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ON DIVISIBILITY BY 2 OF THE RELATIVE CLASS NUMBERS OF IMAGINARY NUMBER FIELDS

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Throughout this paper we shall treat algebraic number fields K and F of finite degree. It was proved by H. Yokoi that if K/F is a cyclic extension such that K and the absolute class field $\widetilde{F}$ of F are disjoint over F and K has only one ramified prime divisor over F, then the class number h_F of F is equal to the ambiguous class number $a_{K/F}$ of K/F. ([3] Theorem 1) First, we prove an analogous result in relation to his result. (§2 Theorem 1)

Next, suppose K is imaginary and $K=K^J$ where J is a substitution from a complex number α to the complex conjugate number $\overline{\alpha}$ and let K_0 be the maximal real subfield of K. Then we shall give necessary conditions to make the relative class number of K/K_0 odd. (§2 Lemma 2) From this Lemma 2, the well known property of cyclotomic field $K=P_{p^n}$ that the relative class number of K/K_0 is odd if and only if the class number of K is odd follows easily, where P_{p^n} is the cyclotomic field generated by a primitive p^n-th root of unity over the rational number field P for a prime number p and a natural number n.

Finally, suppose K is totally imaginary, $K=K^J$, and the maximal real subfield K_0 of K is totally real. Then we shall give necessary conditions to make the relative class number of K/K_0 odd. (§2 Theorem 2) This Theorem 2 is a generalization of H. Hasse's Satz 42 in [2].

In §3 applying Theorem 2 to an absolutely cyclic imaginary number field, we shall give necessary and sufficient conditions to make the relative class number odd.

1. Preliminaries.

Throughout this paper we shall use the following notations:

I_k : the group of ideals in k.
P_k : the group of principal ideals in k.
C_k : the group of absolute ideal classes in k.
$\widetilde{k}$: the absolute class field of k.
h_k : the number of absolute ideal classes in k.
E_k : the group of units in k.

When K/F is a finite Galois extension with Galois group $G=G(K/F)$, we use the following notations:

$\Pi e(\mathfrak{p})$: the product of the ramification exponents of all the finite prime divisors $\mathfrak{p}$ in F with respect to K/F.
$\Pi e(\mathfrak{p}_\infty)$: the product of the ramification exponents of all the infinite prime divisors $\mathfrak{p}_\infty$ in F with respect to K/F.
$\Theta_{K/F}$: the group of numbers in K whose norms are units in F with respect to K/F.
I_K^G : the ambiguous ideal group of K with respect to K/F.
NC_K : the image by the norm homomorphism from C_K into C_F.
${}_NC_K$: the kernel by the norm homomorphism from C_K into C_F.
C_K^G : the group of ambiguous ideal classes in K/F.
$a_{K/F}=[C_K^G]$
P : the rational number field.
P_m : the cyclotomic field generated by a primitive m-th root of unity over P.

Let K/F be a cyclic extension. Then the following formula is well known.

$$a_{K/F}=\frac{h_F\Pi e(\mathfrak{p})\Pi e(\mathfrak{p}_\infty)}{(K:F)[E_F:N_{K/F}\Theta_{K/F}]} \tag{1}$$

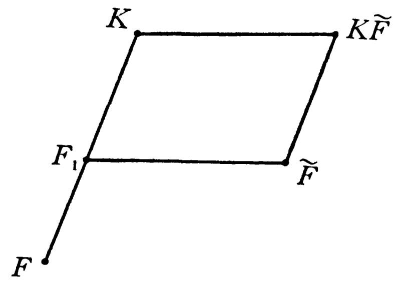
PROPOSITION 1. *Let K/F be a finite extension and let F_1 be the maximal unramified abelian extension field over F contained in K, i.e. $F_1=\widetilde{F}\cap K$. Then we have:*

1. *If K/F is Galois, then h_K is divisible by $h_F/(F_1:F)$.*
2. *If K/F is cyclic, then $a_{K/F}$ is divisible by $h_F/(F_1:F)$.*
3. *If K/F is cyclic and has at most one ramified prime divisor, then $h_F/(F_1:F)$ is equal to $a_{K/F}$.*

PROOF. 1. This is obvious.

2. Let σ be a generator of $G(K/F)$ and let $C_K^{1-\sigma}$ be the image by the homomorphism $1-\sigma$ from C_K to C_K in a natural way. Then $h_K=a_{K/F}\cdot[C_K^{1-\sigma}]$ and $h_K=[{}_NC_K][NC_K]$. Therefore $a_{K/F}=\frac{[{}_NC_K]}{[C_K^{1-\sigma}]}\cdot[NC_K]$. Since ${}_NC_K$ contains $C_K^{1-\sigma}$, $\frac{[{}_NC_K]}{[C_K^{1-\sigma}]}$ is an integer. Hence $a_{K/F}$ is divisible by $[NC_K]$. On the other hand, $[C_F]=(\widetilde{F}:F)=(\widetilde{F}:F_1)(F_1:F)=(K\widetilde{F}:K)(F_1:F)=[C_K:{}_NC_K](F_1:F)=[NC_K](F_1:F)$ by Class Field Theory, therefore $[C_F:NC_K]=(F_1:F)$. From $h_F=[C_F:NC_K][NC_K]$

$=(F_1:F)[NC_K]$, $[NC_K]=h_F/(F_1:F)$ follows. Hence $a_{K/F}$ is divisible by $h_F/(F_1:F)$.



3. By the formula (1), $a_{K/F}=\frac{h_F}{(F_1:F)}\cdot\frac{\Pi e(\mathfrak{p})\Pi e(\mathfrak{p}_\infty)}{(K:F_1)[E_F:N_{K/F}\Theta_{K/F}]}$. $\frac{\Pi e(\mathfrak{p})\Pi e(\mathfrak{p}_\infty)}{(K:F_1)[E_F:N_{K/F}\Theta_{K/F}]}$ is an integer by 2. Since K/F has at most one ramified prime divisor, $(K:F_1)=\Pi e(\mathfrak{p})\Pi e(\mathfrak{p}_\infty)$. Hence $a_{K/F}=h_F/(F_1:F)$.

REMARK. This is a generalization of the theorem of H. Yokoi. In 3. we have $[E_F:N_{K/F}\Theta_{K/F}]=1$.

2. Theorems.

THEOREM 1. *Let K/F be a cyclic extension, suppose that K is totally imaginary, F is real and the maximal real subfied K_0 of K is totally real, and let $F_1=K\cap\widetilde{F}$. If K/F has at most one finite ramified prime divisor, then we have the following facts:*

1. *If the signatures of fundamental units of F are "independent", then $a_{K/F}=h_F/(F_1:F)$.*

2. *If the signatures of fundamental units of F have just one 'relation', then we have the following:*

(a) *If K/F has a finite ramified prime divisor $\mathfrak{p}$ and $\mathfrak{p}$ doesn't ramify in K/K_0, then $a_{K/F}=h_F/(F_1:F)$.*

(b) *If K/F has a finite ramified prime divisor $\mathfrak{p}$ and $\mathfrak{p}$ ramifies in K/K_0, then $a_{K/F}=2\cdot h_F/(F_1:F)$.*

(c) *If K/F hasn't any finite ramified prime divisor, then $a_{K/F}=h_F/(F_1:F)$.*

PROOF. Let J be a substitution from a complex number α to the complex conjugate number $\bar{\alpha}$ and let s be any conjugate substitution of K over the rational number field P. Then we can show $sJ=Js$ by the assumptions that K is totally imaginary and that K_0 is totally real. As J fixes an element of F and K/F is Galois, J induces an automorphism of K. Therefore $K^J=K$. We can put $K=K_0(\theta)$ and $\theta+\theta^J$, $\theta\theta^J\in K$. As $\theta+\theta^J$, $\theta\theta^J$ are real, $\theta+\theta^J$, $\theta\theta^J\in K_0$. Therefore K/K_0 is a quadratic extension. Accordingly we can suppose θ is pure imaginary, i. e. $0>\theta^2=a\in K_0$, $\theta+\theta^J=0$. Hence $\theta^s+(\theta^J)^s=0$, $K^s=K_0^s(\theta^s)$ and $(\theta^s)^2=(\theta^2)^s=a^s\in K_0^s$. As K_0 is totally real, K_0^s is real. As K is totally imaginary, θ^s isn't real. Hence $(\theta^s)^2=a^s<0$. Hence $\theta^s+(\theta^s)^J=0$. From this and $\theta^s+(\theta^J)^s=0$, we have $(\theta^J)^s=(\theta^s)^J$. Hence $(\alpha^J)^s=(\alpha^s)^J$ for all $\alpha\in K$.

If α is any element of $\Theta_{K/F}$, then $N_{K/F}\alpha\in E_F$. Let σ be a generator of $G(K/F)$. Then we have $N_{K/F}\alpha=\alpha^{(1+\sigma+\sigma^2+\cdots+\sigma^{m-1})(1+\sigma^m)}=\beta\beta^J$ for $m=(K_0:F)$, $J=\sigma^m$ in K and $\beta=\alpha^{1+\sigma+\sigma^2+\cdots+\sigma^{m-1}}$. Hence $N_{K/F}\alpha$ is a totally positive unit. If ε is totally positive element of E_F, we have $\left(\frac{\varepsilon, K/F}{\mathfrak{p}_\infty}\right)=1$ for any infinite prime divisor $\mathfrak{p}_\infty$, $\left(\frac{\varepsilon, K/F}{\mathfrak{p}'}\right)=1$ for any finite prime divisor $\mathfrak{p}'$ which doesn't ramify in K/F, for norm residue symbol. From the product formula of norm residue symbol we have $\left(\frac{\varepsilon, K/F}{\mathfrak{p}}\right)=1$ for a finite ramified prime divisor $\mathfrak{p}$. Hasse's Theorem tells us that such a ε is norm of an element of K. Hence $N_{K/F}\Theta_{K/F}$ is the group E_F^+ of totally positive units in F.

Since $E_F^2\subset E_F^+$, we can put $[E_F:E_F^+]=2^R$ and $q=(F:P)-R$. Then we can prove that q is the number of dependent relations between the signatures of fundamental units of F. Let $s_1, s_2, \cdots, s_r$ be all of conjugate substitutions of F over P. We have $r=(F:P)$. Put $\operatorname{sgn}\varepsilon^s=\begin{cases}0 \text{ if } \varepsilon^s \text{ is positive}\\ 1 \text{ if } \varepsilon^s \text{ is negative}\end{cases}$ for a unit ε of F. Cleary we have $\operatorname{sgn}(\varepsilon\eta)^s\equiv\operatorname{sgn}\varepsilon^s+\operatorname{sgn}\eta^s \pmod 2$ for units ε and η of F. Let V be a vector space over a finite field $GF(2)$ which consists of vectors $\{(\operatorname{sgn}\varepsilon^{s_1}, \operatorname{sgn}\varepsilon^{s_2}, \cdots, \operatorname{sgn}\varepsilon^{s_r})\,;\ \varepsilon\in E_F\}$. Then E_F/E_F^+ is isomorphic to V. Let $\varepsilon_1=-1$, and let $\varepsilon_2, \cdots, \varepsilon_r$ be the fundamental units of F, as F is totally real. We have the matrix

$$\begin{pmatrix} \operatorname{sgn}\varepsilon_1^{s_1} & \operatorname{sgn}\varepsilon_1^{s_2}\cdots\cdots\operatorname{sgn}\varepsilon_1^{s_r} \\ \operatorname{sgn}\varepsilon_2^{s_1} & \operatorname{sgn}\varepsilon_2^{s_2}\cdots\cdots\operatorname{sgn}\varepsilon_2^{s_r} \\ \cdots\cdots & \\ \operatorname{sgn}\varepsilon_r^{s_1} & \operatorname{sgn}\varepsilon_r^{s_2}\cdots\cdots\operatorname{sgn}\varepsilon_r^{s_r} \end{pmatrix}$$

Let R' be the rank of this matrix. Then R' is the dimension of V over $GF(2)$. Hence $[E_F:E_F^+]=2^{R'}$. Hence $R'=R$ and $q=r-R$ is the number of dependent relations between the signatures of fundamental units of F and $[E_F^+:E_F^2]=2^{r-R}=2^q$.

By the formula (1) we have

$$a_{K/F} = \frac{h_F}{(F_1:F)}\cdot\frac{e(\mathfrak{p})\Pi e(\mathfrak{p}_\infty)}{(K:F_1)[E_F:N_{K/F}\Theta_{K/F}]}$$

$$= \frac{h_F}{(F_1:F)}\cdot\frac{e(\mathfrak{p})\cdot 2^{[F:P]}}{(K:F_1)[E_F:E_F^+]}$$

$$= \frac{h_F}{(F_1:F)}\cdot\frac{e(\mathfrak{p})\cdot 2^q}{(K:F_1)}$$

$a_{K/F}$ is divisible by $h_F/(F_1: F)$ by Proposition 1.2. $(K: F_1)$ is divisible by $e(\mathfrak{p})$. Accordingly, if $q=0$, then we have $a_{K/F}=h_F/(F_1:F)$ and $e(\mathfrak{p})=(K:F_1)$. If $q=1$ and $\mathfrak{p}$ does't ramify in K/K_0, then we have $a_{K/F}=\frac{h_F}{(F_1:F)}\cdot\frac{e(\mathfrak{p})}{(K_0:F_1)}=\frac{h_F}{(F_1:F)}$ and $e(\mathfrak{p})=(K_0: F_1)$. If $q=1$ and $\mathfrak{p}$ ramifies in K/K_0, then $e(\mathfrak{p})=(K: F_1)$ and $a_{K/F}=\frac{h_F}{(F_1:F)}\times 2$. If $q=1$ and $e(\mathfrak{p})=1$, then $K_0=F_1$, $a_{K/F}=h_F/(K_0: F)$. It is impossible that $q=0$ and $e(\mathfrak{p})=1$.

REMARK. $sJ=Js$ if and only if K is totally imaginary and K is totally real. In 1. K/F has a finite ramified prime divisor $\mathfrak{p}$ and $\mathfrak{p}$ ramifies totally in K/F_1. In 2. (a) $\mathfrak{p}$ ramifies totally in K_0/F_1. In 2. (b) $\mathfrak{p}$ ramifies totally in K/F_1.

LEMMA 1. *Let K/F be a cyclic extension with a prime power degree $n=l^\nu$ and suppose $K\cap\widetilde{F}=F$. If $h^*=h_K/h_F$ is prime to l, then*

1. $a_{R/F}=h_F$,

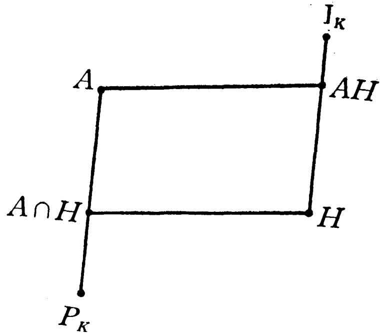
2. *h_F is prime to l if and only if $l^{q'}=Q$(i.e. $N_{K/F}\Theta_{K/F}=N_{K/F}E_K$) and $\mathfrak{A}_1, \mathfrak{A}_2,\cdots,\mathfrak{A}_s\in I_FP_K$, where q' is the number such as $[N_{K/F}\Theta_{K/F}: E_F^n]=l^{q'}$, Q is the index $[N_{K/F}E_K: E_F^n]$, $\mathfrak{A}_1,\mathfrak{A}_2,\cdots,\mathfrak{A}_s$ are such ideals as $\mathfrak{p}_1=\mathfrak{A}_1^{e_1},\cdots,\mathfrak{p}_s=\mathfrak{A}_s^{e_s}$ in K, $\mathfrak{p}_1,\mathfrak{p}_2,\cdots,\mathfrak{p}_s$ are all finite ramified prime divisor in F with respect to K/F, and $e_1,e_2,\cdots,e_s$* are the ramification exponents of $\mathfrak{p}_1,\mathfrak{p}_2,\cdots,\mathfrak{p}_s$ *with respect to K/F respectively.*

PROOF. Let p be any prime number such that $p\neq l$. We can prove easily that a natural mapping $\varphi: C_{F,p}\to C_{K,p}^G$ is an isomorphism where $C_{F,p}$ is the p-class group of F and $C_{K,p}^G$is the ambiguous p-class group of K/F. Let p^m be the order of $C_{K,p}^G$ and let σ be a generator of $G=G(K/F)$. For $\mathfrak{A}\in C_{K,p}^G$, $N_{K/F}\mathfrak{A}=\mathfrak{A}^{1+\sigma+\sigma^2+\cdots+\sigma^{n-1}}$ $=\mathfrak{A}^{\sigma-1}\mathfrak{A}^{\sigma^2-1}\cdots\mathfrak{A}^{\sigma^{n-1}-1}\mathfrak{A}^n$ and $\mathfrak{A}^{\sigma-1}\mathfrak{A}^{\sigma^2-1}\cdots\mathfrak{A}^{\sigma^{n-1}-1}\in P_K$ and so $\mathfrak{A}^n\in(N_{K/F}\mathfrak{A})P_K$. There are such integers x, y as $1=xp^m+yn$, since $(p^m, n)=1$. Therefore $\mathfrak{A}=(\mathfrak{A}^{p^m})^x(\mathfrak{A}^n)^y \in (N_{K/F}\mathfrak{A}^y)P_K$. Therefore φ is 'onto'. Let $p^{m'}$ be the order of $C_{F,p}$. For integers x, y such as $1=xp^{m'}+yn$ and $\mathfrak{b}\in C_{F,p}$, if $\mathfrak{b}=(\alpha)\in P_K$, $\mathfrak{b}=(\mathfrak{b}^{p^{m'}})^x\mathfrak{b}^{yn}\in \mathfrak{b}^{yn}P_F = \mathfrak{b}(N_{K/F}\mathfrak{b}^y)P_F=(N_{K/F}(\alpha)^y)P_F=P_F$. Therefore φ is injective. Suppose that h^* is prime to l. $a_{K/F}/h_F$ is prime to l, for $a_{K/F}$ is divisible by h_F by Proposition 1.2. and h^* is divisible by $a_{K/F}/h_F$. From these facts we have $a_{K/F}=h_F$.

Put $C_K^G=A/P_K$. To any ideal $\mathfrak{A}$ belonging to A, there corresponds a unit η in $N_{K/F}\Theta_{K/F}$ in the following way: since $\mathfrak{A}^{1-\sigma}$ is a principal ideal, there exists a number θ in K such that $\mathfrak{A}^{1-\sigma}=(\theta)$, and $\eta=N_{K/F}\theta$ is cleary a unit in F. By this correspondence, $A/I_K^GP_K\cong N_{K/F}\Theta_{K/F}/N_{K/F}E_K$. Since $E_F\supset N_{K/F}\Theta_{K/F}\supset N_{K/F}E_K \supset E_F^n$, $l^{q'}\geqq Q$. Since $\mathfrak{A}^n=(\alpha)N_{K/F}\mathfrak{A}$ for some $\alpha\in K$ and $\mathfrak{A}\in A$, $[A: I_K^GP_K]$ is the power of l. Hence, if $l^q>Q$, $a_{K/F}=h_F$ is divisible by l.

We consider a natural homomorphism: $C_F\to C_K^G=A/P_K$. If $\mathfrak{A}_1$ doesn't belong to I_FP_K, $\mathfrak{A}_1$ doesn't belong to any class of the image of this homomorphism. $\mathfrak{A}_1$ belongs to A. $\mathfrak{p}_1=\mathfrak{A}_1^{e_1}$ belongs to a class of the image. Hence $a_{K/F}=h_F$ is divisible by l.

Suppose that $l^{q'}=Q$ and $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_\delta\in I_FP_K$. $N_{K/F}\Theta_{K/F}=N_{K/F}E_K$ by $l^{q'}=Q$. Hence $A=I_K^GP_K$. So the homomorphism: $C_F\to C_K^G=I_K^GP_K/P_K$ is 'onto' by $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_\delta\in I_FP_K$. And since $a_{K/F}=h_F$, this homomorphism is an isomorphism. Hence $I_F\cap P_K=P_F$.



Let H be the group of ideals in K whose norms belong to P_F. Then $[A: P_K] = a_{K/F}$, $[I_K: H]=h_F$ and $[H: P_K]=h^*$. And then we have the following facts:

(1°) $[A\cap H: P_K]$ is the power of l.
(2°) $a_{K/F}$ is prime to l if and only if $[A\cap H: P_K]=1$.

Proof of (1°). If $\mathfrak{A}\in A\cap H$, then $\mathfrak{A}^n=N_{K/F}\mathfrak{A}\cdot(\alpha)$ for some $\alpha\in K$. Hence $\mathfrak{A}^n\in P_K$.

Proof of (2°). If $a_{K/F}$ is prime to l, $[A\cap H: P_K]$ is prime to l. By (1°) $[A\cap H: P_K]=1$. If $a_{K/F}$ is divisible by l, then there exists an ideal $\mathfrak{A}\in A$ such that $\mathfrak{A}^l\in P_K$ and $\mathfrak{A}\notin P_K$. Since $N_{F/F}\mathfrak{A}=\mathfrak{A}^n\cdot(\alpha)$ for some $\alpha\in K$, $N_{K/F}\mathfrak{A}\in P_K\cap I_F = P_F$. Hence $\mathfrak{A}\in A\cap H$. Therefore $[A\cap H: P_K]>1$.

Since h^* is prime to l, $A\cap H=P_K$ by (1°). Hence $a_{K/F}=h_F$ is prime to l by (2°).

COROLLARY. *Let K/F be a cyclic extension with a prime power degree $n=l^v$ and suppose $K\cap\widetilde{F}=F$. Then h_K is prime to l if and only if $a_{K/F}=h_F$ and h_F is prime to l.*

PROOF. This follows from Lemma 1 and (3°).

(3°). If $A\cap H=P_K$, then $[H: A\cap H]$ is prime to l.

Proof of (3°). If $C\in H/A\cap H$ and C isn't a unit, then $C^\sigma\neq C$ by $A\cap H=P_K$. Since the order of G is a prime power l^v, the number of distinct G-conjugates of C is a multiple of l. Hence $[H: A\cap H]-1$ is a multiple of l. Hence $[H: A\cap H]$ is prime to l.

LEMMA 2. *Suppose K is imaginary and $K^J=K$ and let K_0 be the maximal real subfield of K. If the relative class number $h^*=h_K/h_{K_0}$ is odd, then we have the following facts:*

1. $a_{K/K_0}=h_{K_0}$.
2. *h_{K_0} is odd if and only if $2^{q'}=Q$ and $\mathfrak{P}_1,\mathfrak{P}_2,\cdots,\mathfrak{P}_\delta\in I_{K_0}P_K$. Here q' is the number such that $[N_{K/K_0}\Theta_{K/K_0}: E_{K_0}^2]=2^{q'}$, Q is the index $[N_{K/K_0}E_K: E_{K_0}^2]$, and $\mathfrak{p}_1=\mathfrak{P}_1^2,\mathfrak{p}_2=\mathfrak{P}_2^2,\cdots,\mathfrak{p}_\delta=\mathfrak{P}_\delta^2$ are all prime divisors which ramify in K/K_0.*

Lemma 2 follows from Lemma 1, since K/K_0 is quadratic. From the Corollary to Lemma 1, we have

COROLLARY 1. *Suppose K is an imaginary and $K^J=K$ and let K_0 be the maximal real subfield of K. Then h_K is odd if and only if $a_{K/K_0}=h_{K_0}$ and h_{K_0} is odd.*

COROLLARY 2. *Let K be the imaginary subfield of the cyclotomic field P_{p^n}, and let K_0 be the maximal real subfield of K. Then the relative class number $h^*=h_K/h_{K_0}$ is odd if and only if h_K is odd.* (This Corollary 2. is induced from the next Theorem 2, too.)

PROOF. A finite ramified prime divisor in K/K_0 is only one and is principal. $r=(K_0: P)$ is the number of all infinite ramified prime divisors in K/K_0. By the formula (1)

$$a_{K/K_0} = h_{K_0} \cdot \frac{2^{1+r}}{(K : K_0) \cdot 2^{r-q'}} = h_{K_0} \cdot 2^{q'}.$$

Suppose that h^* is odd, then $q' = 0$ since a_{K/K_0} is divisible by h_{K_0}. $2^q \geqq Q$. Hence $2^{q'} = Q = 1$. By Lemma 2, h_{K_0} is odd. Hence $h_K = h_{K_0}$. h^* is odd.

THEOREM 2. *Suppose K is totally imaginary, $K^J = K$ and the maximal real subfield K_0 of K is totally real. If the relative class number $h^* = h_K/h_{K_0}$ is odd, then we have the following facts:*

1. *There is* (a) *only one or* (b) *no finite ramified prime divisor in K/K_0.*
2. *In case* (a), *the signatures of fundamental units of K_0 are 'independent'. In case* (b), *they have just one 'relation'.*
3. *In case* (a), *let $\mathfrak{p}$ be the finite ramified prime divisor of K_0 and $\mathfrak{P}$ be the prime factor of $\mathfrak{p}$ in K. Then $h_{K_0}(= a_{K/K_0})$ is odd if and only if $\mathfrak{P} \in I_{K_0}P_K$. In case* (b), *$h_{K_0}(= a_{K/K_0})$ is odd if and only if the index $Q = [N_{K/K_0}E_K : E_{K_0}^2]$ is two.*

PROOF. Let δ be the number of all finite ramified prime divisors in K/K_0 and $r = (K_0 : P)$. By the formula (1) and $[E_{K_0} : E_{K_0}^2] = 2^r$ we have

$$a_{K/K_0} = h_{K_0} \cdot \frac{2^{\delta+r-1}}{[E_{K_0} : N_{K/K_0}\Theta_{K/K_0}]} = h_{K_0} \cdot 2^{\delta+q'-1}$$

where $[N_{K/K_0}\Theta_{K/K_0} : E_{K_0}^2] = 2^{q'}$. Suppose that h^* is odd. Since a_{K/K_0} is divisible by h_{K_0} by Proposition 1.2, $a_{K/K_0} = h_{K_0}$ and $\delta + q' - 1 = 0$. Therefore δ is zero or one and therefore $q' = q$ as we have seen in the proof of Theorem 1. Hence we have 1. and 2. Moreover we have 3 by Lemma 2,

COROLLARY 1. *Suppose K is totally imaginary, $K^J = K$ and the maximal real subfield K_0 of K is totally real. Then h_K is odd if and only if the following three oonditions are satisfied:*

1. *There is* (a) *only one or* (b) *no finite ramified prime divisor in K/K_0.*
2. *In case* (a), *the signatures of fundamental units of K_0 are 'independent'. In case* (b), *they have just one 'relation'.*
3. *h_{K_0} is odd.*

PROOF. If h_K is odd, h^* is odd. Hence 1. 2. 3. are satisfied by Theorem 2.

Conversely suppose 1. 2. 3. are satisfied. $a_{K/K_0} = h_{K_0} \cdot 2^{\delta+q'-1}$ and $q' = q$. By 1. 2. $\delta + q' - 1 = 0$. Hence $a_{K/K_0} = h_{K_0}$ and h_{K_0} is odd. By Corollary 1 to Lemma 2, h_K is odd.

COROLLARY 2. *Suppose K is totally imaginary, $K^J = K$ and the maximal real subfield K_0 of K is totally real.*

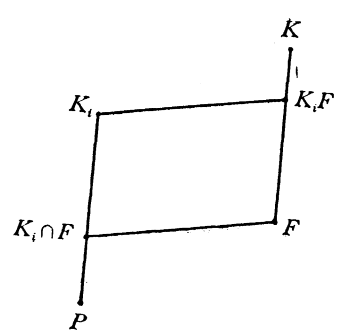
If there exists only one finite ramified prime divisor $\mathfrak{p} = \mathfrak{P}^2$ in K/K_0 and $\mathfrak{P}$ belongs to $I_{K_0}P_K$, then the relative class number h^* is odd if and only if h_K is odd.

If there is no finite ramified prime divisor and $Q = 2$ in K/K_0, then the relative class number h^* is odd if and only if h_K is odd.

3. Absolutely cyclic imaginary number field. In Theorem 2, if K is an absolutely cyclic field, then the case (b) doesn't happen. The next Theorem 3 has been proved by H. Hasse in [2] Satz 45. We shall give another proof by Theorem 2. 3 without using ζ-function.

PROPOSITION 2. *Let K be an absolutely cyclic number field and let F be any proper subfield of K. Then there exists at least one finite ramified prime divisor in K/F.*

PROOF. Let $(K: P) = n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ be a factorization of n into prime numbers. Then there exists an intermediate field K_i of K/F satisfying $(K_i: P) = p_i^{e_i}$. And $K = K_1 K_2 \cdots K_s$. We can prove that there exists a totally ramified finite prime divisor in K_i/P. If K is an imaginary quadratic extension field over P, it is obvious. Suppose K_i isn't an imaginary quadratic extension field over P. If K_i/P hasn't a totally ramified finite prime dlvisor, then there exists no prime divisor which has the rational number field P as the inertia field in K_i/P. Therefore there exists an intermediate field of K_i/P which is nonramified abelian extension over P with the degree at least p_i since K_i/P is cyclic and has prime power degree $p_i^{e_i}$. This is a contradiction for $h_P = 1$. Since F is a proper subfield



of K, there exists such K_i as $K_i \cap F \subsetneqq K_i$ and then the prime divisor which ramifies totally in K_i/P ramifies in K_iF/F because $(K_i : K_i \cap F)$ and $(F : K_i \cap F)$ are relatively prime.

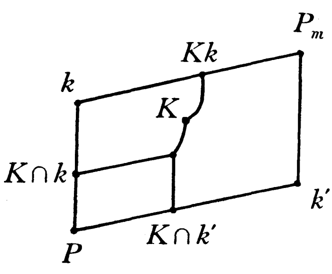
THEOREM 3. *Let K be an absolutely cyclic imaginary number field and let K_0 be the maximal real subfield of K. Then the relative class number $h^* = h_K/h_{K_0}$ is odd if and only if the following three conditions are satisfied*:

1. *There exists one and only one finite ramified prime divisor in K/K_0.*
2. *The signatures of fundamental units of K_0 are 'independent'.*
3. *h_{K_0} is odd.*

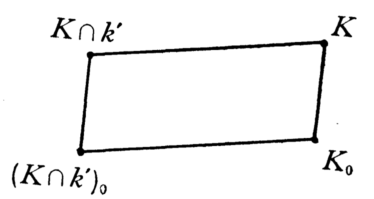
PROOF. Suppose that conditions 1, 2 and 3 are satisffed. $a_{K/K_0} = h_{K_0}$ by 1 and 2 and h_{K_0} is odd by 3. Hence h_K is odd by the Corollary 1 to Lemma 2. So h^* is odd.

Conversely suppose that h^* is odd. We have 1 and 2 by Proposition 2 and Theorem 2. We can show that h_{K_0} is odd by Theorem 2. 3 (a), proving that $\mathfrak{P}$ belongs to $I_{K_0}P_K$ for only one finite ramified prime divisor $\mathfrak{p} = \mathfrak{P}^2$ in K/K_0.

K is contained in P_m. Let $m = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ be a factorization of m into prime numbers and put $k = P_{p_1^{e_1}}$ and $k' = P_{p_1^{e_2} p_3^{e_3} \cdots p_0^{e_s}}$. We can suppose that the finite ramified prime divisor $\mathfrak{p} = \mathfrak{P}^2$ in K/K_0 is a factor of $p = p_1$. Then $K \cap k'$ is real. Because, if $K \cap k'$ is imaginary, then $(K \cap k')_0 = K \cap k' \cap K_0$, $K = (K \cap k') \cdot K_0$ and we have a diagram D.2, p doesn't ramify in K/K_0 since p doesn't ramify in $K \cap k'/(K \cap k')_0$, where $(K \cap k')_0$ is the maximal real subfied of $K \cap k'$, and this is a contradiction. Also we prove that $K \cap k$ is imaginary. If $K \cap k$ is real, then $K \cap k \subset K_0 \subset K$ and we have a diagram D. 3. Let $M = Kk \cap k'$, $M' = K_0 k \cap k'$.

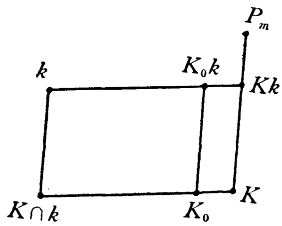


D. 1

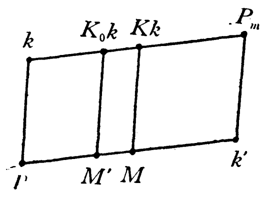


D. 2

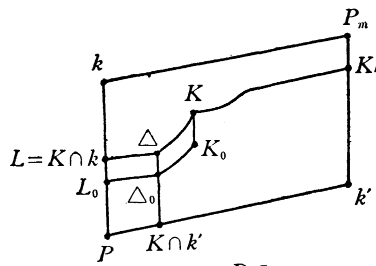
Since $G(K/K \cap k) \cong G(Kk/k) \cong G(M/P)$, M/P is cyclic. Hence M/M has a finite ramified prime divisor by Proposition 2 and it isn't a factor of p. Let it be a factor of prime number l. Since only the factor of p ramifies in K_0k/M' and Kk/M, the factor of l ramifies in Kk/K_0k and so it ramifies in K/K_0 by the diagram D. 3. This is a contradiction.



D. 3



D. 4



D. 5

Since $K\cap k'$ is real and $K\cap k$ is imaginary, we have a diagram D. 5, where put $L=K\cap k$ and let Δ_0 be the maximal real subfield of $\Delta=(K\cap k)(K\cap k')$. Let ζ be a primitive $p_1^{e_1}$ – th root of unity. $N_{k/L}(1-\zeta)$ and $N_{k/L_0}(1-\zeta)$ is the prime factor of p in L and L_0 respectively. Hence the prime factor of p in L and L_0 is principal. Since a finite ramified prime divisor in K/K_0 is only one, p doesn't split in K_0. Therefore p doesn't split in Δ_0. In Δ_0/L_0 and Δ/L, p doesn't ramify. Therefore these prime factors of p in Δ_0 and Δ are principal. Put them $(a), (\alpha)$ and $(a)=(\alpha)^2$. Since the prime factor of p ramifies totally in Kk'/k, it ramifies totally in K/Δ_0. Therefore $(a)=\mathfrak{p}^n$ in K_0 and $(\alpha)=\mathfrak{P}^n$ in K when we put $n=(K_0:\Delta_0)=(K:\Delta)$. Since K/Δ_0 is cyclic and $[\Delta:\Delta_0]=2$, n is odd. Put $n=2m+1$. We have $\mathfrak{P}=\mathfrak{P}^{n-2m}=(\alpha)\cdot\mathfrak{p}^{-m}\in I_{K_0}P_K$.

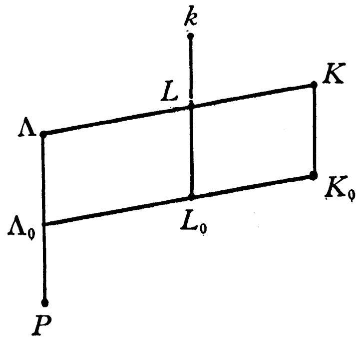
COROLLARY. *Let K be an absolutely cyclic imaginary number field and let K_0 be the maximal real subfield of K. Then h_K is odd if and only if h^* is odd.*

THEOREM 3′. *Let K be an absolutely cyclic imaginary number field and let K_0 be the maximal real subfield of K. Then the relative class number h^* is odd if and only if the following four conditions are satisfied:*

1. *Let Λ be an imaginary subfield of K such that $(\Lambda: P)$ is the power of 2. (There exists only one such a subfield Λ.) A finite ramified prime divisor in Λ/P is only one prime number p.*
2. *This p doesn't split in K.*
3. *The signatures of fundamental units of K_0 are 'independent'.*
4. *h_{K_0} is odd.*

PROOF. We suppose that h^* is odd. As only one prime divisor $\mathfrak{p}$ ramifies in K/K_0 by Theorem 3. 1, p doesn't split in K_0 when $\mathfrak{p}$ is a factor of a prime number p. And 3 and 4 are satisfied, Also we have a diagram D. 5 in the proof of Theorem 3. Put $(K: P)=2^{v}\cdot u$ and $(2, u)=1$. Let Λ be such a subfield as $(\Lambda: P)=2^{v}$, then Λ is imaginary. Since Λ/P is cyclic, there is not such an imaginary subfield Λ' of K as $(\Lambda': P)$ is the power of 2 except Λ. L contains Λ, because $(K: L)$ must be odd. Therefore Λ is contained in k and a ramified prime divisor in Λ/P is only p. So we have 1.

Conversely, suppose that 1, 2, 3 and 4 are satisfied. There exists at least one finite ramified prime divisor in K/K_0 by Proposition 2. Since a finite ramified prime divisor in K/K_0 ramifies in Λ/Λ_0, it is a factor of p by 1. By 2, only one prime factor of p ramifies in K/K_0. Hence h^* is odd by Theorem 3.



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REFERENCES

[1] H. HASSE, Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper. Teil II, Leipzig und Berlin (1930).

[2] H. HASSE, Über die Klassenzahl abelscher Zahlkörper, Berlin (1952).

[3] H. YOKOI, On the class number of a relatively cyclic number field, Nagoya Math. J., 29 (1967), 31-44.

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