Tôhoku Math. Journ.
23(1971), 169-172.

ON NON-RIEMANNIAN SECTIONAL CURVATURE IN RIEMANNIAN HOMOGENEOUS SPACES

ISAAC CHAVEL[1)]

(Rec. May 13, 1970)

In this note we give an M. Berger-L. W. Green type inequality [2] relating non-Riemannian sectional curvature and conjugate points in naturally reductive Riemannian homogeneous spaces (cf [1; 3, Chapter X] for all the necessary details). We first recall the result of Berger and Green: Let M be a compact orientable Riemannian manifold of dimension $\geqq 2$, Γ the scalar curvature and $\pi/\sqrt{\kappa}$, $\kappa>0$, a lower bound of the distance of any point to its first conjugate point along any geodesic. Then

$$\kappa \geqq (1/\text{volume } M)\int_M \Gamma dM, \tag{1}$$

where dM denotes the Riemannian volume element of M, and equality in (1) is achieved if and only if M is a space of constant sectional curvature κ. If M is Riemannian homogeneous, then the scalar curvature Γ is constant and (1) reads as

$$\kappa \geqq \Gamma . \tag{2}$$

We will consider, in this paper, a naturally reductive Riemannian homogeneous space and obtain inequalities of type (1), (2) with the scalar curvature replaced by the *sectional curvature* of the *canonical connection* (which is not the Levi-Civita connection unless M is locally symmetric). We turn to the statement and proof of the inequality.

Henceforth M is a naturally reductive Riemannian homogeneous space of dimension $\geqq 2$, and δ, D denote the Levi-Civita and canonical connections respectively. R will denote the Riemann curvature tensor and T, B the torsion and curvature tensors of the canonical connection. Then for vector fields X, Y on M we have

$$\delta_X Y = D_X Y + (1/2)\, T(X, Y) \tag{3}$$

1) Partially supported by NSF GP 8691

and the two connections therefore have the same geodesics. For any $p \in M$, $x \in M_p$ the tangent space to M at p, we let $\mathcal{R}_x, \mathcal{T}_x, \mathcal{B}_x : M_p \to M_p$ be the linear transformations given by

$$(4) \qquad \begin{aligned} &\mathcal{R}_x y = R(x, y)x\,, \\ &\mathcal{T}_x y = T(x, y)\,, \qquad \mathcal{B}_x y = B(x, y)x\,. \end{aligned}$$

Then relative to the Riemannian inner product $<,>$ on M_p, $\mathcal{R}_x$, $\mathcal{B}_x$ are symmetric and $\mathcal{T}_x$ is skew-symmetric.

THEOREM. *Let $\gamma : [0, +\infty) \to M$ be a geodesic parametrized with respect to arc length such that the first conjugate point of $\gamma(0)$ along γ is at $\gamma(\pi/\sqrt{\kappa})$. Then for every t, and every unit vector $e \in M_{\gamma(t)}$ orthogonal to $\dot\gamma(t)$ (the velocity vector of γ at $\gamma(t)$), we have*

$$(5) \qquad \kappa \geqq <\mathcal{B}_{\dot\gamma(t)}e, e>\,.$$

If for some t_0, there exists an $e \in M_{\gamma(t_0)}$ for which there is equality in (5), then the vector field $\xi(t) = \sin(\sqrt{\kappa}\, t)E(t)$, where E is the δ-parallel field along γ satisfying $E(t_0) = e$, is a Jacobi field along γ. If in addition to equality in (5) we have that $\pi/\sqrt{\kappa}$ is the minimum distance of any point to its first conjugate point along any geodesic, then the surface, geodesic at $\gamma(0)$, generated by $E(0)$, $\dot\gamma(0)$ is a totally geodesic submanifold of M of constant Riemannian sectional curvature κ.

PROOF. We first note that since $\mathcal{T}_x$ is skew symmetric, D-parallel transport preserves inner products; also, one knows that $DT \equiv DB \equiv 0$, and that

$$(6) \qquad \mathcal{R}_x = \mathcal{B}_x - (1/4)\mathcal{T}_x^2,$$

from which one easily shows

$$(7) \qquad <B(x, y)x, y> = <B(y, x)y, x>\,.$$

Thus for orthonormal x and y, (7) defines the *D-sectional curvature* of the 2-section spanned by x and y.

We now turn to our geodesic γ and write δ, $\mathcal{R}$, D, $\mathcal{T}$, $\mathcal{B}$ for $\delta_{\dot\gamma}$, $\mathcal{R}_{\dot\gamma}$, $D_{\dot\gamma}$, $\mathcal{T}_{\dot\gamma}$, $\mathcal{B}_{\dot\gamma}$ respectively. Let $E(t)$ be a D-parallel vector field along δ of unit length and orthogonal to γ and set

$$\xi(t) = \sin(\sqrt{\kappa}\, t) \cdot E(t)\,,$$

then

$$\begin{aligned}\delta\xi(t) &= D\xi(t) + (1/2)\mathcal{T}\xi(t)\\ &= \sqrt{\kappa}\cos(\sqrt{\kappa}\ t)\cdot E(t) + (1/2)\sin(\sqrt{\kappa}\ t)\cdot \mathcal{T}E(t)\ .\end{aligned}$$

By the skew-symmetry of $\mathcal{T}$ we have $<E, \mathcal{T}E>(t) = 0$ for all t and therefore

$$\|\delta\xi\|^2(t) = \kappa\ \cos^2(\sqrt{\kappa}\, t) + (1/4)\sin^2(\sqrt{\kappa}\, t)\|\mathcal{T}E\|^2(t)\ ,$$

where $\|\ \ \|$ denotes the norm associated with the Riemannian metric. Note that $\mathcal{T}E$, $\mathcal{B}E$ are D-parallel vector fields along γ and hence of constant length. The second variation of arc length (with fixed endpoints) of $\gamma([0, \pi/\sqrt{\kappa}\,]$ is non-negative for all variations of γ and is zero if and only if the induced vector field along γ is a Jacobi field. Therefore

$$\begin{aligned}0 &\leqq \int_0^{\pi/\sqrt{\kappa}} \{\|\delta\xi\|^2 - <\mathcal{R}\xi, \xi>\}(t)dt\\ &= \int_0^{\pi/\sqrt{\kappa}} \{\kappa\cos^2(\sqrt{\kappa}\ t) - \sin^2(\sqrt{\kappa}\ t)(<\mathcal{R}E, E> - (1/4)\|\mathcal{T}E\|^2)\}dt\\ &= \int_0^{\pi/\sqrt{\kappa}} \{\kappa\cos^2(\sqrt{\kappa}\ t) - \sin^2(\sqrt{\kappa}\ t)<\mathcal{B}E, E>\}dt\\ &= (\pi/2\sqrt{\kappa}\,)\{\kappa - <\mathcal{B}E, E>\}\ ,\end{aligned}$$

which implies (5) for all $t \in [0, \pi/\sqrt{\kappa}\,]$. However since $DB \equiv 0$, if (5) is true for one point of γ then it is true for all of γ.

If $\kappa = <\mathcal{B}E, E>$ then $\xi(t)$ is a Jacobi field along γ. In the canonical connection (which has the same geodesics as the Levi-Civita connection) Jacobi's equations read as

$$(8) \qquad D^2\eta + \mathcal{T}(D\eta) + \mathcal{B}\eta = 0$$

for vector fields η along γ. By (5), E is a critical point of the quadratic form $X \to <\mathcal{B}X, X>$ for each t, hence $\mathcal{B}E = \nu E$. One now easily sees that since ξ is a solution of (8) and $<E, \mathcal{T}E> = 0$, we have $\mathcal{T}E = 0$ – and by (3) E is parallel in the Levi-Civita connection.

In particular, $T(\dot\gamma(0), E(0)) = 0$. For the final statement we note that the hypotheses imply that $\kappa \geqq <\mathcal{B}_x y, y>$ for any orthonormal vectors x and y tangent to M at any point of M. If $\kappa = <\mathcal{B}_x y, y>$ then y is a critical point of $z \to <\mathcal{B}_x z, z>$

and hence $\mathscr{B}_x y = \kappa y$ — by switching the roles of x and y, (7) implies $\mathscr{B}_y x = \kappa x$. The theorem now follows from

LEMMA (A. A. Sagle [4]). *Let M be a naturally reductive Riemannian homogeneous space, $p \in M$, and V a subspace of M_p such that for all $x, y, z \in V$*

$$T(x,y) \in V, \qquad B(x,y)z \in V.$$

Then $\mathrm{Exp}_p V$ *is a totally geodesic naturally reductive Riemannian homogeneous submanifold of* M.

REMARK. We note that by a lemma of J. L. Synge [5] if $E(t)$ is a D-parallel vector field along a geodesic $\gamma(t)$, then $\langle \mathscr{B}E, E \rangle$ is the Gaussian curvature of any surface containing γ and tangent to the plane spanned by $\dot{\gamma}(t)$, $E(t)$ for every t.

REFERENCES

[1] I. CHAVEL, Isotropic Jacobi fields and Jacobi's equations on Riemannian homogeneous spaces, Comment. Math. Helv., 42(1967), 237–248.

[2] L. W. GREEN, Aufwiedersehensflächen, Ann. of Math., 78(1963), 289–299.

[3] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Vol II, Interscience, New York, 1969.

[4] A. A. SAGLE, A note on triple systems and totally geodesic submanifolds in a homogeneous space, Nagoya Math. J., 32(1968), 5–20.

[5] J. L. SYNGE, The first and second variations of length in Riemannian space, Proc. London Math. Soc., 25(1926), 247–264.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA, U. S. A.