$$\mathrm{Hom}_R(G(N), M) \cong \mathrm{Hom}_S(N, F(M))$$

$$\mathrm{Hom}_S(F(M), N) \cong \mathrm{Hom}_R(M, H(N))$$

for $M \in \mathfrak{M}_R$ and $N \in \mathfrak{M}_S$. There also exist natural transformations

$$\alpha : 1_{M_S} \to FG, \quad \alpha \ : FH \to 1_{M_S},$$

$$\beta : 1_{M_R} \to HF \text{ and } \beta' : GF \to 1_{M_R}$$

and since $S = \mathrm{End}_R(P_R)$ both α and α' are natural equivalences. (See [10]).

There exists a natural (R, R)-homomorphism $\mathcal{J}: P^* \otimes_S P \to R$ and a natural (S, S)-homomorphism $\theta : P \otimes_R P^* \to S$ defined via $\mathcal{J}(f \otimes x) = f(x)$ and $\theta(x \otimes f) = xf(\text{-})$. These maps play a key role in Bass' exposition of the Morita theorems. (See [3] or [4].) We need only a few of their properties which follow easily from the dual basis lemma.

1.1 Lemma. *$\theta : P \otimes_R P^* \to S$ is an (S, S)-isomorphism.*

In general $\mathcal{J}$ is not an isomorphism. Indeed, $\mathcal{J}$ is an epimorphism if and only if P_R is a generator in $\mathfrak{M}_R$. The image T of $\mathcal{J}$ is an ideal of R called the trace ideal of P. Since $T = \sum_{f \in P^*} Im\, f$, $PT = P$.

We note finally that ${}_R P^*$ is also finitely generated and projective with T as trace ideal, $TP^* = P^*$, and $S = \mathrm{End}_R({}_R P^*)$. (We write homomorphisms opposite scalars.)

We shall continue the notation of this section throughout this note.

Results. We begin this section with several lemmas. To simplify their statement we let

$$\mathrm{Ker}(F) = \{M \in \mathfrak{M}_R \mid F(M) = M \otimes_R P^* = 0\} .$$

2.1 Lemma. *Let $M \in \mathfrak{M}_R$. Then $M \in \mathrm{Ker}(F)$ if and only if $MT = 0$.*

Proof. If $M \otimes P^* = 0$, then $M \otimes P^* \otimes P = 0$. But $M \otimes P^* \otimes P$ maps onto MT via the map $m \otimes f \otimes x \to m\mathcal{J}(f \otimes x)$, so $MT = 0$.

If $MT = 0$, then $M \otimes P^* = 0$ since $TP^* = P^*$.

2.2 Lemma. *For all $M \in \mathfrak{M}_R$ the exact sequence*

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PF-MODULES

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Azumaya [1] and, independently, Utumi [16] studied rings with the property that every faithful right R-module is a generator and obtained essentially the same structure theorem for this class of rings which Utumi called (right) PF-rings. About the same time Osofsky [12] obtained a structure theorem for the class of right self injective cogenerator rings which shows that they are precisely the (right) PF-rings. This class of rings and another related generalization of quasi-Frobenius rings, the (two-sided) cogenerator rings, have been studied recently by a number of authors notably Kato [7], [8] and [9] and Onodera [11].

The purpose of this note is to consider a module theoretic generalization of (right) PF-rings. We say that a (right) R-module P_R is a *PF-module* if P_R is a finitely generated, projective and injective module with the property that every simple homomorphic image of P_R is isomorphic to a submodule of P_R. We prove that if P_R is a PF-module then $S=\mathrm{End}_R(P_R)$ is a (right) PF-ring and deduce a structure theorem for P_R similar to the Azumaya-Osofsky-Utumi Theorem for (right) PF-rings. Then we consider briefly the question of when the endomorphism ring of a finitely generated projective module is a (two-sided) cogenerator ring. Our work also generalizes a result of Rosenberg and Zelinsky [15] who proved that the endomorphism ring of a PF-module over a quasi-Frobenius ring is quasi-Frobenius as well as some recent results of Wagoner [17].

Preliminaries. We assume throughout that all rings have identities and all modules are unitary.

Let P_R be a finitely generated projective right R-module and $S=\mathrm{End}_R(P_R)$. Then ${}_SP_R$ is a bimodule and letting ${}_RP_S^*=\mathrm{Hom}_R({}_SP_R, R)$ we have functors

$$F = \text{——} \underset{R}{\otimes} P^* : \mathfrak{M}_R \to \mathfrak{M}_S \quad \text{and}$$

$$G = \text{——} \underset{S}{\otimes} P, \quad H = \mathrm{Hom}_S(P^*, \text{——}) : \mathfrak{M}_S \to \mathfrak{M}_R ,$$

where $\mathfrak{M}_R$ and $\mathfrak{M}_S$ denote the categories of right R-modules and right S-modules respectively. The functors (G, F, H) form an adjoint triple. That is there are natural isomorphisms

$$0 \to \mathrm{Ker}\beta_M \to M \xrightarrow{\beta_M} HF(M) \to \mathrm{Cok}\beta_M \to 0$$

has $\mathrm{Ker}\beta_M$, $\mathrm{Cok}\beta_M \in \mathrm{Ker}(F)$.

PROOF. By [10, (53), p. 55] we have $\alpha'_{F(M)}F(\beta_M) = 1_{F(M)}$. Thus since $\alpha_{F(M)}$ is an isomorphism and F is an exact functor the conclusion follows.

2.3 LEMMA. *If* $Q \in \mathfrak{M}_R$ *is injective and* $\mathrm{Ann}_T(Q) = \{x \in Q \mid xT = 0\} = 0$, *then* $F(Q)$ *is injective.*

PROOF. We first note that $\beta_Q: Q \to HF(Q)$ is an isomorphism. β_Q is a monomorp hism since $\mathrm{Ker}\beta_Q \subseteq \mathrm{Ann}_T(Q)$ by Lemmas 1 and 2. Furthermore, $\mathrm{Ann}_T(HF(Q)) = 0$ since

$$\mathrm{Hom}_R(R/T, HF(Q)) \cong \mathrm{Hom}_S(F(R/T), F(Q)) = 0$$

as $F(R/T) = 0$ by Lemma 1. It, therefore, follows easily from Lemmas 1 and 2 that $HF(Q)$ is an essential extension of Im β_Q and so β_Q is an isomorphism.

Now

$$\begin{aligned}\mathrm{Hom}_S(\text{——}, F(Q)) &\cong \mathrm{Hom}_S(FH(\text{-}), F(Q)) \\ &\cong \mathrm{Hom}_R(H(\text{-}), HF(Q) \cong \mathrm{Hom}_R(H(\text{-}), Q)\end{aligned}$$

where the first equivalence follows because $FH \cong 1_{\mathfrak{M}_S}$, the second from adjointness and the third from the isomorphism established above. Thus since H is left exact and Q is injective, $\mathrm{Hom}_S(\text{——}, F(Q))$ takes monomorphisms into epimorphisms and so $F(Q)$ is injective.

2.4 LEMMA. *If N is a simple right S-module,* $N \cong F(M)$ *where* M_R *is a simple epimorphic image of* P_R.

PROOF. Since N is simple, P_R is finitely generated and G is right exact, $G(N) = N \otimes P_R$ is finitely generated and so contains a maximal submodule K. Let $M = G(N)/K$. Since $PT = P$, $G(N)T = G(N)$ and so $MT = M$. Thus M is an epimorphic image of T and so also of P. Since F is exact the natural epimorphism of $G(N)$ onto M induces an epimorphism of $F(G(N))$ onto $F(M)$. Since $F(G(N)) \cong N$ via α_N and $F(M) \neq 0$ by Lemma 1, we conclude $N \cong F(M)$.

Recall that a module M_R is a *cogenerator* in $\mathfrak{M}_R$ if and only if every right R-module can be imbedded in a direct product of copies of M. It is well known that M is a cogenerator if and only if M contains a copy of the injective hull $E(U_R)$ of U_R for every simple right R-module U_R. (See [12].)

In order to prove our main theorem we would like to apply Osofsky's structure theorem [12, Thm. 1] for right self injective cogenerator rings. Unfortunately, we cannot apply it directly. However, a straightforward modification of her proof serves to establish the following theorem.

2.5 THEOREM. (Osofsky) *Let R_R be a cogenerator and R/J be a right self injective ring where J is the radical of R. Then R is right self injective and $R=\sum_{i=1}^{n} e_i R$ where $\{e_i|i=1,\cdots,n\}$ is a set of orthogonal idempotents with e_iR/e_iJ simple for each $i=1,\cdots,n$.*

2.6 THEOREM. *Let P_R be a (right) PF-module and $S=\mathrm{End}_R(P_R)$. Then S is a (right) PF-ring and $P=P_1\oplus\cdots\oplus P_n$ with P_i/P_iJ simple for $i=1,\cdots,n$ where J is the radical of R.*

PROOF. Let N be any simple right S-module. Then by Lemma 4, there exists a simple epimorphic image M of P such that $F(M)=M\otimes P^*\cong N$. Since P is right *PF*, it contains a copy of $E(M)$, the injective hull of M. Furthermore, M essential in $E(M)$ implies $\mathrm{Ann}_T(E(M))=0$ since $\mathrm{Ann}_T(E(M))\cap M=\mathrm{Ann}_T(M)=0$ as M is simple and $MT=M$.

Applying F to the exact sequence $0\to E(M)\to P$ gives an exact sequence $0\to E(M)\otimes P^*\to P\otimes P^*$. However, $P\underset{R}{\otimes} P^*\cong S_S$ by Lemma 1.1 and $E(M)\otimes P^*$ is injective by Lemma 3 and contains a copy of $M\otimes P^*\cong N$. Thus S_S contains a copy of $E(N)$, the injective hull of N. Since N is an arbitrary simple right S-module, S_S is a cogenerator.

Since P_R is injective, $S/J(S)$ is a right self injective ring by [13, Thm. 12], where $J(S)$ is the radical of S. It, therefore, follows from Theorem 5 that S is (right) *PF*.

Now by Theorem 5 there exist orthogonal primitive idempotents $\{e_i|i=1,\cdots,n\}$ in S such that $1=e_1+\cdots+e_n$. Then $P=P_1\oplus\cdots\oplus P_n$ where $P_i=e_iP$. Since each P_i is an indecomposable projective and injective R-module P_i is isomorphic to a direct summand of R_R[6, Cor. 2.5]. Thus P_i/P_iJ is isomorphic to a right ideal of R/J and since $\mathrm{End}_R(P_i/P_iJ)\cong e_iSe_i/e_iJ(S)e_i$ is a division ring and R/J is a semi-prime ring, P_i/P_iJ is a simple R-module [14, p. 65].

2.7 COROLLARY. *If P_R is a (right) PF-module which satisfies the ascending or descending chain condition on submodules, then $S=\mathrm{End}_R(P_R)$ is a quasi-Frobenius ring.*

PROOF. By [4, Cor. 2, p. 35] there exists a 1-to-1 lattice homomorphism from the lattice of right ideals of S into the lattice of R-submodules of P_R. Thus S satisfies either the ascending or descending chain condition on right ideals and so is quasi-Frobenius [5, Thm. 1].

This Corollary generalizes results of Rosenberg and Zelinsky [15, Cor. 3. 8] and Wagoner [17, Thm. 1. 3].

2. 8 REMARK. The proof of Theorem 6 also shows that if P_R contains a copy of the injective hull of ach of its simple epimorphic images then S is a cogenerator in $\mathfrak{M}_S$ and by symmetry if ${}_RP^*$ also has this property then S is a (two-sided) cogenerator ring.

If R is a (two-sided) cogenerator ring, i. e., ${}_RR$ and R_R are cogenerators in ${}_R\mathfrak{M}$ and $\mathfrak{M}_R$, respectively, then R is both left and right PF(See [11].) and hence is also semi-perfect [2].

Thus any finitely generated projective right R-module P_R is isomorphic to a finite direct sum $\bigoplus_{i=1}^{n} e_iR$ where the e_i are primitive idempotents. Hence each e_iR is injective, contains a unique minimal right ideal and e_iR/e_iJ is simple. Thus P_R is injective and contains an essential socle. Since the correspondence $U \to E(U)/E(U)J$ induces a 1-to-1 correspondence between the isomorphism classes of simple submodules of P_R and the isomorphism classes of simple epimorphic images of P_R, a simple counting argument shows that P_R is right PF if and only if every simple submodule of P_R is an epimorphic image of P_R. This occurs if and only if $\mathrm{Ann}_T(P) = 0$. By symmetry the same is true for finitely generated projective left R-modules.

2. 9 COROLLARY. *If R is a (two-sided) cogenearator ring and P_R is a finitely generated projective module such that every simple epimorphic image of P_R is isomorphic to a submodule of P_R, then $S = \mathrm{End}_R(P_R)$ is a (two-sided) cogenerator ring.*

PROOF. By Remark 2. 8 and the comments just preceeding, it suffices to show that $\mathrm{Ann}_T(P^*) = 0$.

Since P_R is right PF, we have as in the proof of Lemma 3 that $P_R \cong HF(P_R) = \mathrm{Hom}_S({}_RP_S{}^*, P \otimes_R P_S{}^*)$ via β_P. Now since $P \otimes_R P^* \cong S_S$ by Lemma 1. 1 and S_S is a cogenerator, there exists for each $0 \neq f \in P^*$ an $h \in \mathrm{Hom}_S(P_S{}^*, P \otimes_R P_S{}^*)$ such that $h(f) \neq 0$. But there exists $p \in P$ such that $\beta(p) = h$ and so $0 \neq \beta(p)(f) = p \otimes f$. Since $PT = P$, we conclude that $f \in \mathrm{Ann}_T(P^*)$ and so $\mathrm{Ann}_T(P^*) = 0$.

This result has also been obtained by Wagoner [17, Thm. 1. 15] using an entirely different technique of proof.

REFERENCES

[1] G. AZUMAYA, Completely faithful modules and self-injective rings, Nagoya Math. J., 27(1966), 697-708.
[2] H. BASS, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc., 95(1960), 466-488.
[3] H. BASS, The Morita Theorems. Lecture Notes, University of Oregon, 1962.
[4] P. M. COHN, Morita Equivalence and Duality. Lecture Notes, Queen Mary College, 1966.
[5] C. FAITH, Rings with ascending chain condition on annihilators, Nagoya Math. J., 27 (1966), 179-191.
[6] C. FAITH AND E. A. WALKER, Direct-sum representations of injective modules, J. Algebra, 5(1967), 203-220.
[7] T. KATO, Self-injective rings, Tôhoku Math. J., 19(1967), 485-495.
[8] T. KATO, Torsionless modules, Tôhoku. Math. J., 20(1968), 234-243.
[9] T. KATO, Some generalizations of QF-rings, Proc. Japan Acad. 44 (1968), 114-119.
[10] K. MORITA, Adjoint pairs of functors and Frobenius extensions, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 9(1965), 40-71.
[11] T. ONODERA, Über Kogeneratoren, Arch. Math. Vol. XIX (1968), 402-410.
[12] B. L. OSOFSKY, A generalization of quasi-Frobenius rings, J. Algebra, 4(1966), 373-387.
[13] B. L. OSOFSKY, Endomorphism rings of quasi-injective modules, Canad. J. Math., 20(1968), 895-903.
[14] N. JACOBSON, Structure of Rings, Amer. Math. Soc. Colloq. Pub. Vol. 36, Providence, R. I. (1964).
[15] A. ROSENBERG AND D. ZELINSKY, Annihilators, Portugal. Math., 20(1961), 53-65.
[16] Y. UTUMI, Self-injective rings, J. Algebra, 6(1967), 56-64.
[17] R. WAGONER, Endomorphism rings of projective RZ modules, Doctoral Dissertation, The University of Oregon, 1969.

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