorem 1.6.1, we have the desired result. Q.E.D.

COROLLARY 1.6.1. *Under the assumption in Theorem 1.6.2, if, in addition, n is even, then $M^n = S^n$ or RP^n, where S^n is the n-dimensional sphere of radius $\frac{1}{\sqrt{k}}$ and RP^n is the projective space given by identifying antipodal points in S^n above, which also has constant sectional curvature k.*

PROOF. See p. 294 in [10] combined with Theorem 1.6.2.

1.7. The relative nullity and the index of relative nullity. Chern and Kuiper in [2] introduced these notions to study differential geometry of submanifolds imbedded in Euclidean spaces.

The following definitions are extensions of those given in [2] to more general cases.

As in Section 1.5, let M^n be a Riemannian submanifold of a Riemannian space form of constant sectional curvature c. Then we have the bilinear form $\alpha_x: TM_x \times TM_x \to TM_x^{\perp}$ at each x in M and symmetric tensors A_ξ for any ξ in $TM_x^{\perp}$, for more details, see K-N [10].

DEFINITION 1.7.1. *The relative nullity space at x is defined to be the subspace $\{X \in TM_x : \alpha(X, Y) = 0$ for all Y in $TM_x\}$. Denote it by $RN(x)$.*

It is easy to check the following proposition.

PROPOSITION 1.7.1. *The following three subspaces of TM_x are same:*

(1.7.1) $RN(x) = \{X \in TM : \alpha(X, Y) = 0$ *for all* $Y \in TM_x\}$;

(1.7.2) $\{X \in TM_x : A_\xi(X) = 0$ *for all* $\xi \in TM_x^{\perp}\}$;

(1.7.3) $\{X \in TM_x : A_{\xi_i}(X) = 0$, *where* ξ_i*'s*, $1 \leqq i \leqq N-n$, *form an orthonormal base of* $TM_x^{\perp}\}$.

DEFINITION 1.7.2. *$\nu(x)$ is defined to be* dim $RN(x)$ *and is called the relative nullity at x. The minimum among $\nu(x)$ over M is called the index of relative nullity of the submanifold M.*

REMARK 1.7.1. Gauss equation (1.5.3) shows some relation between the nullity space and the relative nullity space in the followiug sense.

Let $\widetilde{M}$ be a space form of sectional curvature c and let M be a Riemannian submanifold of $\widetilde{M}$. Then as in Section 1.3, we can define a relative nullity curvature tensor with constant c by $D(X,Y)=R(X,Y)-\widetilde{R}(X,Y)$ for X and Y in TM.

By (1.5.3), we have $DN(x)\supset RN(x)$. However, the equality may not hold. For example, let $\widetilde{M}$ be R^3 and let M be the cylinder $\{(x,y,z)\in R^3 : y^2+z^2=1\}$. Then, as we know, $R(X,Y)=0$ and $\widetilde{R}(X,Y)=0$, therefore $DN=TM$, but RN are generators.

1.8. Completeness of the relative nullity distribution. Denote by G the set $\{x\in M\colon \nu(x)=\nu\}$. Then by upper semicontinuity of ν, G is open. Define the relative nullity distribution RN by assigning $RN(x)$ to x in G and call it the relative nullity distribution of M.

PROPOSITION 1.8.1. *The relative nullity distribution RN is of C^∞, involutive and each leaf, say L, of the foliation is totally geodesic in M and $\widetilde{M}$.*

PROOF. This proposition is well known. One can refer to [2] for the proof.
Q. E. D.

THEOREM 1.8.1.* *If M is complete, then the leaves of the foliation defined by RN are complete and have the same constant sectional curvature as the ambient space form under the induced Riemannian structure from that of M.*

[8] showed Theorem 1.8.1 when M^n is a submanifold in the Euclidean space with an arbitrary codimension. For non-zero c, O'Neill and Stiel [16] have this theorem under the restriction that M^n also has the same constant sectional curvature as that of the ambient space form.

PROOF. Let $\gamma(t)$ be the geodesic in a leaf L as in the proof of Theorem 1.4.1, where $\gamma([0,t_*))$ is in G, but $\gamma(t_*)=p_*$ is not.

(*) After submitting this paper, the author was informed that R. Maltz has the same result as Theorem 1.8.1

Define $(U, x^1, \cdots, x^\nu, x^{\nu+1}, \cdots, x^n, \zeta)$ be a Frobenius coordinate system around $\gamma(0)=p$ as in Lemma 1.4.1 with respect to the relative nullity distribution. Again as in Lemma 1.4.2, define $F: R^\nu \times W \to M^n$ by

$$F((x^1, \cdots, x^\nu, (y))) = \mathrm{Exp}_{\phi^{-1}(y)}\left(\sum_{i=1}^{\nu} x^i e_i(\phi^{-1}(y))\right),$$

where W is the transversal slice defined by $\zeta^{-1}(W) = \{x \in M: \zeta(x) = (\overbrace{0, \cdots, 0}^{\nu}, x^{\nu+1}, \cdots, x^n\}$ and ϕ is the restriction of ζ to $\zeta^{-1}(\{(0, \cdots, 0, x^{\nu+1}, \cdots, x^n)\})$, and e_i's, $1 \leqq i \leqq \nu$, are vector fields on $\zeta^{-1}(W)$ such that $e_1(p) = \gamma(0)$ and e_i's, $1 \leqq i \leqq \nu$, form orthonormal base of RN at each point on $\zeta^{-1}(W)$.

By Gauss equation (1.5.3), for any X in RN and Y in TM,

$$R(X, Y) = \widetilde{R}(X, Y) + \sum_{i=1}^{N-n} A_i X \wedge A_i Y.$$

Since X is in RN and $\widetilde{M}$ is a space form of sectional curvature c,

$$R(X, Y) = c(X \wedge Y).$$

Thus let X_α, $\nu+1 \leqq \alpha \leqq n$, be the variation vector field along $\gamma(t)$ as in the proof of Lemma 1.4.2. Then we have

$$\nabla^2_{\dot\gamma(t)} X_\alpha(t) + c(X_\alpha \wedge \dot\gamma(t))\dot\gamma(t) = 0.$$

By the same arguments as in the proof of Lemma 1.4.2, we have a parameter value, say h_*, of γ such that F is regular on the set $H = \{(t, 0, \cdots, 0) \in R^\nu \times W: h_* < t < t_*\}$.

Let $N_1, \cdots, N_\nu, N_{\nu+1}, \cdots, N_n$ be the canonical rectangular coordinate fields of $R^\nu \times W \subset R^n$ such that $X_\alpha(\gamma(t)) = F_*(N_\alpha(t, 0, \cdots, 0))$, $\nu+1 \leqq \alpha \leqq n$. Moreover, we have a neighborhood N_x of x in H such that $\partial/\partial x^1 = F_*(N_1), \cdots, \partial/\partial x^n = F_*(N_n)$ form coordinate fields on $F(N_x)$, i.e., $\partial/\partial x^1, \cdots, \partial/\partial x^n$ are linearly independent on $F(N_x)$ and $[\partial/\partial x^i, \partial/\partial x^j] = 0$ for all $1 \leqq i, j \leqq n$.

For the proof of Theorem 1.8.1, we claim that $\widetilde{\nabla}_{\dot\gamma(t)}(\alpha(X_\alpha, Y)) = 0$, $\nu+1 \leqq \alpha \leqq n$ for all parallel vector fields Y along $\gamma(t)$ with respect to ∇, hence with respect to $\widetilde{\nabla}$.

Let Z be any tangent vector field to M. Then

$$\widetilde{g}(\widetilde{\nabla}_{\dot\gamma(t)}(\alpha(X_\alpha, Y)), Z) + \widetilde{g}(\alpha(X_\alpha, Y), \widetilde{\nabla}_{\dot\gamma(t)} Z) = \widetilde{\nabla}_{\dot\gamma(t)} \widetilde{g}(\alpha(X_\alpha, Y), Z) = 0.$$

On the other hand, because $\dot\gamma(t)$ is in $RN(\gamma(t))$,

$$\tilde\nabla_{\dot\gamma(t)}Z = \nabla_{\dot\gamma(t)}Z + \alpha(\dot\gamma(t), Z) = \nabla_{\dot\gamma(t)}Z\,.$$

So

$$\tilde g(\nabla_{\dot\gamma}(\alpha(X_\alpha, Y)), Z) = -\tilde g(\alpha(X_\alpha, Y), \tilde\nabla_{\dot\gamma} Z) = -\tilde g(\alpha(X_\alpha, Y), \nabla_{\dot\gamma} Z) = 0\,.$$

Thus the tangential component of $\nabla_{\dot\gamma}(\alpha(X_\alpha, Y)) = 0$ along $\gamma(t)$, $h_* < t < t_*$.

To show that the normal component of $\nabla_{\dot\gamma}(\alpha(X_\alpha, Y)) = 0$ as well, on each N_x, we have

$$\begin{aligned}
c(\partial/\partial x^1 \wedge \partial/\partial x^\alpha)Y &= \tilde R(\partial/\partial x^1, \partial/\partial x^\alpha)Y \\
&= \tilde\nabla_{\partial/\partial x^1}\tilde\nabla_{\partial/\partial x^\alpha}Y - \tilde\nabla_{\partial/\partial x^\alpha}\tilde\nabla_{\partial/\partial x^1}Y - \tilde\nabla_{[\partial/\partial x^1, \partial/\partial x^\alpha]}Y \\
&= \tilde\nabla_{\partial/\partial x^1}(\nabla_{\partial/\partial x^\alpha}Y + \alpha(\partial/\partial x^\alpha, Y)) - \tilde\nabla_{\partial/\partial x^\alpha}(\nabla_{\partial/\partial x^1}Y + \alpha(\partial/\partial x^1, Y)) \\
&\quad - \nabla_{[\partial/\partial x^1, \partial/\partial x^\alpha]}Y - \alpha([\partial/\partial x^1, \partial/\partial x^\alpha], Y) \\
&= \nabla_{\partial/\partial x^1}\nabla_{\partial/\partial x^\alpha}Y + \alpha(\partial/\partial x^1, \nabla_{\partial/\partial x^\alpha}Y) + \tilde\nabla_{\partial/\partial x^1}\alpha(\partial/\partial x^\alpha, Y) - \nabla_{\partial/\partial x^\alpha}\nabla_{\partial/\partial x^1}Y \\
&\quad - \alpha(\partial/\partial x^\alpha, \nabla_{\partial/\partial x^1}Y) - \tilde\nabla_{\partial/\partial x^\alpha}\alpha(\partial/\partial x^1, Y) - \nabla_{[\partial/\partial x^1, \partial/\partial x^\alpha]}Y \\
&\quad - \alpha([\partial/\partial x^1, \partial/\partial x^\alpha], Y)\,.
\end{aligned}$$

Since F is a diffeomorphism on N_x, $[\partial/\partial x^i, \partial/\partial x^j] = F_*[N_i, N_j] = 0, 1 \leqq i, j \leqq n$. Also $\partial/\partial x^1$ is in RN in the neighborhood N_x, so the normal component of $\tilde R(\partial/\partial x_1, \partial/\partial x^\alpha)Y$ is equal to the normal component of $\tilde\nabla_{\partial/\partial x^1}\alpha(\partial/\partial x^\alpha, Y) - \alpha(\partial/\partial x^\alpha, \nabla_{\partial/\partial x^1}Y)$.

However, $\tilde R(\partial/\partial x^1, \partial/\partial x^\alpha)Y = c(\partial/\partial x^1 \wedge \partial/\partial x^\alpha)Y$ is tangent to M, therefore the normal component of $\tilde\nabla_{\partial/\partial x^1}(\alpha(\partial/\partial x^\alpha, Y)) - \alpha(\partial/\partial x^\alpha, \nabla_{\partial/\partial x^1}Y) = 0$. Since $\nabla_{\partial/\partial x^1}Y$ restricted on $\gamma(t)$ is $\nabla_{\dot\gamma(t)}Y = 0$ on $F(N_x)$, we have the normal component of $\tilde\nabla_{\partial/\partial x^1}\alpha(\partial/\partial x^\alpha, Y) = \alpha(\partial/\partial x^\alpha, \nabla_{\partial/\partial x^1}Y) = 0$ on $\gamma(t) \cap F(N_x)$. Since $\alpha(\partial/\partial x^\alpha, Y)$ is well defiend on $\gamma((h_*, t_*))$, $\tilde\nabla_{\dot\gamma(t)}(\alpha(X_\alpha, Y)) = 0, \nu + 1 \leqq \alpha \leqq n$, together with the fact that the tangential component of $\tilde\nabla_{\dot\gamma(t)}(\alpha(X_\alpha, Y)) = 0$.

Let $Y(t_*) \neq 0$ be in $RN(t_*)$. If $Y(\bar t)$ were not in $RN(\gamma(t))$, where $\bar t$ is a fixed value in (h_*, t_*), then for some X_α, we would have to conclude that $\alpha(X_\alpha(\bar t), Y(\bar t)) \neq 0$. By continuity of α and by the fact that $\alpha(X_\alpha(t), Y(t))$ is parallel along $\gamma(t)$, $h_* \leqq t \leqq t_*$, we conclude that $\alpha(X(t_*), Y(t_*)) \neq 0$. This is a contradiction. Thus $Y(\bar t)$ must be in $RN(\gamma(\bar t))$. Therefore the parallel displacement along $\gamma(t)$ from $\gamma(\bar t)$ to $\gamma(t_*)$ sends $RN(\gamma(\bar t))$ onto a subspace in $TM_{\gamma(t_*)}$ containing $RN(\gamma(t_*))$. So $\nu(\gamma(\bar t)) = \nu \geqq \nu(\gamma(t_*))$, but ν is the minimum among $\nu(x)$ for x in M. Thus $\gamma(t_*)$

is in G. Now by the argument in the proof of Theorem 1.4.1, we see that p_* $= \gamma(t_*)$ is in L. This contradicts the choice of $\gamma(t)$ and p_*. Q. E. D.

COROLLARY 1.8.1. *Under the notations and conditions in Theorem* 1.8.1, *if M has constant sectional curvature and if $\nu \geqq 2$, then it must have the same sectional curvature as $\tilde{M}$, i.e., c.*

PROOF. Obvious by Gauss equation (1.5.3). Q. E. D.

Chapter 2. Completeness of the nullity and relative nullity of complex submanifolds in Kählerian manifolds.

2.1. Relative curvature tensors of Kählerian manifolds. In Section 1.1, we defined curvature-like tensors on Riemannian manifolds. The definition of such tensors on Kählerian manifold M is the same. Thus the rest of Definitions and Propositions in Sections 1.1 and 1.2 are valid for the Kählerian case. Especially, as an example of curvature-like tensor, we introduced the relative curvature tensor on Riemannian manifolds in Section 1.3. Now we shall define the Kählerian analogue in this section.

DEFINITION 2.1.1. *The relative curvature tensor field D relative to a real number k on a Kählerian manifold M is defined to be*

$$D(X,Y) = R(X,Y) - k/4\{X \wedge Y + JX \wedge JY + 2g(X,JY)J\}\,, \tag{2.1.1}$$

where R, J and g are the curvature tensor, the complex structure and the metric tensor of M, respectively. We call D the relative curvature tensor with constant k on M.

PROPOSITION 2.1.1. *D in Definition 2.1.1 is a curvature-like tensor on M and the nullity space $DN(x)$ at x in M is invariant under the complex structure J.*

PROOF. Verify (1.1.1.), $\cdots$, (1.1.5.) for the first half, and for the last half, check $J(DN) \subset DN$. Q. E. D.

Proposition 2.1.1 tells us that the nullity space is a complex space. Thus the nullity $\mu(x)$ is an even number. From now on, we define *the nullity at x* for DN to be the complex dimension of $DN(x)$ and denote it by $\mu(x)$. Automatically, the index of nullity is to be the minimum complex dimension among $\mu(x)$ and is denoted by μ as well.

2. 2. Completeness of the nullity distribution of the relative curvature tensor on a Kählerian manifold.

THEOREM 2. 2. 1. *Under the notations in Section* 2. 1, *if* M *is complete, then all leaves in* G *are also complete.*

PROOF. The proof of this theorem is quite similar to that of Theorem 1. 4. 1.
By the same reason as in proof of Lemma 1. 4. 2, we are only interested in the normal components X_a's, say Z_a^l, $1 \leqq l \leqq n-2\mu$, $2\mu+1 \leqq \alpha \leqq n$.
Thus for $k>0$,

(2. 2. 7) $$Z_\alpha^l(t) = a_\alpha^l \cos \frac{\sqrt{k}}{2} t + b_\alpha^l \sin \frac{\sqrt{k}}{2} t, 2\mu+1 \leqq \alpha \leqq n$$

and $1 \leqq l \leqq n-2\mu$.

Considering the initial condition of X_α's at q,

(2. 2. 8) $$Z_\alpha^l(t) = \delta_\alpha^{l+2\mu} \cos \frac{\sqrt{k}}{2} t + b_\alpha^l \sin \frac{\sqrt{k}}{2} t, 2\mu+1 \leqq \alpha \leqq n$$

and $1 \leqq l \leqq n-2\mu$.

Similarly for $k=0$,

(2. 2. 9) $$Z_\alpha^l(t) = \delta_\alpha^{l+2\mu} + b_\alpha^l t, 2\mu+1 \leqq \alpha \leqq n \text{ and } 1 \leqq l \leqq n-2\mu,$$

and for $k<0$,

(2. 2. 10) $$Z_\alpha^l(t) = \delta_\alpha^{l+2\mu} \sinh \frac{\sqrt{-k}}{2} t + b_\alpha^l \cosh \frac{\sqrt{-k}}{2} t, 2\mu+1 \leqq \alpha \leqq n$$

and $1 \leqq l \leqq n-2\mu$.

Here all coefficients appearing in the above differential equations do not depend on the variable t.

By analyticity of the solutions, we have a parameter value h_* such that F is regular on the set $H = \{(t, 0, \cdots, 0) \in R^{2\mu} \times W : h_* < t < t_*\}$ as in the proof of Lemma 1. 4. 2.

Applying the method used in the proof of Theorem 1. 4. 1 to this case, we can conclude the desired result in Theorem 2. 2. 1. Q. E. D.

THEOREM 2. 2. 2. *Under the same assumption as Theorem* 2. 2. 1, *if* M

has positive sectional curvature and if $2\mu \geqq n$, *then for* $k>0$, *M has constant sectional curvature k and also has the complex projective space of constant holomorphic sectional curvature k as its universal covering manifold. Especially, M is compact.*

PROOF. Use Lemma 1.6.1 and imitate the proof of Theorem 1.6.2.

Q. E. D.

Recently, Goldberg and Kobayashi [6] showed the following result which is a slight generalization of the complex version of Lemma 1.6.1 by Frankel [5].

LEMMA 2.2.1 [Goldberg and Kobayashi]. *Let* M^n *be a complete connected Kählerian manifold with positive holomorphic bisectional curvature and let* V^k *and* W^l *be compact complex submanifolds with dimensions k and l, respectively. If* $k+l \geqq n$, *then V and W have non-empty intersection.*

About holomorphic bisectional curvature, see [6] or [10].

Thus, we have the following additional theorem.

THEOREM 2.2.3. *Let* M^n *be a connected complete Kählerian manifold with positive holomorphic bisectional curvature. If* $2\mu \geqq n$ *and* $k>0$, *then* M^n *has constant holomorphic sectional curvature k and also has the projective n-space of constant holomorphic sectional curvature k as its universal covering manifold. Especially,* M^n *is compact.*

In [7], A. Gray made a conjecture that if M is an n-dimensional manifold with the relative curvature with positive constant k, then its index of relative nullity is either 0 or n.

Our Theorems 1.6.2, 2.2.2 and 2.2.3 are partial answers to this conjecture under the conditions mentioned in each theorem.

2.3. Completeness of the relative nullity distributions on Kählerian manifolds. In Section 1.7, we introduced the notion of relative nullity for submanifolds and in Section 1.8, we proved completeness of the relative nullity submanifolds.

Here we shall take care of the Kählerian analogue of Section 1.8.

Let M^n be a Kählerian submanifold of a Kählerian manifold M. Then as in Section 1.7, we have the relative nullity space, the index of relative nullity and the relative nullity distribution. Denote them by $RN(x)$, ν and RN, respectively.

PROPOSITION 2.3.1. $RN(x)$ *at* x *in* M *is invariant under the complex structure J, i.e., the leaves of relative nullity distribution are complex* *submanifolds of M and $\widetilde{M}$, as well.*

PROOF. Let X be in RN, i. e., $\alpha(X, Y)=0$ for all Y in TM. Then we have a well known equation, $\alpha(JX, Y)=\alpha(X, JY)=J(\alpha(X, Y))$ so $\alpha(JX, Y)=0$. Thus JX is in RN. Q. E. D.

PROPOSITION 2. 3. 2. *If M^n is a Kählerian submanifold in a complex space form $\widetilde{M}(c)$ with constant holomorphic sectional curvature c, then the relative nullity distribution RN is the same as the nullity distribution defined by the relative curvature $D(X, Y)=R(X, Y)-\widetilde{R}(X, Y)=R(X, Y)-c/4\{X\wedge Y+JX\wedge JY+2g(X, JY)J\}$, where R and $\widetilde{R}$ are the curvature tensors of M and $\widetilde{M}$, respectively.*

PROOF. The following proof was given by Professor K. Nomizu in his lecture. By Gauss equation, for the complex case,

$$R(X, Y)-\widetilde{R}(X, Y)=\sum_i A_iX\wedge A_iY+\sum_i JA_iX\wedge JA_iY\,,$$

we have $RN(x)\subset DN(x)$ for each x in M.

On the other hand, if X is in $DN(x)$, i. e., $D(X, Y)=\sum_i A_iX\wedge A_iY+\sum_i JA_iX\wedge JA_iY=0$ for all Y in TM_x, then clearly $2\sum_i JA_iX\wedge A_iX=0$ by replacing JX for Y. Thus $g\left(\left(\sum JA_iX\wedge A_iX\right)Y, JY\right)=0$.

On the other hand, the left-hand side of this equation is equal to $\sum_i\{g(Y, A_iX)^2+g(JY, A_iX)^2\}$, therefore, $A_iX=0$ for all i. So X is in $RN(x)$ by Proposition 1. 7. 2. Q. E. D.

THEOREM 2. 3. 1. *Under the notation mentioned above, in addition, if M is complete, then the leaves of the relative uullity distribution are complete. Also the leaves are Kählerian submanifolds of M and $\widetilde{M}$ under the induced Kählerian structure and have the same constant holomorphic sectional curvature as the ambient space form.*

PROOF. Combine Theorem 2. 2. 1, Proposition 2. 3. 1 and Proposition 2. 3. 2. Q. E. D

COROLLARY 2. 3. 1. *Under the same assumption as Theorem 2. 3. 1, if M has constant holomorphic sectional curvature and if $\nu\geqq 1$, then M has the* *same holomorphic sectional curvature as the ambient space form.*

PROOF. Obvious by Gauss equation. Q. E. D.

Chapter 3. A characterization of totally geodesic submanifolds in $S^N(c)$ by an inequality.

3.1. Preliminary lemmas. Let M^n be an n-dimensional complete connected Riemannian submanifold of $S^N(c)$, i.e., the standard N-dimensional sphere with radius $\frac{1}{\sqrt{c}}$ with constant sectional curvature $c>0$. Let M^n be immersed into $S^N(c)$ by f isometrically.

LEMMA 3.1.1. *Let f be a continuous mapping of a locally compact metric space M into a locally compact metric space M'. Assume*

(3.1.1) *f is a local homeomorphism,*

(3.1.2) *f is one-to-one on a compact subset L of M.*

Then f is one-to-one on a neighborhood of L.

PROOF. First of all, we notice that we can prove the same conclusion under weaker conditions, but Lemma 3.1.1 is good enough for our purpose here.

Assume that there exists no neighborhood of L where f gives a homeomorphism. Then there exist two sequences of points, say $\{a_i\}$ and $\{b_i\}, i=1,2,\cdots$, such that

(3.1.3) $\{a_i\}$ and $\{b_i\}$ converge to points, say a and b, in L, respectively.

(3.1.4) For each i, $a_i \neq b_i$, but $f(a_i)=f(b_i)$.

It is just a simple matter to construct such sequences as above.

For $\{a_i\}$ and $\{b_i\}, i=1,2,\cdots$, we can show that $a=\lim_{i\to\infty}\{a_i\} \neq \lim_{i\to\infty}\{b_i\}=b$, because if $a=b$, then by (3.1.1), we have a neighborhood of a where f is one-to-one, and for sufficient large i, both a_i and b_i must be in the neighborhood. This is a contradiction to (3.1.1) and (3.1.4). Thus $f(a) \neq f(b)$ by (3.1.2). However, continuity of f assures that $f(a)=\lim_{i\to\infty} f(a_i)=\lim_{i\to\infty} f(b_i)=f(b)$, again a contradiction.

Q. E. D.

Let ν be the index of relative nullity of M^n. We follow the notations in Chapter I.

LEMMA 3.1.2. *Let L be a leaf in M with respect to the relative nullity distribution as in Sections* 1.7 *and* 1.8.

If $\nu>1$, then L is imbedded isomorphically by f restricted to L as some great sphere of dimension ν in $S^N(c)$, where L has the Riemannian structure induced by that of M.

If $\nu=1$, then f restricted to L is an isometric covering projection with L as the covering manifold and with some great circle as its base space.

PROOF. Let x_0 be a fixed point in the leaf L. Then by Proposition 1.8.1, a neighborhood of x_0 in L is isometrically imbedded by $f|L$ as an open sebset in a great sphere of dimension ν, say $S^\nu(c)$. If γ is any geodesic starting at x_0 in L, then $f(\gamma)$ must stay in $S^\nu(c)$. We know that $f|L$ is an isometric immersion of L into $S^\nu(c)$ by completeness of L. Theorem 4.6 in K-N [10], for example, tells us that $f|L$ is a covering map. As we know, S^ν is simply connected if $\nu>1$, thus $f|L$ must be an isometry. Q.E.D.

LEMMA 3.1.3. *Let x be a point in a leaf L. Then there exists a Frobenius coordinate system around x which contains at most one slice of each leaf. This is nothing but regularity of the distribution in the sense of Palais* [17].

PROOF. If $S^m(c)$ is any great sphere of $S^N(c)$, then for any two points x and y in $S^m(c)$, the distance between x and y in S^m is equal to the distance in S^N.

Let $(U, x^1, \cdots, x^\nu, x^{\nu+1}, \cdots, x^n, \zeta)$ be a Frobenius coordinate neighborhood around x in L such that $U \subset \bar{B}(x, 2\pi/\sqrt{c})$, an open ball with radius $2\pi/\sqrt{c}$ and with x as its center, where any two points are joined by a unique geodesic segment whose length gives the distance between them.

Define $U_i, i=1,2,\cdots$, to be a sequence of neighborhoods of x as follows:

$$U_i = \left\{\zeta(x^1, \cdots, x^\nu, x^{\nu+1}, \cdots, x^n)) \in U : |x_j| < \frac{1}{i}, 1 \leqq j \leqq n\right\}, i = 1, 2, \cdots$$

where $(x^1, \cdots, x^n)$ is the coordinate system and ζ is the coordinate mapping.

Assume that for any i, U_i contains two slices belonging to a leaf, say L_i. Take points x_i and y_i from each of two such slices, respectively. Then $\lim_{x\to\infty} x_i = x = \lim_{i\to\infty} y_i$, and the distance $d_i(x_i, y_i)$ in L_i is greater than a constant c for large i's.

LEMMA 3.1.2 tells us that $f|L_i$ is a local isometry, thus combining it with the above fact, we have $d^N(f(x_i), f(y_i))$, i.e., the distance in S^N is equal to $d_i(x_i, y_i)$ which is greater or equal to c for large i. Since $\lim_{i\to\infty} x_i = x = \lim_{i\to\infty} y_i$, $\lim_{i\to\infty} f(x_i) = f(x) = \lim_{i\to\infty} f(y_i)$.

So $d^N(f(x_i),f(y_i)) \leqq d^N(f(x_i),f(x))+d^N(f(x),f(y_i)) < c/2$ for sufficiently large i. Now we have a contradiction. Q. E. D.

LEMMA 3. 1. 4. *If L is a leaf of RN in M, then for any neighborhood of L, say U, there exists a leaf L' in U different from L for $\nu < n$.*

PROOF. Actually, we could prove much stronger result, but this lemma is enough for our purpose here.

Professor Nomizu pointed out that R. Palais had proved the stronger version of Lemma 3. 1. 4 in a fashion quite similar to that of ours in his thesis at Harvard. So we refer to it for the proof of Lemma 3. 1. 4. See Theorem VI, p. 16, in Palais [17].

3. 2. The statements of Theorems and their proofs.

THEOREM 3. 2. 1. *Let M^n be a complete Riemannian submanifold of $S^N(c)$ immersed by f isometrically. If $2\nu \geqq N$ and if $\nu > 1$, then M^n is isometric to the standard n-dimensional sphere of radius $\frac{1}{\sqrt{c}}$ and is imbedded as a great sphere $S^n(c)$ of $S^N(c)$ by f.*

PROOF. Theorem 3. 2. 1 will be proved by Lemmas 3. 1. 1, $\cdots$, 3. 1. 4 and Frankel's result, i. e. Lemma 1. 6. 1, since $S^N(c)$ has positive curvature. However, we shall here present a more elementary proof.

Let L be a leaf in G. By Lemma 3. 1. 2, the immersion f restricted to L is one-to-one. Applying Lemma 3. 1. 1, we have a neighborhood U of L where f is one-to-one. Lemma 3. 1. 4 tells us that there exists a leaf L' different from L in U. We shall claim that under the condition $2\nu \geqq N, f(L) \cap f(L') \neq \phi$, i. e., f is not one-to-one unless $\nu = n$.

It is well known that every k-dimensional great sphere in $S^N(c)$ corresponds to a $(k+1)$-dimensional real linear subspace in R^{N+1} in the canonical fashion, and that two such great spheres have a common point in $S^N(c)$ if and only if the corresponding linear subspaces have an intersection whose dimension is greater or equal to 1.

By elementary linear algebra, we have the following formula:

$$\dim K + \dim H = \dim(K+H) + \dim(H \cap K), \tag{3. 2. 1}$$

where K and H are two linear subspaces in R^{N+1}.

Since $f(L)$ and $f(L')$ are ν-dimensional great spheres, they correspond to $(\nu+1)$-dimensional subspaces, say $\overline{L}$ and $\overline{L}'$, respectively. By (3. 2. 1) and the given condition $2\nu \geqq N$,

$$(3.2.2) \qquad (\nu+1)+(\nu+1) \leqq N+1+\dim(\overline{L} \cap \overline{L}'),$$

because $\dim(\overline{L}+\overline{L}')$ is at most $N+1$.

From 3.2.2, we have

$$(3.2.3) \qquad \dim(\overline{L} \cap \overline{L}') \geqq 2\nu+2-(N+1) = 2\nu-N+1 \geqq 1.$$

Thus $f(L) \cap f(L') \neq 0$. This is a contradiction unless $\nu = n$, i.e. $M^n = S^n(c)$.

Q. E. D.

LEMMA 3.1.2 and LEMMA 1.6.1 give us the following:

THEOREM 3.2.2. *Let M^n be a complete connected Riemannian submanifold of $S^N(c)$ immersed by f isometrically. If M^n has positive sectional curvature and if $2\nu \geqq n$ and $n > 1$, then M^n is isomorphic to the standard sphere of radius $\frac{1}{\sqrt{c}}$ and of dimension n, and is imbedded by f as an n-dimensional great sphere $S^n(c)$ of $S^N(c)$.*

Chapter 4. a characterization of totally geodesic submanifolds in $CP^N(c)$ by an inequality.

In this chapter, we shall study the complex analogue of Theorems in Chapter III. Naturally, most of the proofs of lemmas and theorems in Chapter IV will be quite similar to those in Chapter III. Here we shall present only those which are not obvious from the real case.

4.1. Preliminary lemmas. Let M^n be a complex n-dimensional complete, connected Kählerian submanifold of $CP^N(c)$, i.e., the N-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature c.

Let M^n be immersed into $CP^N(c)$ by f isometrically and holomorphically. As in Chapter II, we denote by ν the index of relative nullity in the complex sense.

LEMMA 4.1.1. *Let L be a leaf of the relative nullity distribution. Then L is isometric to the ν-dimensional complex projective space with constant holomorphic sectional curvature c and is imbedded by $f|L$ as a ν-dimensional projective subspace of $CP^N(c)$, where L is given the Kählerian structure induced by that of M^n.*

PROOF. Using the fact that L is complete and totally geodesic in $CP^N(c)$, we can prove Lemma 4.1.1. as in the proof of Lemma 3.1.2. Notice here that every complex projective space is simply connected, so we do not have an additional restriction as in Lemma 3.1.2. Q. E. D.

LEMMA 4.1.2. *Let x be a point in a leaf L. Then there exists a Frobenius coordinate neighborhood around x which contains at most one slice of each leaf in M.*

PROOF. One can apply the same argument as the real case. The proof is left to the reader. Q. E. D.

4.2. Statements of Theorems and their proofs.

THEOREM 4.2.1. *Let M^n be a complex n-dimensional complete connected Kählerian submanifold of $CP^N(c)$. Let M^n be immersed by f into $CP^N(c)$ isometrically and holomorphically. If $2\nu \geqq N$, then M^n is isometric to the complex projective space of dimension n and of constant holomorphic sectional curvature c, and is imbedded by f as an n-dimensional complex projective subspace $CP^n(c)$ in $CP^N(c)$.*

K. Nomizu [13] proved that if M^n is a compact Kählerian hypersurface imbedded to $CP^{n+1}(c)$ and if $2\nu \geqq n+1$, then $M^n = CP^n(c)$.

Thus Theorem 4.2.1 is an extension of his theorem. Nomizu's proof uses Chow's theorem, but our proof is more differential geometric as we saw in the real case.

PROOF. The same argument as the real case can be applied to this case. Of course, every dimension here should be read as complex dimension. The details are left to the reader. Q. E. D.

THEOREM 4.2.2. *If M^n has positive sectional curvature and if $2\nu \geqq n$ then M^n is isometric to the complex projective space of constant holomorphic sectional curvature c and is imbedded as a complex projective subspace $CP^n(c)$ in $CP^N(c)$.*

PROOF. Use Lemma 1.6.1 and Lemma 4.1.1.

COROLLARY 4.2.1. *If M^n is a complete Kählerian hypersurface of $CP^{n+1}(c)$ immersed by f into $CP^{n+1}(c)$ isometrically and holomorphically and if the sectional curvature is $\geqq c/4$, then under $n \geqq 3$, M^n is imbedded as a projective hyperplane $CP^n(c)$ in $CP^{n+1}(c)$.*

PROOF. This is a generalization of Theorem 2 in [13].

To get $\nu \geqq n-1$, we can refer to the proof of Theorem 2 of [13]. Then apply

Theorem 4. 2. 2. Q. E. D.

THEOREM 4. 2. 3. *Let M^n be as in Theorem* 4. 2. 1. *If M^n has positive holomorphic bisectional curvature and if $2v \geqq n$, then M^n is isometric to the n-dimentional complex projective space of constant holomorphic sectional curvature c and is imbedded as a complex projective subspace $CP^n(c)$ in $CP^N(c)$.*

PROOF. Combine Lemma 2. 2. 1 with Lemma 4. 1. 1. Q. E. D.

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