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HOMOMORPHISMS OF TENSOR ALGEBRAS

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Introduction. In the study of spectral synthesis and symbolic calculus for tensor algebras, certain homomorphisms of tensor algebras play a crucial rôle ([21], [23]; see also [8]). Thus the problem of determining homomorphisms of tensor algebras naturally arises. C. C. Graham [7] has recently characterized all automorphisms of tensor algebras under certain topological conditions. On the other hand, some authors (e. g. [5], [11], [12], and [14]) have obtained several interesting results on the homomorphism problem for restriction algebras of the Fourier algebra $A(T)$. Their works are, however, incomplete as compared with Cohen's theorem on homomorphisms of group algebras ([2]; see also [16]). Since every tensor algebra can be regarded as a restriction algebra of a Fourier algebra ([19], [21], and [23]), it thus seems reasonable that one should treat the homomorphism problem for tensor algebras as a step in the direction of identifying homomorphisms of restriction algebras of Fourier algebras.

In this paper, we shall consider that problem for tensor algebras over two compact spaces, and, in particular, entirely describe all homomorphisms of such algebras with norm 1.

In § 1, we determine the structures of all idempotent functions and unimodular functions in such algebras with norm smaller than certain constants.

§ 2 contains our main results on the homomorphism problem. We introduce the notion of "piecewise product mappings", and characterize by it all homomorphisms with norm smaller than a certain constant.

In § 3, we determine the isomorphisms between two tensor algebras over compact connected spaces, which improves a result of C. C. Graham [7].

§ 4 is devoted to obtain certain properties of tensor algebras as restriction algebras of Fourier algebras.

Finally, in § 5, we make the calculus of some constant and estimate another constant obtained in § 1.

1. Idempotent and unimodular functions.

Let X_1 and X_2 be two compact (Hausdorff, and nonempty) spaces, and let

$$V = V(X_1, X_2) = C(X_1) \widehat{\otimes} C(X_2)$$

be the tensor algebra over the spaces X_1 and X_2 with the projective norm (see [23]). We shall always regard $V = V(X)$ as a linear subalgebra of the Banach algebra $C(X)$, where $X = X_1 \times X_2$. Let us denote by π_j the canonical projection from X onto $X_j (j = 1, 2)$, and by $I_E = I[E]$ the indicator of any set E. If f is a function on X, and if E is a subset of X, we define

$$\|f\|_{V(E)} = \inf\{\|g\|_V : g \in V, g|_E = f\} .$$

For any point p of X, p_j denotes the j-*th* coordinate of $p(j = 1, 2) : p = (p_1, p_2)$. Finally, $V' = V'(X)$ denotes the conjugate space of the Banach space V, each element of which is called a bimeasure on X.

Let now f be any idempotent function in $C(X)$. It is then easy to see that f admits a decomposition of the form

$$f = I[E_1] + \cdots + I[E_n] ,$$

where the sets E_k are pairwise disjoint, clopen, and rectangular subsets of X; in particular, we see that every idempotent function in $C(X)$ belongs to $V(X)$.

It is known ([17]; see also [18]) that $\|\mu\|_M > 1$ implies $\|\mu\|_M \geqq (1 + 2^{1/2})/2$ for any idempotent measure μ on a locally compact abelian group. An analogous result also holds for idempotent functions in $C(X)$. In fact, $\|f\|_V > 1$ implies $\|f\|_V \geqq 2/3^{1/2}$ for any such functions. To show this, we need a lemma.

LEMMA 1.1. *Let $D = \{p, q, r, s\}$ be a set of four points of X such that*

$$p_1 = r_1 \neq q_1 = s_1, \text{ and } p_2 = q_2 \neq r_2 = s_2 .$$

Suppose also that f is any function on X such that $f = 0$ at a point of D and $|f| = 1$ at the other three points of D, then we have $\|f\|_{V(D)} = 2/3^{1/2}$.

PROOF. In general, we shall denote by $C_1(K)$, for any compact space K, the multiplicative group consisting of all unimodular functions in $C(K)$ (that is, $g \in C(K)$ with $|g| = 1$). Let $\mu \in M(D)$ be any measure on D, and put

$$\mu(\{p\}) = a, \mu(\{q\}) = b, \mu(\{r\}) = c, \mu(\{s\}) = d .$$

It is then easy to see that

$$\|\mu\|_V = \sup\left\{\left|\int_D g_1 \cdot g_2 \, d\mu\right| : g_j \in C_1(X_j), j = 1, 2\right\}$$

$$= \sup\{|a + bz| + |c + dz| : |z| = 1\} = A(a, b, c, d).$$

Suppose now that f is any function on X satisfying the condition in the above statement; without loss of generality, we may assume that $f(s)=0$. Then we have

$$\|f\mu\|_{V'} = A(a, b, c, 0) = |a| + |b| + |c| .$$

Since $\|f\mu\|_{V'} \leqq \|f\|_{V(D)} \cdot \|\mu\|_{V'}$, it follows that

$$A(a, b, c, 0) \leqq \|f\|_{V(D)} \cdot A(a, b, c, d) .$$

Therefore, setting

$$u_0 = \sup\{A(a, b, c, 0)/A(a, b, c, d) : abcd \neq 0\} , \tag{1.1}$$

we have $\|f\|_{V(D)} \geqq u_0$. But, as is easily seen from the Hahn-Banach theorem, the equality $\|f\|_{V(D)} = u_0$ holds. We have also

$$u_0 \geqq A(2, 2, 2, 0)/A(2, 2, 2, -1) = 2/3^{1/2} .$$

The equality $u_0 = 2/3^{1/2}$ will be proved in § 5, and this establishes our lemma.

Throughout the remainder parts of this paper, u_0 denotes the constant $2/3^{1/2}$. Following Graham [7], we say that two subsets E and F of X are *bidisjoint* if $\pi_j(E)$ and $\pi_j(F)$ are disjoint for $j=1$ and 2.

THEOREM 1.2. *For any idempotent function f in $C(X)$, $\|f\|_V > 1$ implies $\|f\|_V \geqq u_0$, and we have $\|f\|_V = 1$ if and only if f has the form*

$$f = I[E_1] + \cdots + I[E_n] \qquad (n \geqq 1) , \tag{1.2}$$

where the sets E_k are pairwise bidisjoint, clopen, nonempty, and rectangular subsets of X.

PROOF. Suppose that $\|f\|_V < u_0$, and let E be any maximal rectanglar subset of $S(f) = \{x \in X : f(x) = 1\}$. We then claim that

$$\pi_j(E) \cap \pi_j(S(f)\backslash E) = \emptyset \qquad (j = 1, 2) . \tag{1}$$

To get a contradiction, suppose the contrary; we have, say, $\pi_1(E) \cap \pi_1(S(f)\backslash E) \neq \emptyset$. Let r be any point of $S(f)\backslash E$ such that r_1 is in $\pi_1(E)$, and choose an arbitrary point s of the set $\pi_1(E) \times \{r_2\}$. Taking any point p_2 of $\pi_2(E)$, we see that all the points $p = (r_1, p_2)$, $q = (s_1, p_2)$, and r are in $S(f)$. It follows from Lemma 1.1

and the assumption $\|f\|_V < u_0$ that s must be in $S(f)$. Therefore we have $\pi_1(E) \times \{r_2\} \subset S(f)$, and so

$$E \subsetneqq \pi_1(E) \times (\pi_2(E) \cup \{r_2\}) \subset S(f),$$

which contradicts the maximality of E, and hence (1) holds. Note now that E is clopen since $S(f)$ is both open and compact, and that the family of such sets E covers $S(f)$. Thus we can easily conclude that f has the desired form, provided that f is nonzero.

Conversely, suppose that f admits a decomposition of the form (1.2). Setting $F_k = \pi_1(E_k)$ and $G_k = \pi_2(E_k)$, we define

$$g_k = \frac{1}{2}\{1 + (I[F_k] - I[F_k^c])(I[G_k] - I[G_k^c])\}$$

for $k = 1, 2, \cdots, n$; it is easy to see that every g_k is idempotent and has V-norm 1, and that $f = g_1 \cdots g_n \cdot I_E$, where $E = \pi_1(S(f)) \times \pi_2(S(f))$. Therefore $f(\neq 0)$ is an idempotent function with V-norm 1, and this completes the proof.

LEMMA 1.3. *Let D be a subset of X as in Lemma* 1.1, *and let f be a function in $C_1(X)$ such that*

$$f(p) = f(q) = f(r) = 1 \neq f(s),$$

then we have

$$\limsup_{n\to\infty} \|f^n\|_{V(D)} = \sup_n \|f^n\|_{V(D)} \geqq u_1, \tag{1.3}$$

where u_1 is an absolute constant larger than u_0.

PROOF. As in the proof of Lemma 1.1, we have

$$\|f^n\|_{V(D)} = \sup\{A(a, b, c, z_0^n d)/A(a, b, c, d)\} = B(z_0^n)$$

where $z_0 = f(s)$ and the supremum is taken over all complex numbers a, b, c, and d with $abcd \neq 0$. Therefore, setting

$$u_1 = \inf\{\sup_n B(z_0^n) : |z_0| = 1, z_0 \neq 1\}$$

we have

$$\sup_n \|f^n\|_{V(D)} \geqq u_1,$$

and, as is proved in § 5, $u_1 > u_0$. The equality in (1.3) is obvious, and the proof is complete.

We now obtain an analogue of a theorem of Beurling and Helson [1].

THEOREM 1.4. *Suppose that f is a function in $C_1(X)$. Then we have*

$$\limsup_{n\to\infty} \|f^n\|_{V(D)} < u_1 \tag{1.4}$$

for all subsets D of X as in Lemma 1.1 if and only if f has the form $f = f_1 \otimes f_2$ for some $f_j \in C_1(X_j)$ $(j = 1, 2)$. In this case f is in $V(X)$ and its V-norm is 1.

PROOF. The last assertion and the sufficient condition are obvious, and it suffices to show only the necessary condition. Fix any point p of X; replacing f by $\overline{f(p)}f$, we may assume that $f(p) = 1$. We then define

$$f_1(x_1) = f(x_1, p_2), \quad f_2(x_2) = f(p_1, x_2) \qquad (x_j \in X_j\,;\ j = 1, 2)$$

and claim that $f = f_1 \otimes f_2$. Otherwise, there would exist a point s of X such that $f(s) \neq f_1(s_1) \cdot f_2(s_2)$. Choose any function g_j in $C_1(X_j)$ so that

$$g_j(p_j) = 1, \text{ and } g_j(s_j) = \overline{f_j(s_j)} \qquad (j = 1, 2)\,.$$

Setting $h = f \cdot (g_1 \otimes g_2)$, we then see that (1.4) holds with f replaced by h, and that

$$h(p) = h(q) = h(r) = 1 \neq h(s)\,,$$

where $q = (s_1, p_2)$ and $r = (p_1, s_2)$. Thus, by Lemma 1.3, we have

$$\limsup_{n\to\infty} \|h^n\|_{V(D)} \geqq u_1\,,$$

where $D = \{p, q, r, s\}$. This contradiction implies $f = f_1 \otimes f_2$, and the proof is complete.

COROLLARY 1.5. *Suppose that f is any non-zero function in $C(X)$ such that $|f|^2 = |f|$. Then we have*

$$\limsup_{n\to\infty} \|f^n\|_{V(D)} < u_0 \tag{1.5}$$

for all subsets D of X as in Lemma 1.1 *if and only if f has the form*

$$f = \sum_{k=1}^{n} (f_1^{(k)} \otimes f_2^{(k)}) \cdot I[E_k], \tag{1.6}$$

where $f_j^{(k)} \in C_1(X_j)$ and the sets $E_k (1 \leqq k \leqq n)$ are pairwise bidisjoint, clopen, nonempty, and rectangular subsets of X.

PROOF. If f satisfies (1.5), it follows from Lemma 1.1 and Theorem 1.2 that $|f|$ is an idempotent function whose V-norm is smaller than u_0. Thus Theorem 1.2 assures that $|f|$ admits a decomposition of the form (1.2). Applying Theorem 1.4 for X replaced by E_k, we see that f has the form $f = f_1^{(k)} \otimes f_2^{(k)}$ on each se E_k for some $f_j^{(k)}$ in $C_1(X_j)$ $(j = 1, 2; k = 1, \cdots, n)$, which yields the required decomposition of f.

Conversely, suppose that f has the form (1.6). Defining

$$f_j = \sum_{k=1}^{n} f_j^{(k)} \cdot I[\pi_j(E_k)] \qquad (j = 1, 2),$$

we see that

$$f = (f_1 \otimes f_2) \cdot \sum_{k=1}^{n} I[E_k],$$

which together with Theorem 1.2 shows that $\|f\|_V = 1$. The proof is now complete.

REMARK. That (1.6) implies $\|f\|_V = 1$ is also contained in [24] (see also [6] and [7]).

2. Homomorphisms of tensor algebras. In this section let us fix four compact spaces X_j and $Y_j (j = 1, 2)$, and put $X = X_1 \times X_2, Y = Y_1 \times Y_2$. By the same notation π_j we denote both of the canonical projections from X onto X_j and from Y onto $Y_j (j = 1, 2)$. If φ is a mapping from a subset E of X into Y, and if f is a function on $Y, f \circ \varphi$ denotes the function on X defined by

$$(f \circ \varphi)(x) = \begin{cases} f(\varphi(x)) & (x \in E) \\ 0 & (x \in X \backslash E). \end{cases}$$

Let also $\varphi_j = \pi_j \circ \varphi : E \to Y_j$, and so $\varphi = (\varphi_1, \varphi_2)$. We say that φ is a piecewise *product mapping*, provided that:

(A) The domain E of φ has the form $E=\cup_1^n E_k$, where the sets E_k are pairwise disjoint, clopen, nonempty, and rectangular subsets of X.

(B) For each $j=1,2$, and $k=1,\cdots,n$, the mapping φ_j restricted to E_k depends only on one of the variables x_1 and x_2.

Suppose now that φ is a continuous piecewise product mapping from X into Y with domain E, and let $\{E_k\}_1^n$ be as in (A) and (B). Defining

$$H(f)=f\circ\varphi \qquad (f\in V(Y)), \tag{2.1}$$

we easily see that

$$\|H(f)\cdot I[E_k]\|_{V(X)}\leqq\|f\|_{V(Y)} \qquad (f\in V(Y)\,;\ k=1,\cdots,n)\,.$$

Therefore H is an algebra homomorphism from $V(Y)$ to $V(X)$, and its operator norm does not exceed n. As is proved below, all homomorphisms of tensor algebras satisfying a certain norm condition are of the above type.

Let now H be any non-zero homomorphism from $V(Y)$ to $V(X)$, then $H(1)$ is an idempotent function in $C(X)$, and so the set

$$E=E_H=\{x\in X:\ H(1)(x)=1\}$$

is clopen. By a familiar argument [13] there exists a continuous mapping $\varphi=\varphi_H$ from E into Y for which (2.1) holds. Hereafter, we shall fix H arbitrarily, and associate with it the set E and the mapping φ as above.

LEMMA 2.1. *Let D be a subset of X as in lemma 1.1, and let f be a function on D such that $f=1$ at some three points of D and $f=-1$ at the remainder point of D, then we have $\|f\|_{V(D)}\geqq 2^{1/2}$.*

PROOF. Using the notation $A(a,b,c,d)$ in the proof of Lemma 1.1, we have

$$\begin{aligned}\|f\|_{V(D)}&=\sup\{A(a,b,c,-d)/A(a,b,c,d):\ abcd\neq 0\}\\&\geqq A(1,1,1,1)/A(1,1,1,-1)=2^{1/2}\,.\end{aligned}$$

LEMMA 2.2. *Let $\{E_k\}_1^n$ be pairwise bidisjoint, clopen, and rectangular subsets of X. Then for every bimeasure P in $V'(X)$, we have*

$$\|P\|_{V'}\geqq\|I[E]P\|_{V'}=\sum_{k=1}^{n}\|I[E_k]P\|_{V'}\,, \tag{2.2}$$

where $E=\bigcup_{k=1}^{n} E_k$.

PROOF. For any functions $f_j^{(k)}$ in $C_1(X_j)$ $(j=1,2;\ k=1,\cdots,n)$, we have by Corollary 1.5

$$\|P\|_{V'} \geqq \|I[E]P\|_{V} \geqq \left| P\left(\sum_{k=1}^{n} (f_1^{(k)} \otimes f_2^{(k)}) I[E_k]\right)\right|$$

$$= \left| \sum_{k=1}^{n} (I[E_k]P)(f_1^{(k)} \otimes f_2^{(k)}) \right| .$$

Taking the supremum over all such $f_j^{(k)}/S$, we obtain (2.2).

THEOREM 2.3. *If* $\|H(1)\|_V < u_0$, *and if* $\|H\| < 2^{1/2}$, *then* φ *is a piecewise product mapping such that the sets* $\{E_k\}_1^n$ *as in* (A) *and* (B) *can be so chosen as to be bidisjoint. In this case the operator norm of* H *is* 1.

PROOF. Since $H(1)$ is a non-zero idempotent function whose V-norm is smaller than u_0, we have $E=\cup_1^n E_k (n\geqq 1)$, where the sets $\{E_k\}_1^n$ are as in Theorem 1.2. We then claim that (B) holds for these sets $\{E_k\}_1^n$. To prove this, fix j and $k(j=1,2;\ k=1,\cdots,n)$, and assume that there exist two points p and r in E_k such that $p_1=r_1$ but $\varphi_j(p)\neq\varphi_j(r)$. We can then verify that φ_j restricted to E_k does not depend on the variable z_1 as follows.

Step 1. To get a contradiction, suppose that there exists a point q in E_k such that $q_2=p_2$ but $\varphi_j(q)\neq\varphi_j(p)$. Setting $s=(q_1, r_2)$, note that s is in E_k since E_k is rectangular. If $\varphi_j(p)\neq\varphi_j(s)$, we choose an $f\in C_1(Y_j)$ so that $f\circ\varphi_j(p) = -1$ and $f=1$ on $\varphi_j(\{q,r,s\})$; if $\varphi_j(p)=\varphi_j(s)$, we choose an $f\in C_1(Y_j)$ so that $f\circ\varphi_j(p)=1$ and $f=i$ on $\varphi_j(\{q,r\})$. Then we have

$$\|H\| \geqq \|H(f)\|_{V(X)} \geqq 2^{1/2}$$

by Lemma 2.2, where we have regarded $C(Y_j)$ as a subalgebra of $V(Y)$. This contradiction assures that φ_j is constant on $\pi_1(E_k)\times\{p_2\}$.

Step 2. Replacing p and r in Step 1, we see that φ_j is constant on $\pi_1(E_k)\times\{r_2\}$, too. Let t be any point of E_k with $t_1=p_1$. Since $\varphi_j(p)\neq\varphi_j(r)$, we then have either $\varphi_j(t)\neq\varphi_j(p)$ or $\varphi_j(t)\neq\varphi_j(r)$. Thus the preceeding argument applies, and hence φ_j is constant on $\pi_1(E_k)\times\{t_2\}$.

Similarly, we can show that, if there exist two points p and q of E_k such that $p_2=q_2$ but $\varphi_j(p)\neq\varphi_j(q)$, then φ_j restricted to E_k does not depend on the variable x_2. Therefore φ is a piecewise product mapping by definition.

We now prove that $\|H\| = 1$ if H has the form described in our theorem. To do this, it suffices to verify for any $f \in V(X)$

$$\|I[E]\cdot f\|_V = \sup\{\|I[E_k]\cdot f\|_V : k = 1, \cdots, n\}; \tag{2.3}$$

but this follows from Lemma 2.2 and the Hahn-Banach theorem (cf. [24], [6], and [7]):

$$\begin{aligned}\|I[E]\cdot f\|_V &= \sup\{|P(I[E]\cdot f)| : P \in V, \|P\|_{V'} \leqq 1\} \\ &= \sup\left\{\sum_{k=1}^{n} |P_k(f)| : P_k \in V'(E_k), \sum_{k=1}^{n} \|P_k\|_{V'} \leqq 1\right\} \\ &= \sup\{\|I[E_k]f\|_V : k = 1, \cdots, n\}.\end{aligned}$$

The proof is now complete.

COROLLARY 2.4 (cf. [14]). *Every isometric homomorphism H with $H(1) = 1$ is essentially of the type*

$$H(f) = f \circ (\psi_1 \times \psi_2) \qquad (f \in V(Y)), \tag{2.4}$$

where each ψ_j is a continuous mapping from X_j onto $Y_j (j = 1, 2)$. Conversely, a pair of such mappings (ψ_1, ψ_2) defines by (2.4) an isometric homomorphism H with $H(1) = 1$. In this case, the range of H consists of all functions g in $V(X)$ such that $g = f \circ (\psi_1 \times \psi_2)$ for some f in $C(Y)$.

PROOF. Note that $E = X$, since $H(1) = 1$. By Theorem 2.3, each mapping φ_j depends on only one of the variables x_1 and x_2. Suppose first that both mappings φ_1 and φ_2 depend on (only) the some variable, say, x_1. We then have

$$\|f\|_{V(Y)} = \|H(f)\|_{V(X)} = \|H(f)\|_{C(X)} \leqq \|f\|_{C(Y)} \qquad (f \in V(Y)),$$

because H is isometric. But this is the case only if at least one of the spaces Y_1 and Y_2, say Y_2, consists of a single point (see Lemma 2.1). Therefore, if we define

$$\psi_1(x_1) = \varphi_1(x_1, x_2), \text{ and } \psi_2(x_2) = \varphi_2(x_1, x_2) \quad (x_j \in X_j : j = 1, 2), \tag{1}$$

it is easy to see that each ψ_j continuously maps X_j onto $Y_j (j = 1, 2)$, and that H is given by (2.4). Suppose next that each φ_j depends on (only) the variable

$x_j (j=1,2)$. It then suffices to define ψ_1 and ψ_2 by (1), again. Thus the first statement is established. The remainder two statements are contained in [20], and the proof is complete.

COROLLARY 2.5. *Every isomorphism H from $V(Y)$ onto $V(X)$ with max $(\|H\|, \|H^{-1}\|) < 2^{1/2}$, is isometric, and essentially of the type* (2.4), *where each ψ_j is a homeomorphism from X_j onto $Y_j (j=1,2)$.*

PROOF. Trivial from the proof of Corollary 2.4.

REMARKS. (a) The identity (2.3) is also a consequence of Lemma 2 in [24]. But the author cannot understand the proof given there.

(b) It is not true that every homomorphism of tensor algebras is induced by a piecewise product mapping.

Let $\{I_n\}_1^\infty$ be a sequence of disjoint closed intervals $I_n = [a_n, b_n]$ of the real line such that

$$0 = a_1 < a_2 < \cdots, \text{ and } \lim a_n = \lim b_n = 1.$$

Let K be the union of all I_n with the limit point $\{1\}$. We define a mapping ψ from $K \times K$ to K by

$$\psi(x) = \begin{cases} x_1 \text{ if } x \in I_n \times I_n \text{ and } n \text{ is odd,} \\ x_2 \text{ if } x \in I_n \times I_n \text{ and } n \text{ is even,} \\ 1 \text{ otherwise.} \end{cases}$$

Then, using (2.3), we can easily prove that, if $\varphi(x) = (\psi(x), \psi(x))$, φ induces a homomorphism H of $V(K, K)$ (into itself) by (2.1) (see [24]).

3. Isomorphisms of tensor algebras.

Throughout this section, we shall assume that H is an isomorphism from $V(Y)$ onto $V(X)$, and that both the spaces X_1 and X_2 are perfect. It follows, in particular, that φ is a homeomorphism from X onto Y.

Let now E be any subset of X. We say that E is *diagonal* if the sets $\{x\}$, x in E, are pairwise bidisjoint, and that E is *parallel to* X_1 (resp. X_2) if $\pi_2(E)$ (resp. $\pi_1(E)$) consists of a single point. Finally, let

$$\mathcal{D}(E) = \sup\{\text{Card } D : D \text{ is a diagonal subset of } E\}.$$

LEMMA 3.1. *Let K_1 and K_2 be two finite spaces each of which consists of n distinct points, and let $K = K_1 \times K_2$. Then there is a function f in $V(K)$ such that*

$$|f(x)| = 1 \ (x \in K), \text{ and } \|f\|_{V(K)} \geqq n^{1/2}.$$

PROOF. It is known [23; pp. 87-88] that there exists a measure μ in $M(K)$ such that

$$|\mu(\{x\})| = n^{-2} \ (x \in K), \text{ and } \|\mu\|_{V} \leqq n^{-1/2}.$$

Define $f(x) = \operatorname{sgn} \mu(\{x\})$ for x in K; then we have

$$1 = \int_K \bar{f}\, d\mu \leqq \|f\|_V \cdot \|\mu\|_V \leqq \|f\|_V \cdot n^{-1/2},$$

which completes the proof.

LEMMA 3.2. *We have*

$$\mathcal{D}[\varphi(\{x_1\} \times X_2)] \leqq (\|H\| \cdot \|H^{-1}\|)^2 \qquad (x_1 \in X_1),$$

and

$$\mathcal{D}[\varphi(X_1 \times \{x_2\})] \leqq (\|H\| \cdot \|H^{-1}\|)^2 \qquad (x_2 \in X_2).$$

PROOF. Suppose that n is a positive integer larger than 1, and that there exists a point x_1 in X_1 such that

$$\mathcal{D}[\varphi(\{x_1\} \times X_2)] \geqq n.$$

Let $K_2 = \{x_2^{(1)}, x_2^{(2)}, \cdots, x_2^{(n)}\}$ be any subset of X_2 such that $\varphi(\{x_1\} \times K_2)$ is diagonal. Since x_1 is an accumulation point of X_1, and since φ is continuous, we can then find a subset K_1 of X_1 with Card $K_1 = n$ so that the n subsets of Y

$$\varphi(K_1 \times \{x_2\}) \qquad (x_2 \in K_2)$$

are pairwise bidisjoint. By Lemma 3.1, there is a function f in $V(K_1 \times K_2)$ such that

$$|f(x)| = 1 \ (x \in K_1 \times K_2), \text{ and } \|f\|_V \geqq n^{1/2}.$$

For each $k=1,2,\cdots,n$, choose a neighborhood $U^{(k)}$ of $K_1\times\{x_2^{(k)}\}$ so that the sets $\varphi(U^{(k)})$ are pairwise bidisjoint. It is then easy to find n functions $f^{(k)}$ in $V(X)$ such that:

$$f^{(k)}=f \text{ on } K_1\times\{x_2^{(k)}\}\ ;\ \|f^{(k)}\|_{V(X)}=1\ ;\ \operatorname{supp} f^{(k)}\subset U^{(k)}\ .$$

It follows that the supports of $H^{-1}(f^{(k)})$ are pairwise bidisjoint and hence (see (2.3), [24], and [6])

$$\left\|H^{-1}\left(\sum_{k=1}^{n}f^{(k)}\right)\right\|_{V(Y)}=\left\|\sum_{k=1}^{n}H^{-1}(f^{(k)})\right\|_{V(Y)}$$
$$=\sup\{\|H^{-1}(f^{(k)})\|_{V(Y)}:k=1,2,\cdots,n\}\leqq\|H^{-1}\|\ .$$

Therefore we have

$$n^{1/2}\leqq\left\|\sum_{k=1}^{u}f^{(k)}\right\|_{V(X)}\leqq\|H\|\cdot\left\|H^{-1}\left(\sum_{k=1}^{n}f^{(k)}\right)\right\|_{V(Y)}$$
$$\leqq\|H\|\cdot\|H^{-1}\|\ ,$$

which is trivial when $n=1$. We thus obtain

$$\mathscr{D}[\varphi(\{x_1\}\times X_2)]\leqq(\|H\|\cdot\|H^{-1}\|)^2\qquad(x_1\in X_1)\ ,$$

and similarly

$$\mathscr{D}[\varphi(X_1\times\{x_2\})]\leqq(\|H\|\cdot\|H^{-1}\|)^2\qquad(x_2\in X_2)\ .$$

This completes the proof.

LEMMA 3.3. *Both the spaces Y_1 and Y_2 are perfect.*

PROOF. Suppose that y_1 is an isolated point of Y_1, and so $\{y_1\}\times Y_2$ is clopen in Y. Since φ is a homromorphism, it follows that $\varphi^{-1}(\{y_1\}\times Y_2)$ is clopen, and hence $\varphi^{-1}(\{y_1\}\times Y_2)$ contains a non-empty, clopen rectangle E. If Card $\pi_j(E)=\infty\ (j=1,2)$, we could show that $\|H\|=+\infty$, using the fact that $V(E)\neq C(E)$ for such an E. Since $\pi_j(E)$ is clopen in $X_j(j=1,2)$, it follows that at least one of the spaces X_1 and X_2 contains an isolated point. This contradiction completes the proof.

LEMMA 3.4. *For any point $x=(x_1,x_2)$ of X, there exists a neighborhood*

U of x such that

$$\varphi[U\cap((\{x_1\}\times X_2)\cup(X_1\times\{x_2\}))]=\varphi(U)\cap((\{y_1\}\times Y_2)\cup(Y_1\times\{y_2\})),$$

where $y=(y_1,y_2)=\varphi(x_1,x_2)$.

PROOF. By Lemma 3.2, there is a neighborhood V_j of $x_j, j=1,2$, such that

$$(1)\qquad \varphi[(\{x_1\}\times V_2)\cup(V_1\times\{x_2\})]\subset(\{y_1\}\times Y_2)\cup(Y_1\times\{y_2\}).$$

On the other hand, Lemma 3.3 assures that an analogous conclusion as in Lemma 3.2 also holds for φ^{-1}, and hence there is a neighborhood W_j of $y_j, j=1,2$, such that

$$(2)\qquad \varphi^{-1}[(\{y_1\}\times W_2)\cup(W_1\times\{y_2\})]\subset(\{x_1\}\times X_1)\cup(X_1\times\{x_2\}).$$

Using (1) and (2), we can easily show the existence of a neighborhood U of x with the required property.

LEMMA 3.5. *Let $x=(x_1,x_2)$ be any point of X, and let y and U be as in Lemma 3.4. Then every point x_2' in $\pi_2[U\cap(\{x_1\}\times X_2)]$, possibly except x_2, has a neighborhood V_2 such that $\varphi[\{x_1\}\times V_2]$ is parallel to Y_2(or Y_1) and such that $\pi_2(\varphi[\{x_1\}\times V_2])$ (or $\pi_1(\varphi[\{x_1\}\times V_2])$) is open in Y_2(or Y_1). A similar assertion holds for every point x_1' in $\pi_1[U\cap(\{x_1\}\times X_2)]$, possibly except x_1.*

PROOF. It is easy to see that $\varphi[U\cap(\{x_1\}\times\{x_2\}^c)]$ is a relatively open subset of the set $(\{y_1\}\times Y_2)\cup(Y_1\times\{y_2\})$, from which our lemma follows.

We can now improve Lemma 3.2 as follows.

LEMMA 3.6. *We have*

$$\mathscr{D}[\varphi(\{x_1\}\times X_2)]\leqq\|H\|^2, \text{ and } \mathscr{D}[\varphi(X_1\times\{x_2\})]\leqq\|H\|^2$$

for all points x_j in $X_j(j=1,2)$.

PROOF. Let n, x_1, and K_2 be as in the proof of Lemma 3.2. By using Lemma 3.5 and replacing each $x_2^{(k)}$ by a point of X_2 sufficiently near to $x_2^{(k)}$, we may assume that every point x_2 of K_2 satisfies the following condition; there is a neighborhood V_2 of x_2 such that $\varphi(\{x_1\}\times V_2)$ is parallel to Y_2(or Y_1) and relatively open in $\{y_1\}\times Y_2$ (or $Y_1\times\{y_2\}$), where $(y_1,y_2)=\varphi(x_1,x_2)$. It then follows from Lemma 3.4 that to every point x_2 of K_2 corresponds a neighborhood V_1 of x_1 such that $\varphi(V_1 \times \{x_2\})$ is parallel to Y_1 or Y_2. Thus the set K_1 as in the proof of Lemma 3.2 can be taken so that the set $\varphi(K_1 \times \{x_2\})$ is parallel to Y_1 or Y_2 for every point x_2 of K_2. Let f be any function in $V(K_1 \times K_2)$ as in the proof of Lemma 3.2; it is easy to see that the function $f \circ \varphi^{-1}$ can be extended to a function g in $V(Y)$ such that $\|g\|_{V(Y)} = 1$. It follows that

$$n^{1/2} \leqq \|H(g)\|_{V(X)} \leqq \|H\| .$$

Thus we have

$$\mathcal{D}[\varphi(\{x_1\} \times X_2)] \leqq \|H\|^2 \qquad (x_1 \in X_1),$$

and similarly

$$\mathcal{D}[\varphi(X_1 \times \{x_2\})] \leqq \|H\|^2 \qquad (x_2 \in X_2).$$

THEOREM 3.7. *If the spaces X_1 and X_2 are connected, then φ is essentially of the form*

$$\varphi(x_1, x_2) = (\varphi_1(x_1), \varphi_2(x_2)) \qquad (x_j \in X_j;\ j = 1, 2), \tag{3.1}$$

where each φ_j is a homeomorphism from X_j onto $Y_j (j = 1, 2)$.

PROOF. We may assume that both of X_1 and X_2 are perfect, since otherwise the required conclusion is trivial. Let

$$N = \sup\{\mathcal{D}[\varphi(\{x_1\} \times X_2)];\ x_1 \in X_1\},$$

which is finite by Lemma 3.6. Let us fix any point p_1 of X_1 so that

$$\mathcal{D}[\varphi(\{p_1\} \times X_2)] = N,$$

and take a subset K_2 of X_2 so that

$$\text{Card } K_2 = N = \mathcal{D}[\varphi(\{p_1\} \times K_2)].$$

To get a contradiction, we shall assume that $N \neq 1$. Since X_2 is connected, so is $\varphi(\{x_1\} \times X_2)$. Thus there must be a point $p = (p_1, p_2)$ such that the sets

$$\varphi(\{p_1\} \times X_2) \cap (\{y_1\} \times W_2) \text{ and } \varphi(\{p_1\} \times X_2) \cap (W_1 \times \{y_2\})$$

are infinite for any rectangular neighborhood $W_1 \times W_2$ of $(y_1, y_2) = \varphi(p)$. (In such a case, we shall say that (y_1, y_2) is a *corner* of $\varphi(\{p_1\} \times X_2)$). It follows from the definition of N and the choice of K_2 that $\{p_1\} \times K_2$ contains two (distinct) points $q = (p_1, q_2)$ and $r = (p_1, r_2)$ such that

(1) $$\varphi(q) \in Y_1 \times \{y_2\}, \text{ and } \varphi(r) \in \{y_1\} \times Y_2 .$$

Let $K_2' = K_2 \cap \{q_2, r_2\}^c$; there is a rectangular neighborhood $U_1 \times U_2$ of p such that the $N-1$ sets

(2) $$\varphi(U_1 \times U_2) \text{ and } \varphi(U_1 \times \{x_2\}),\ x_2 \in K_2' ,$$

are pairwise bidisjoint. Taking $U_1 \times U_2$ sufficiently small and replacing q and r by some other points of $\{p_1\} \times U_2$, we may assume that:

(3) $$\varphi[(\{p_1\} \times U_2) \cup (U_1 \times \{p_2\})] = \varphi(U_1 \times U_2) \cap ((\{y_1\} \times Y_2) \cup (Y_1 \times \{y_2\}));$$

(4) $\varphi(U_1 \times \{q_2\})$ and $\varphi(U_1 \times \{r_2\})$ are bidisjoint;

(5) $\varphi(U_1 \times \{q_2\})$ is parallel to Y_2 and $\pi_2(\varphi(U_1 \times \{q_2\}))$ is open in Y_2;

(6) $\varphi(U_1 \times \{r_2\})$ is parallel to Y_1 and $\pi_1(\varphi(U_1 \times \{r_2\}))$ is open in Y_1.

These requirements are guaranteed by Lemma 3.4 and 3.5.

The proof now proceeds in five steps.

(I) The point $\varphi(p)$ is a corner of $\varphi(X_1 \times \{p_2\})$. To show this, suppose the contrary. We can then take a neighborhood $V_1 (\subset U_1)$ of p_1 so that $\varphi(V_1 \times \{p_2\})$ is parallel to either Y_1 or Y_2. Without loss of generality, suppose that it is parallel to Y_1. We then claim that V_1 contains a point x_1 such that

$$\pi_1[\varphi(x_1, r_2)] \neq \pi_1[x_1, p_2)] .$$

Otherwise, $\varphi(V_1 \times \{p_2\})$ would be a neighborhood of $\varphi(p)$ in the relative topology of $Y_1 \times \{y_2\}$, by (6); hence, $\varphi(p)$ could not be a corner of $\varphi(\{p_1\} \times X_2)$. It follows from (4), (5) and (6) that the set

$$\varphi(\{x_1\} \times \{p_2, q_2, r_2\})$$

is diagonal. Since the sets in (2) are pairwise bidisjoint, and since x_1 is in U_1, we see that

$$\mathscr{D}[\varphi\{x_1\} \times (\{p_2, q_2, r_2\} \cup K_2))] = N+1,$$

which contradicts the definition of N.

(II) There is an infinite connected subset C_1 of X_1 such that

$$(7) \qquad p_1 \in C_1 \subset U_1, \text{ and } \varphi(C_1 \times \{p_2\} \subset Y_1 \times \{y_2\}.$$

To show this, let

$$F = \varphi(U_1 \times \{p_2\}) \cap (Y_1 \times \{y_2\}),$$

and suppose that the connected component of $\varphi(p)$ in F is $\{\varphi(p)\}$. Then $\varphi(p)$ has a basis of neighborhoods (in F) each of which is open and compact in the relative topology of F (see the proof of [9; (3.5)]). It follows from (I) that there are (relatively) open and compact neighborhoods S and T of $\varphi(p)$ with $S \cap T^c \neq \emptyset$. Then $S \cap T^c$ is compact in F, and so in $\varphi(X_1 \times \{p_2\})$. On the other hand, since $S \cap T^c$ does not contain $\varphi(p)$, (3) assures that $S \cap T^c$ is open in $\varphi(X_1 \times \{p_2\})$. But then, $\varphi^{-1}(S \cap T^c)$ is a non-empty set which is both open and closed in $X_1 \times \{p_2\}$, which contradicts the connectedness of X_1. It follows that the connected component C of $\varphi(p)$ in F is infinite. Thus it suffices to set

$$C_1 = \pi_1(\varphi^{-1}(C)).$$

(III) There is a point $s = (p_1, s_2)$ with s_2 in $U_2 \cap \{p_2\}^c$ such that $\varphi(s)$ is in $Y_1 \times \{y_2\})$ but $\varphi(C_1 \times \{s_2\})$ is not parallel to Y_2. In fact, let

$$V_2 = \{s_2 \in U_2 : s_2 \neq p_2, \varphi(p_1, s_2) \in Y_1 \times \{y_2\}\},$$

and observe that p_2 is in the closure of V_2, since $\varphi(p)$ is a corner of $\varphi(\{p_1\} \times X_2)$. It follows at once from (7) that if s_2 in V_2 is sufficiently near to p_2, then $\varphi(C_1 \times \{s_2\})$ is not parallel to Y_2.

(IV) Let $s = (p_1, s_2)$ be any point as in (III). (3) assures that s_2 has a neighborhood $V(s_2)$ such that $\varphi(\{p_1\} \times V(s_2))$ is contained and open in $Y_1 \times \{y_1\}$. It follows from Lemma 3.4 that there is a neighborhood V_1 of $p_1 (V_1 \subset U_1)$ such that $\varphi(V_1 \times \{s_2\})$ is parallel to Y_2. This, combined with the facts that C_1 is a connected, infinite set containing p_1 and that $\varphi(C_1 \times \{s_2\})$ is not parallel to Y_2, guarantees that there is a point $s' = (s_1', s_2)$ with s_1' in C_1 such that $\pi_1(\varphi(s')) = \pi_1(\varphi(s))$ and $\varphi(s')$ is a corner of $\varphi(C_1 \times \{s_2\})$. Let $p = (s_1', p_2)$, $q = (s_1', q_2)$, and note that

$$\varphi(q),\ \varphi(s),\ \text{and}\ \varphi(p')$$

are distinct points in $Y_1 \times \{y_2\}$.

Suppose first that $\pi_2(\varphi(q')) \neq \pi_2(\varphi(s))$. Then it is readily seen that the set $\varphi(\{p', q', s'\})$ is diagonal. Thus we have

$$\mathscr{D}[\varphi(\{s_1'\} \times (\{p_2, q_2, s_2\} \cup K_2))] = N+1,$$

which contradicts the definition of N.

Suppose next that $\pi_2(\varphi(q')) = \pi_2(\varphi(s'))$. There is a neighborhood V_1 of s_1' such that $\varphi(V_1' \times \{p_2\}) \subset Y_1 \times \{y_2\}$. Since $\varphi(s')$ is a corner of $\varphi(C_1 \times \{s_2\})$, there is a point v_1' in V_1' such that $v_1' \neq s_1'$ and $\pi_2(\varphi(v_1', s_2)) = \pi_2(\varphi(s'))$. Note then that $\pi_2(\varphi(v_1', s_2)) \neq \pi_2(\varphi(v_1, q_2))$ by (5). Since such a point v_1' can be taken arbitrarily near to s_1', we have for a suitable point v_1' in U_1

$$\mathscr{D}[\varphi(\{v_1'\} \times (\{p_2, q_2, s_2\} \cup K_2'))] = N+1,$$

which again contradicts the definition of N.

(V) Summarizing up, we have concluded that $N=1$, i.e., that

$$\mathscr{D}[\varphi(\{x_1\} \times X_2)] = 1 \qquad (x_1 \in X_1).$$

Similarly, we have

$$\mathscr{D}[\varphi(X_1 \times \{x_2\})] = \mathscr{D}[\varphi^{-1}(Y_1 \times \{y_2\}) = \mathscr{D}[\varphi^{-1}(\{y_1\} \times Y_2)] = 1$$

for all points x_2 of X_2 and y_j of $Y_j (j=1,2)$. From these facts, we can easily show that φ is essentially of the form (3.1), which completes the proof.

THEOREM 3.8. *Suppose that each X_j is the union of finitely many, pairwise disjoint, compact, connected subsets $C_{j,k} (j=1,2;\ k=1, \cdots, N_j)$. Then, on each rectangle $C_{1,m} \times C_{2,n}$, φ is essentially of the form* (3.1).

PROOF. Since $D_{m,n} = \varphi(C_{1,m} \times C_{2,n})$ is an open, compact, connected subset of Y, it is easy to see that $D_{m,n}$ is rectangular. Thus the required conclusion follows from Theorem 3.7.

4. Certain propertier of tensor algebras. In this section we shall use, without explanation, some well-established and standardized notations; most of them are adopted from [23] and [19].

Let G be a non-discrete locally compact abelian group with dual $\widehat{G}$, and let

X_1 and X_2 be two disjoint compact subsets of G. We set

$$X^* = X_1 \cup X_2,\ \widetilde{X} = X_1 + X_2, \text{ and } X = X_1 \times X_2 .$$

If X^* is a $\mathcal{K}$-set (that is, either a Kronecker set or a K_p-set for some natural number $p \geqq 2$), it is known ([21], [23], [19]; see also [3] and [4]) that $\widetilde{X}$ is an S-set (for the algebra $A(G)$), and that $\widetilde{X}$ is an SR-set if and only if at least one of the sets X_j does not contain any perfect subset. We have also $A(\widetilde{X}) = V(X)$ isometrically and algebraically if X^* is a Kronecker set; and topologically if X^* is a K_p-set for scme $p \geqq 2$; furthermore, in the later case, we have

$$(4.1) \qquad \|f\|_{V(X)} \leqq \|f\|_{A(\widetilde{X})} \leqq 4\|f\|_{V(X)} \qquad (f \in A(\widetilde{X})).$$

Varopoulos [23] has proved these facts for compact groups, but the conclusions are still true for general locally compact abelian groups. We can verify these using the principal structure theorem of locally compact abelian groups [9] and the well-known relationship between $A(R^n)$ and $A(T^n)$ [16].

THEOREM 4.1. *Suppose that X^* is a Kronecker set (resp. a K_p-set for some $p \geqq 2$), then we have:*

(a) *If f is a non-zero idempotent function in $C(X)$, and if $\|f\|_{A(D)} < u_0$ for all subsets D of $\widetilde{X}$ with* Card $(D) = 4$, *then* $\|f\|_{A(X)} = 1$ *(resp.* $1 \leqq \|f\|_{A(X)} \leqq 4$).

(b) *If f is a unimodular function in $C(\widetilde{X})$, and if*

$$\limsup_{n\to\infty} \|f^n\|_{A(D)} < u_1 \qquad (D \subset \widetilde{X}: \text{Card}(D) = 4),$$

then we have

$$f(x_1 + x_2) = f_1(x_1) \cdot f_2(x_2) \qquad (x_j \in X_j;\ j = 1, 2)$$

for some f_j in $C_1(X_j)$ $(j = 1, 2)$. Conversely, every function f on X expressible in the above form is in $A(\widetilde{X})$ and has $A(\widetilde{X})$-norm 1 *(resp.* $1 \leqq \|f\|_{A(\widetilde{X})} \leqq 4$).

(c) *If f is a non-zero function in $C(\widetilde{X})$ such that $|f|^2 = |f|$, and if*

$$\limsup_{n\to\infty} \|f^n\|_{A(D)} < u_0 \qquad (D \subset \widetilde{X}: \text{Card}(D) = 4),$$

then f has the form

$$f(x_1+x_2)=\sum_{k=1}^{n} f_1^{(k)}(x_1)\cdot f_2^{(k)}(x_2)\cdot I[E_1^{(k)}+E_2^{(k)}](x_1+x_2),$$

where the sets $E_j^{(k)}(1\leqq k\leqq n)$ are pairwise disjoint, non-empty, clopen (in X_j) subsets of X_j and the functions $f_j^{(k)}(1\leqq k\leqq n)$ are in $C_1(X_j)$ $(j=1,2)$. Conversely, every function f on $\widetilde{X}$ expressible in the above form is in $A(\widetilde{X})$ and has $A(\widetilde{X})$-norm 1 (resp. $1\leqq\|f\|_{A(\widetilde{X})}\leqq 4$).

PROOF. Every statement follows from the results in §1 and the above observations.

LEMMA 4.2 (cf. [5] and [11]). *Let K be any compact subset of G, and suppose that there exists a pseudomeasure P in $N(K)=(I(K))^{\perp}$ such that*

$$\|P\|_{PM}>\limsup_{\gamma\to\infty}|\hat{P}(\gamma)| = \inf\{\sup_{\gamma\notin E}|\hat{P}(\gamma)|\}, \tag{4.2}$$

the infimum being taken over all compact sebsets E of $\hat{G}$. Then every function f in $A(K)$ such that $\|f\|_{A(K)}=1$ and $|f(x)|\equiv 1$, has the form $f=c\gamma$ on K for some complex number c with $|c|=1$ and some character γ in $\hat{G}$.

PROOF. Let P be as above. Without loss of generality, we may assume that

$$\|P\|_{PM}=\hat{P}(0)=1$$

(note that $\hat{P}$ is uniformly continuous). Then it is easy to see that

$$\limsup_{\gamma\to\infty}|(\widehat{gP})(\gamma)|\leqq\|g\|_{A(K)}\cdot\limsup_{\gamma\to\infty}|\hat{P}(\gamma)|$$

for all functiong g in $A(K)$. Therefore if f is a function in $A(K)$ such that $\|f\|_{A(K)}=1$ and $|f|\equiv 1$, we have

$$\|\bar{f}P\|_{PM}\leqq 1, \text{ and } \limsup_{\gamma\to\infty}|(\bar{f}P)^{\wedge}(\gamma)|<1,$$

and also

$$(\bar{f}P)(f) = P(1) = \hat{P}(0) = 1 .$$

Applying a slightly modified form of Proposition 4.1 in [5], we obtain the required conclusion.

THEOREM 4.3. *Suppose that X^* is a Kronecker set, then we have*

$$\|P\|_{PM} = \limsup_{\gamma\to\infty} |\hat{P}(\gamma)| \qquad (P \in PM(\widetilde{X})), \tag{4.3}$$

provided that there exist points x_j' in X_j such that $G_1 \cap G_2 \neq \{0\}$, where G_j is the closed subgroup generated by the set $X_j - x_j'$ $(j = 1, 2)$.

PROOF. Suppose that (4.3) does not hold: we can find a pseudomeasure P in $PM(\widetilde{X})$ for which we have (4.2) (note that $N(\widetilde{X}) = PM(\widetilde{X})$ since X is an S-set). Let γ_1 and γ_2 be any characters in $\hat{G}$; then the function f in $C(\widetilde{X})$ defined by

$$f(x_1 + x_2) = \gamma_1(x_1) \cdot \gamma_2(x_2) \qquad (x_j \in X_j;\ j = 1, 2)$$

is in $A(\widetilde{X})$ and has $A(\widetilde{X})$-norm 1 by Theorem 4.1. Thus Lemma 4.2 applies, and we see that there exist a complex number c and a character γ in $\hat{G}$ such that

$$\gamma_1(x_1) \cdot \gamma_2(x_2) = c\gamma(x_1 + x_2) \qquad (x_j \in X_j;\ j = 1, 2).$$

It follows at once that

$$\gamma_j(x_j - x_j') = \gamma(x_j - x_j') \qquad (x_j \in X_j;\ j = 1, 2),$$

and so we see that $\gamma_1 = \gamma_2$ on $G_1 \cap G_2$. Since γ_1 and γ_2 were arbitrary characters in $\hat{G}$, this implies $G_1 \cap G_2 = \{0\}$. The proof is now complete.

LEMMA 4.4. *Let E_j be any dense subset of $X_j (j = 1, 2)$, $E = E_1 \times E_2$, and $M_F(E)$ the space of all measures on E whose supports are finite. Then there exists a directed family of linear operators $\mathcal{L}_\Delta$, $\Delta \in \mathcal{J}$, from $V'(X)$ into $M_F(X)$ such that:*

(a) *The range of each operator $\mathcal{L}_\Delta$ is finite dimensional.*

(b) *For every P in $V(X)$, we have*

$$\sup_\Delta \|\mathcal{L}_\Delta(P)\|_{V'(X)} \leqq \|P\|_{V'(X)} ; \tag{b.1}$$

(b. 2) $$\mathrm{supp}(\mathcal{L}_\Delta(P)) \to \mathrm{supp}(P);$$

(b. 3) $$\mathcal{L}_\Delta(P) \to P \text{ in the weak-* topology of } V'(X).$$

(c) *The statement* (b) *holds even if $V(X)$ is repalced by $M(X)$.*

PROOF. Fix $j = 1, 2$, and let $\mathcal{U}_j$ be the directed family of all finite open coverings of X_j (for any Δ and Δ' in $\mathcal{U}_j$, $\Delta \prec \Delta'$ implies that Δ' is a refinement of Δ). For every covering $\Delta = \{U_1, U_2, \cdots, U_n\}$ in $\mathcal{U}_j$, we can find a subset $\{g_1, g_2, \cdots, g_n\}$ of $C(X_j)$ so that [9];

(1) $$\sum_{k=1}^{n} g_k = 1 \text{ on the whole space } X_j;$$

(2) $$0 \leqq g_k, \text{ and } g_k = 0 \text{ outside } U_k \ (k = 1, 2, \cdots, n).$$

Choosing any points p_k in $U_k \cap E_j (k = 1, 2, \cdots, n)$, we define a linear operator $\mathcal{G}_\Delta$ on $C(X_j)$ by

(3) $$\mathcal{G}_\Delta(f) = \sum_{k=1}^{n} f(p_k) g_k \qquad (f \in C(X_j)).$$

It is then easy to see that

(4) $$\sup_\Delta \|\mathcal{G}_\Delta\| \leqq 1, \text{ and } \lim_\Delta \|f - \mathcal{G}_\Delta(f)\|_\infty = 0 \qquad (f \in C(X_j)).$$

Put now $\mathcal{I} = \mathcal{U}_1 \times \mathcal{U}_2$ the product space with the product order, and for any $\Delta = (\Delta_1, \Delta_2)$ in $\mathcal{I}$, let $\mathcal{H}_\Delta = \mathcal{G}_{\Delta} \widehat{\otimes} \mathcal{G}_{\Delta_2}$ be the operator on $V(X)$ canonically induced by $\mathcal{G}_{\Delta_1}$ and $\mathcal{G}_{\Delta_2}$ (see [23]). It follows at once from (4) that

$$\sup_\Delta \|\mathcal{H}_\Delta\| \leqq 1, \text{ and } \lim \|f - \mathcal{H}_\Delta(f)\|_{V(X)} = 0$$

for all f in $V(X)$. It is also easy to see from (3) that

(5) $$\mathcal{H}_\Delta(f) = \sum_k \sum_l f(p_k, q_l) g_k \otimes h_l \qquad (f \in V(X)),$$

where the p_k (resp. q_l) are points in E_1 (resp. E_2) and the g_k (resp. h_l) are functions in $C(X_1)$ (resp. $C(X_2)$) associated with Δ_1 (resp. Δ_2) as before.

Define now $\mathcal{L}_\Delta : V'(X) \to V'(X)$ to be the conjugate operator of $\mathcal{H}_\Delta$, $\Delta \in \mathcal{I}$. It is easy to verify that so-defined operators $\mathcal{L}_\Delta$ have all the required properties, which establishes our lemma.

Let us now consider the Banach algebra $\widetilde{V}(X)$ introduced by Varopoulos [24]. He proved that the natural imbedding $V(X)\subset\widetilde{V}(X)$ is isometric if the spaces X_j are either totally disconnected or homemorphic to compact metrizable groups ([24]; see also [25]), and Graham asked in [6] whether this is true for every tensor algebra. As is shown below, the answer is Yes.

THEOREM 4.5. *Let E be any rectangular dense subset of X. Then, for any function f in $V(X)$ (resp. in $C(X)$), we have*

$$\|f\|_{V(X)}(resp.\ \|f\|_{\widetilde{V}(X)}) = \sup\|f\|_{V(F)},$$

where the supremum is taken over all finite rectangular subsets F of E. In particular, the imbedding $V(X)\subset\widetilde{V}(X)$ is isometric.

PROOF. Let f be any function in $V(X)$. Then, by Lemma 4.4 we have

$$\begin{aligned}\|f\|_{V(X)} &= \sup\{|P(f)| : P\in V(X), \|P\|_V \leqq 1\}\\ &= \sup\left\{\left|\int_X f\,d\mu\right| : \mu\in M_F(E), \|\mu\|_{V'}\leqq 1\right\}\\ &= \sup\{\|f\|_{V(F)} : F\subset E, \text{Card } F<\infty\}.\end{aligned}$$

Similarly, for any f in $C(X)$, we have the required equality, and this completes the proof.

Let now K be any compact subset of G, and consider two Banach algebras $B(K)$ and $B'(K)$ introduced by Katznelson and McGehee [10]. We have

$$\|f\|_{A(K)}\geqq\|f\|_{B(K)}\geqq\|f\|_{B'(K)}\geqq\|f\|_{C(K)} \qquad (f\in C(K)),$$

and so $A(K)\subset B(K)\subset B'(K)\subset C(K)$. It is known ([15], [22]) that every non-discrete locally compact abelian group contains a compact set K for which we have

$$A(K)\neq B(K) = C(K).$$

THEOREM 4.6 (cf. [10], [24]). *Suppose that X^* is a $\mathcal{K}$-set, and that K is a clopen subset of $\widetilde{X}$, then we have*

$$\|f\|_{A(K)} = \|f\|_{B'(K)} \qquad (f\in A(K)),$$

and

$$\|f\|_{B(K)} = \|f\|_{B'(K)} \qquad (f \in C(K)).$$

In particular, we have $B(K) = B'(K)$ *isometrically.*

PROOF. If X^* is a Kronecker set, our statements follow from Lemma 4.4 and the fact that the canonical identification $A(\widetilde{X}) = V(X)$ is isometric. If X^* is a K_p-set for some $p \geqq 2$, then the spaces X_j are totally disconnected. Therefore the functions g_k and h_l used in (5) of the proof of Lemma 4.4 can be chosen to be idempotent. Using this, we obtain the required conclusions.

REMRK. Under the assumption of part (a) in Theorem 4.1, we have $\|f\|_{A(\widetilde{X})} = 1$ even if X^* is a K_p-set. This can be easily proved, and we omit the details.

5. The constants u_0 and u_1. Let u_0 be the constant defined by (1.1) in §1. Professor Leblanc calculated the exact value of u_0 to obtain $u_0 = 2/3^{1/2}$. We should like to thank him for allowing us to use his calculus.

There are only modulus in this calculus, and so, we can suppose that a and c are real positive, and so is b if we change the origin of z. Thus, it is obvious that u_0 is obtained when d is real and negative, and we can choose $d = -1$ because the result is independent of a constant factor. Then we have

$$A(a, b, c, 0) = a + b + c;$$

$$A(a, b, c, -1) = \sup_t \{(a^2 + b^2 + 2abt)^{1/2} + (c^2 + 1 - 2ct)^{1/2}\}$$

where $-1 \leqq t \leqq 1$. We now define two functions u and s by

$$u = A(a, b, c, 0)/A(a, b, c, -1);$$

$$s = \{a^2b^2(c^2 + 1) - c^2(a^2 + b^2)\}/2abc(ab + c),$$

and divide the positive cone $\{(a, b, c) : a > 0,\ b > 0, c > 0\}$ into three parts: $D_1 = \{|s| \leqq 1\}, D_2 = \{s \geqq 1\}$, and $D_3 = \{s \leqq -1\}$.

(I) Suppose here that the point (a, b, c) ranges over D_1. It is then easy to see that

$$A(a, b, c, -1) = (a^2 + b^2 + 2abs)^{1/2} + (c^2 + 1 - 2cs)^{1/2}$$

$$= ab\left\{\frac{a^2c+b^2c+ab(c^2+1)}{abc(ab+c)}\right\}^{1/2} + c\left\{\frac{ab(c^2+1)+c(a^2+b^2)}{abc(ab+c)}\right\}^{1/2}$$

$$= \{(ab+c)(abc^2+a^2c+a^2b+ab)/(abc)\}^{1/2}$$

$$= \{(ab+c)(bc+a)(ca+b)/(abc)\}^{1/2}.$$

Therefore, setting

$$x = a+b+c, y = ab+bc+ca, z = abc,$$

we have

$$u = x\{z/(z^2+zx^2-2zy+y^2-2zx+z)\}^{1/2}.$$

But $\partial u/\partial x = \partial u/\partial y = \partial u/\partial z = 0$ implies $x = 1$ and $y = z$, that is

$$a+b+c = a^{-1}+b^{-1}+c^{-1} = 1,$$

which is impossible. So

$$\begin{cases} \dfrac{\partial u}{\partial a} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}(b+c) + \dfrac{\partial u}{\partial z}bc = 0 \\ \dfrac{\partial u}{\partial b} = \dfrac{\partial u}{\partial c} = 0 \end{cases}$$

is possible only if

$$\begin{vmatrix} 1 & b+c & bc \\ 1 & c+a & ca \\ 1 & a+b & ab \end{vmatrix} = (a-b)(b-c)(a-c) = 0,$$

and we can suppose for instance $a = b$. Then

$$u = \frac{a(2a+c)c^{1/2}}{(a^2+c)^{1/2}(ac+a)},$$

and we see that $\partial u/\partial a = \partial u/\partial c = 0$ implies

$$\begin{cases} 2c = ac \\ (2a+c)^{-1}+(2c)^{-1} = 2^{-1}(a^2+c)^{-1}+(c+1)^{-1} \end{cases}$$

that is, $a=c=2$. Then we have $s=\frac{1}{2}$ and $u(2,2,2)=2/3^{1/2}$, which is the largest value of u in the region D_1.

(II) Suppose now that the point (a,b,c) ranges over D_2. The inequality $s\geqq 1$ implies

$$\{a^2b^2-(a+b)^2\}c^2-2a^2b^2c+a^2b^2\geqq 0$$

that is

$$|1/c-1|\geqq 1/a+1/b\,.$$

Suppose first that $c\geqq 1$; we then have

$$1/a+1/b+1/c\leqq 1$$

and

$$u=(a+b+c)/(a+b+c-1)\,.$$

It follows that the maximum value of u in $D_2\cap\{c\geqq 1\}$ is obtained when $a=b=c=3$ and equals $9/8(<2/3^{1/2})$. Suppose next that $c\leqq 1$, and so we have

$$\begin{cases}1/c-1\geqq 1/a+1/b\,,\\ u=(a+b+c)/(a+b+1-c)\,.\end{cases}$$

Since u is an increasing function of c for fixed a and b, we may assume that $1/c-1=1/a+1/b$. Therefore, it is easy to check that the maximum value of u in $D_2\cap\{c\leqq 1\}$ is obtained when $a=b=5$ and $c=5/7$, and equals $25/24$.

(III) Suppose finally that the point (a,b,c) ranges over D_3. We then have

$$|1/a-1/b|\geqq 1+1/c\,,$$

and

$$u=(a+b+c)/(|a-b|+c+1)\,.$$

As in (II), we can show that the maximum value of u in D_3 is

$$u(5,5/7,5)=u(5/7,5,5)=25/24\,.$$

If follows from (I), (II), and (III) that $u_0 = 2/3^{1/2}$, the required conclusion.

We now estimate the constant u_1 defined in the proof of Lemma 1.3. Let θ be any real number with $0 \leqq \theta \leqq \pi$. Then we have

$$A(1,1,1,-e^{i\theta}) = A(1,1,e^{i\theta},-1)$$
$$= 2|1+e^{i(\pi-\theta)/2}| = 2^{3/2}(1+\sin\theta/2)^{1/2}$$

and so

$$A(1,1,1,-e^{i\theta})/A(1,1,1,-1) = (1+\sin\theta/2)^{1/2}.$$

Suppose that z_0 is any complex number with $|z_0| = 1 \neq z_0$ and take an integer n so that $2\pi/3 \leqq \arg z_0{}^n \leqq \pi$; we have

$$A(1,1,1,-z_0{}^n)/A(1,1,1,-1) \geqq (1+\sin\pi/3)^{1/2},$$

and hence

$$u_1 \geqq (1+\sin\pi/3)^{1/2} > u_0 = 2/3^{1/2}.$$

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