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ON MULTIPLIER TRANSFORMATIONS

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1. **Introduction.** Let λ be a real number such that $-\frac{1}{2}<\lambda<\frac{1}{2}$. Let T denote the set of real numbers modulo one and Z the additive group of integers. For $1\leqq p<\infty$, we denote by $l^{p,\lambda}(z)$ the vector space of complex-valued functions f defined on Z such that

$$N_{p,\lambda}[f]=\left\{\sum_{n\in Z}|f(n)|^p(|n|+1)^{p\lambda}\right\}^{1/p}<\infty,$$

while $L^{p,\lambda}(T)$ denotes the space of those complex-valued functions f defined on T for which

$$\|f\|_{p,\lambda}=\left(\int_T |f(\theta)\theta^{\lambda}|^p\,d\theta\right)^{1/p}<\infty.$$

If $f\in l^{2,0}(Z)$, its Fourier transform

$$f^\wedge(\theta)=\sum_{n\in Z}f(n)e^{2\pi in\theta},\qquad \theta\in T,$$

exists as a limit in the mean, of order 2, of the partial sums of the series on the right, and the inversion formula

$$f(n)=\int_T f^\wedge(\theta)\,e^{-2\pi in\theta}\,d\theta$$

is valid. Let $h^\wedge$ be a bounded measurable function defined on T. Set

$$Hf(n)=\int_T f^\wedge(\theta)h^\wedge(\theta)\,e^{-2\pi in\theta}\,d\theta$$

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for $n \in Z$, $f \in l^{2,0}(Z)$. Such a transformation H, determined by $h^\wedge$, is called a multiplier transformation. If

$$N_{p,\lambda}[H] = \text{l. u. b. } \{N_{p,\lambda}[Hf]/N_{p,\lambda}[f], f \in l^{2,0}(Z) \cap l^{p,\lambda}(Z), f \not\equiv 0\}$$

is finite, then H has a unique extension, as a bounded linear transformation of $l^{p,\lambda}(Z)$ into itself, with norm $N_{p,\lambda}[H]$, since $l^{2,0}(Z) \cap l^{p,\lambda}(Z)$ is dense in $l^{p,\lambda}(Z)$.

Similarly for $f \in L^{2,0}(T)$, we set

$$f^\wedge(n) = \int_T f(\theta)\, e^{-2\pi i n\theta}\, d\theta\,.$$

Let $h^\wedge$ be a bounded function defined on Z. Then the multiplier transformation H, associated with $h^\wedge$, is defined by

$$Hf(\theta) = \sum_{n \in Z} h^\wedge(n) f^\wedge(n)\, e^{2\pi i n\theta}\,.$$

If

$$\|H\|_{p,\lambda} = \text{l. u. b. } \{\|Hf\|_{p,\lambda}/\|f\|_{p,\lambda}, f \in L^{2,0}(T) \cap L^{p,\lambda}(T), f \not\equiv 0\}\,,$$

is finite, then H has a unique extension as a bounded linear transformation of $L^{p,\lambda}(T)$ into itself.

An important problem in this connection is to find sufficient conditions on the multiplier function $h^\wedge$ so that the multiplier transformation H associated with $h^\wedge$ is bounded. In [4] Hirschman has investigated this problem when $\lambda = 0$. In [6] he considered the problem for $l^{2,\lambda}(Z)$ and obtained the following result in terms of bounded β-variation of a function.

THEOREM A. *Let $h^\wedge$ be defined on T and let H be the corresponding multiplier transformation. If $V_\beta[h^\wedge]$ is finite* ($\beta > 2$) *then*

$$N_{2,\lambda}[H] < \infty \qquad \textit{if } |\lambda| < \frac{1}{\beta}\,,$$

where $V_\beta[h^\wedge]$ denotes the β-variation of $h^\wedge$.

In this paper we extend the results of Hirschman to $l^{p,\lambda}(Z)$. These results are given in section 3. In section 2, the result analogous to Theorem A is given for $L^{2,\lambda}(T)$. The authors wish to express their gratitude to Professor Igari for his useful comments, particularly for the improvement on the proof of Theorem 2.6.

2. Multipliers on $\mathbf{L^{2,\lambda}(T)}$. Let $h^\wedge$ be a bounded function defined on Z and H the corresponding multiplier transformation on $L^{2,\lambda}(T)$. If $I(H)$ is the set of all indices λ for which $\|H\|_{2,\lambda}$ is finite, then it is easy to verify that

(a) if $\lambda_1, \lambda_2 \in I(H)$ and if $\gamma = (1-\eta)\lambda_1 + \eta\lambda_2, 0<\eta<1,$

$$\text{then } \gamma \in I(H) \text{ and } \|H\|_{2,\gamma} \leqq \|H\|_{2,\lambda_1}^{1-\eta}\|H\|_{2,\lambda_2}^{\eta},$$

(b) if $\lambda \in I(H)$, then $-\lambda \in I(H)$, and $\|H\|_{2,\lambda} = \|H\|_{2,-\lambda}$.

The first of these results is a consequence of the Riesz-Thorin convexity theorem, see [7], while the second results from the fact that the conjugate space of $L^{2,\lambda}(T)$ is $L^{2,-\lambda}(T)$.

We shall now give two lemmas that we need.

Lemma 2.1. *If $f(\theta) \sim \sum_{n\in Z} f^\wedge(n)\, e^{2\pi i n\theta}$, then for $0 \leqq \lambda < \frac{1}{2}$,*

(a) $$\sum_{n\in Z} |f^\wedge(m+n)|^2(|n|+1)^{-2\lambda} \leqq A'(\lambda)\|f\|_{2,\lambda}^2$$

(b) $$\sum_{n\in Z} |f^\wedge(m+n)|^2(|n|+1)^{2\lambda} \geqq A''(\lambda)\|f\|_{2,-\lambda}^2$$

for all $m \in Z$ where $A'(\lambda)$ and $A''(\lambda)$ are positive constants depending only on λ.

This can be easily deduced from Hirschman [3, p. 51].

Lemma 2.2. *If $f \in L^{2,\lambda}(T)$ and if $a_n = \int_T f(\theta)\, e^{-2\pi i n\theta}\, d\theta$. then for $0<\lambda < \frac{1}{2}$*

$$A'\int_T |f(\theta)\theta^\lambda|^2 d\theta \leqq \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} |a_{n+m} - a_m|^2 n^{-1-2\lambda} \leqq A''\int_T |f(\theta)\theta^\lambda|^2\, d\theta$$

where A and A'' are positive constants depending only on λ. (See Hirschman [3, p. 52]).

Let $\mathfrak{M}_\lambda$ denote the set of all bounded multiplier transformations on $L^{2,\lambda}(T)$.

THEOREM 2.3. *Suppose* $0<\lambda<\frac{1}{2}$ *and* $H\in\mathfrak{M}_{\lambda}$. *Then there exists a constant* $A'(\lambda)$ *such that for any* $f\in L^{2,\lambda}(T)$,

$$(1)\qquad \sum_{n=1}^{\infty}n^{-1-2\lambda}\sum_{m=-\infty}^{\infty}|f^{\wedge}(m)|^{2}|h^{\wedge}(m+n)-h^{\wedge}(m)|^{2}\leqq A'(\lambda)\|H\|_{2,\lambda}^{2}\|f\|_{2,\lambda}^{2}.$$

PROOF. From the relation

$$f^{\wedge}(m)[h^{\wedge}(m+n)-h^{\wedge}(m)]=[f^{\wedge}(m+n)h^{\wedge}(m+n)-f^{\wedge}(m)h^{\wedge}(m)]$$
$$+[f^{\wedge}(m)-f^{\wedge}(m+n)]h^{\wedge}(m+n)$$

it follows that, since $|h^{\wedge}(m+n)|\leqq\|H\|_{2,\lambda}$, as can be easily verified,

$$|f^{\wedge}(m)|^{2}|h^{\wedge}(m+n)-h^{\wedge}(m)|^{2}\leqq 2|f^{\wedge}(m+n)h^{\wedge}(m+n)-f^{\wedge}(m)h^{\wedge}(m)|^{2}$$
$$+2\|H\|_{2,\lambda}^{2}|f^{\wedge}(m+n)-f^{\wedge}(m)|^{2}.$$

Multiplying by $n^{-1-2\lambda}$ and summing over m and n, we get the desired result, using Lemma 2.2.

THEOREM 2.4. *Let* $0<\lambda<\frac{1}{2}$. *There exists a constant* $A''(\lambda)$ *such that if* $h^{\wedge}$ *is defined on* Z *satisfying*

$$|h^{\wedge}(m)|\leqq C \qquad m\in Z$$

and

$$\sum_{n=1}^{\infty}n^{-1-2\lambda}\sum_{m=-\infty}^{\infty}|f^{\wedge}(m)|^{2}|h^{\wedge}(m+n)-h^{\wedge}(m)|^{2}\leqq C^{2}.\ \|f\|_{2,\lambda}^{2},$$

for every $f\in L^{2,\lambda}(T)$, *then* $H\in\mathfrak{M}_{\lambda}$, *and* $\|H\|_{2,\lambda}\leqq A''(\lambda)C$.

PROOF. We have

$$f^{\wedge}(m+n)h^{\wedge}(m+n)-f^{\wedge}(m)h^{\wedge}(m)=f^{\wedge}(m)[h^{\wedge}(m+n)-h^{\wedge}(m)]$$
$$+[f^{\wedge}(m+n)-f^{\wedge}(m)]h^{\wedge}(m+n)$$

so that

$$|f^\wedge(m+n)h^\wedge(m+n)-f^\wedge(m)h^\wedge(m)|^2$$
$$\leqq 2|f^\wedge(m)|^2.|h^\wedge(m+n)-h^\wedge(m)|^2+2C^2|f^\wedge(m+n)-f^\wedge(m)|^2.$$

Multiplying by $n^{-1-2\lambda}$ and summing over m and n, the desired result follows by virtue of Lemma 2.2.

THEOREMS 2.3 and 2.4 correspond to the results of Devinatz and Hirschman [1, Lemmas 3d, 3e].

Before we prove our main result in this section, we need the following definition.

DEFINITION 2.5. If $g^\wedge$ is a function defined on Z, then we define

$$V_\beta[g^\wedge]=\text{l. u. b.}\left\{\sum_{k=0}^{N-1}|g^\wedge(n_{k+1})-g^\wedge(n_k)|^\beta\right\}^{1/\beta},$$

the least upper bound being taken over all sets of integers $n_0<n_1<n_2<\cdots<n_N$ and it is called the β-variation of $g^\wedge$.

First we prove a result analogous to the lemma of Hirschman [6].

THEOREM 2.6. *Suppose that* $0<\lambda<\frac{1}{2}$. *Let* $h^\wedge$ *be of bounded* 1-*variation on* Z. *Then, if* H *is the corresponding multiplier transformation, we have*

$$\|H\|_{2,\lambda}^2\leqq B(\lambda)\{\|h^\wedge\|_\infty^2+\|h^\wedge\|_\infty V_1[h^\wedge]\}$$

where $B(\lambda)$ *is a finite constant depending only on* λ *and*

$$\|h^\wedge\|_\infty=\sup_{n\in Z}|h^\wedge(n)|.$$

PROOF. By virtue of theorem 2.4, we need only to estimate the quantity $M=\sum_{n=1}^{\infty}n^{-1-2\lambda}\sum_{m\in Z}|f^\wedge(m)|^2|h^\wedge(m+n)-h^\wedge(m)|^2$.
Now

$$M\leqq 2\|h^\wedge\|_\infty\sum_{n=1}^{\infty}n^{-1-2\lambda}\sum_{m=-\infty}^{\infty}|f^\wedge(m)|^2\sum_{k=1}^{n}|h^\wedge(m+k)-h^\wedge(m+k-1)|$$

$$= 2\|h^\wedge\|_\infty \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{k=1}^{n} \sum_{m=-\infty}^{\infty} |f^\wedge(m-k)|^2 |h^\wedge(m) - h^\wedge(m-1)|$$

$$= 2\|h^\wedge\|_\infty \sum_{m=-\infty}^{\infty} |h^\wedge(m) - h^\wedge(m-1)| \sum_{k=1}^{\infty} |f^\wedge(m-k)|^2 \sum_{n=k}^{\infty} n^{-1-2\lambda}$$

$$\leqq \frac{1}{\lambda} \|h^\wedge\|_\infty \sum_{m=-\infty}^{\infty} |h^\wedge(m) - h^\wedge(m-1)| \sum_{k=1}^{\infty} |f^\wedge(m-k)|^2 k^{-2\lambda}$$

$$\leqq C(\lambda) \|h^\wedge\|_\infty \, V_1[h^\wedge] \|f\|_{2,\lambda}^2$$

using Lemma 2.1.

LEMMA 2.7. *Let $h^\wedge$ be a real valued function defined on Z. For each $\beta > 1$, there exists a constant $C(\beta)$, depending only on β, such that for each $h^\wedge$ for which $V_\beta[h^\wedge] < \infty$ and $\varepsilon > 0$, there exists $h^\wedge_\varepsilon$ with the following properties:*

(a) $\|h^\wedge - h^\wedge_\varepsilon\|_\infty \leqq \varepsilon$,

(b) $V_1[h^\wedge_\varepsilon] \leqq C(\beta) V_\beta[h^\wedge]^\beta \varepsilon^{1-\beta}$,

where $\|\cdot\|_\infty$ is defined as in Theorem 2.6.

This lemma corresponds to Lemma 3 of Hirschman in [6] and can be proved by the arguments used in [4].

We now come to the main result in this section and it is the analogue of Theorem A stated in the introduction.

THEOREM 2.8. *Let $h^\wedge$ be defined on Z and let H be the corresponding multiplier transformation on $L^{2,\lambda}(T)$. If $V_\beta[h^\wedge]$ is finite, where $\beta > 2$, then*

$$\|H\|_{2,\lambda} < \infty \qquad \text{if } |\lambda| < \frac{1}{\beta}.$$

PROOF. First we obtain a sequence of functions $g^\wedge_m$ such that

$$h^\wedge = \lim_{m \to \infty} g^\wedge_m$$

pointwise on Z. This construction is given by Hirschman [4] (see also Edwards [2, Vol. 2, p. 270]). We shall not give the details here. Assuming without loss of generality that $h^\wedge(0)=0$, a real valued function h^* on the entire real line is obtained by interpolating linearly between successive values of $h^\wedge(n)$ so that $h^*(x)|_{x=n}=h^\wedge(n)$. Then for each positive integer m, a function $g^\wedge_m$ is constructed satisfying

$$V_1[g^\wedge_m] \leqq 2^{(\beta-1)m} V_\beta[h^\wedge]^\beta \tag{2}$$

and

$$\|h^\wedge - g^\wedge_m\|_\infty \leqq 2^{-m}. \tag{3}$$

Furthermore

$$V_\beta[g^\wedge_m] \leqq V_\beta[h^\wedge].$$

The proof of our theorem is completed following the arguments of Hirschman [6]. Define a sequence of functions $\{h^\wedge_m\}_{m=1}^\infty$ on Z as follows:

$$h^\wedge_1(n) = g^\wedge_1(n)$$

$$h^\wedge_m(n) = g^\wedge_m(n) - g^\wedge_{m-1}(n).$$

Then

$$h^\wedge(n) = \sum_{m=1}^{\infty} h^\wedge_m(n)$$

and

$$V_1[h^\wedge_m] \leqq C.\, 2^{(\beta-1)m} V_\beta[h^\wedge]^\beta,$$

$$\|h^\wedge_m\|_\infty \leqq C\, 2^{-m}.$$

If H_m is the multiplier transformation associated with $h^\wedge_m$, then

$$\|H\|_{2,\lambda} \leqq \sum_{m=1}^{\infty} \|H_m\|_{2,\lambda}.$$

Choose $\alpha, \lambda < \alpha < \frac{1}{2}$. By Theorem 2.6,

$$\|H_m\|_{2,\alpha} = O[(2^{-m})^2 + 2^{-m}.\, 2^{m(\beta-1)}]^{1/2} = O(2^{m(\beta/2-1)}).$$

On the other hand, by Parseval's equality

$$\|H_m\|_{2,0} = \|h^\wedge{}_m\|_\infty = O(2^{-m}) .$$

Putting $\lambda = (1-\theta)0 + \theta\alpha$, $0<\theta<1$, we obtain by the Riesz-Thorin convexity theorem,

$$\|H_m\|_{2,\lambda} = O(2^{m(-1+\beta\lambda/2\alpha)}) .$$

The series $\sum_{m=1}^{\infty} \|H_m\|_{2,\lambda}$ is convergent if $\lambda<\frac{2\alpha}{\beta}$. Since α is arbitrary such that $\lambda<\alpha<\frac{1}{2}$, it is always possible to choose α so that $\lambda<\frac{2\alpha}{\beta}$ if $0<\lambda<\frac{1}{\beta}$. Thus we have proved our theorem if $0<\lambda<\frac{1}{\beta}$. The case when $\lambda=0$ being trivial, the theorem follows from the duality argument given at the beginning of this section.

3. Multipliers on $l^{p,\lambda}(Z)$. We shall now consider the problem for $l^{p,\lambda}(Z)$ and obtain some results similar to those given by Hirschman [4] for the case $\lambda=0$. Let $f\in l^{2,0}(Z)$. If

$$h(k) = \int_T h^\wedge(\theta)e^{-2\pi ik\theta}\,d\theta \qquad k\in Z$$

then

$$Hf(n) = \sum_{k\in Z} f(n-k)\,h(k) .$$

The series on the right converges absolutely for each n, by Parseval's relation. If $1/p+1/q=1$, then it is easy to verify that if H is a multiplier transformation on $l^{p,\lambda}(Z)$ then H is also a multiplier transformation on $l^{q,-\lambda}(Z)$ associated with the same $h^\wedge$ and $N_{p,\lambda}[H]=N_{q,-\lambda}[H]$.

THEOREM 3.1. *If*

(a) $|h^\wedge(\theta)|\leqq A \qquad \theta\in T$

(b) $|h^\wedge(\theta)-h^\wedge(\theta+t)|\leqq A|t|^\alpha \qquad 1/2<\alpha\leqq 1$

then H is a bounded linear transformation of $l^{p,\lambda}(Z)$ into itself where $1<p$

$<\infty$ *and* $\frac{1}{2}-\alpha<\lambda<\alpha-\frac{1}{2}$.

PROOF. Let

$$s^\wedge_k(\theta) = \sum_{|n|\leqq 2^k} h(n)\, e^{2\pi i n\theta}$$

be the partial sum of order 2^k of the Fourier series for $h^\wedge$. Given $\varepsilon>0$, it is easily seen that

$$\|s^\wedge_k - h^\wedge\|_\infty \leqq AC(\alpha, \varepsilon) 2^{-k(\alpha-\varepsilon)}$$

(Zygmund [7, p. 61], Hirschman [4, p. 223]) so that if

$$h^\wedge_k = s^\wedge_k - s^\wedge_{k-1}$$

then

$$\|h^\wedge_k\|_\infty \leqq AC(\alpha, \varepsilon)\, 2^{-k(\alpha-\varepsilon)} \tag{4}$$

where $\|\cdot\|_\infty$ is on T. Let H_k be the multiplier transformation associated with $h^\wedge_k$. Then

$$H_k f(n) = \int_T f^\wedge(\theta)\, h^\wedge_k(\theta)\, e^{-2\pi i n\theta}\, d\theta = \sum_{j\in Z_k} f(n-j)\, h(j)$$

where $Z_k = \{n \in Z, 2^{k-1} < |n| \leqq 2^k\}$ and

$$N_{r,\lambda}[H_k] \leqq \left\{ \sum_{j\in Z_k} |h(j)|^r (1+|j|)^{r\lambda} \right\}^{1/r} \qquad r = 1, 2. \tag{5}$$

Using the relation

$$\sum_{j\in Z_k} |h(j)| \leqq AC(\alpha, \varepsilon)\, 2^{k(1/2-\alpha+\varepsilon)}$$

it easily follows that

$$N_{1,\lambda}[H_k] \leqq AC(\alpha, \varepsilon)\, 2^{k(1/2-\alpha+\varepsilon+|\lambda|)}. \tag{6}$$

From (4) and (5) using Schwartz inequality and Parseval's relation, it follows that

$$N_{2,\lambda}[H_k] \leqq AC(\alpha, \varepsilon)\, 2^{k(1/2+|\lambda|+\varepsilon-\alpha)} . \tag{7}$$

Suppose $1<p\leqq 2$. Putting $\frac{1}{p}=\frac{1-\omega}{1}+\frac{\omega}{2}$, $0<\omega<1$, we obtain from (6) and (7) by virtue of Riesz-Thorin convexity theorem

$$N_{p,\lambda}[H_k] \leqq AC(\alpha, \varepsilon)\, 2^{k(1/2+|\lambda|+\varepsilon-\alpha)} .$$

If $|\lambda| <\alpha - \frac{1}{2}$, we can choose ε so small that

$$\sum_{k=0}^{\infty} N_{p,\lambda}[H_k] < \infty .$$

Further since $h^{\wedge}(\theta)=\sum_{k=0}^{\infty} h^{\wedge}_k(\theta)$, the convergence being uniform in θ it is easy to see that $Hf(n)=\sum_{k=0}^{\infty} H_k f(n)$ and $N_{p,\lambda}[H] \leqq \sum_{k=0}^{\infty} N_{p,\lambda}[H_k] < \infty$. The regular conjugacy argument gives the result for $2 \leqq p < \infty$.

Now we state two results of Devinatz and Hirschman [1] as lemmas.

LEMMA 3.2. *If* $0<\lambda<1/2$, *then there exist positive constants* $A_1(\lambda)$ *and* $A_2(\lambda)$ *depending only on* λ *such that*

$$(N_{2,\lambda}[f])^2 - |f(0)|^2 \leqq A_1(\lambda) \int_0^1 \int_0^1 \{|f^{\wedge}(\theta)-f^{\wedge}(\phi)|^2 (\sin \pi |\theta-\phi|)^{-1-2\lambda}\} d\theta\, d\phi$$

and

$$(N_{2,\lambda}[f])^2 - |f(0)|^2 \geqq A_2(\lambda) \int_0^1 \int_0^1 \{|f^{\wedge}(\theta)-f^{\wedge}(\phi)|^2 (\sin \pi |\theta-\phi|)^{-1-2\lambda}\} d\theta\, d\phi .$$

LEMMA 3.3. *Let* $0<\lambda<\frac{1}{2}$. *There exists a constant* $A''(\lambda)$ *such that if* $h^{\wedge}$ *is a measurable function on* T *satisfying* $h(0)=0$,

$$\|h^{\wedge}\|_{\infty} \leqq C$$

and

$$\int_T |f^\wedge(\theta)|^2 d\theta \int_T |h^\wedge(\theta)-h^\wedge(\phi)|^2(\sin\pi|\theta-\phi|)^{-1-2\lambda}d\phi \leqq C^2(N_{2,\lambda}[f])^2$$

for every $f\in l^{2,\lambda}(Z)$, *then* $N_{2,\lambda}[H]\leqq A''(\lambda)\ C$.

We now prove

THEOREM 3.4. *Suppose* $h^\wedge$ *satisfies the condition* (a) *of Theorem* 3.1 *and*

(b′) $$|h^\wedge(\theta)-h^\wedge(\theta+t)|\leqq B|t|^\alpha \qquad 0<\alpha\leqq 1.$$

Then there exists a constant C such that

$$\int_T |f^\wedge(\theta)|^2 d\theta \int_T |h^\wedge(\theta)-h^\wedge(\varphi)|^2(\sin\pi|\theta-\varphi|)^{-1-2\lambda}d\theta d\varphi \leqq CAB(N_{2,\lambda}[f])^2$$

where $0<\lambda<\alpha/2$.

PROOF. We consider the quantity

$$M=\int_T |f^\wedge(\theta)|^2\int_T |h^\wedge(\theta)-h^\wedge(\phi)|^2(\sin\pi|\theta-\phi|)^{-1-2\lambda}d\theta d\phi$$

$$\leqq 2\|h^\wedge\|_\infty \int_T |f^\wedge(\theta)|^2\int_T |h^\wedge(\theta)-h^\wedge(\phi)|(\sin\pi|\theta-\phi|)^{-1-2\lambda}d\theta d\phi.$$

It is easy to establish that there exists a constant C which depends on λ and α such that

$$\int_T |h^\wedge(\theta)-h^\wedge(\phi)|(\sin\pi|\theta-\phi|)^{-1-2\lambda}d\phi\leqq C.$$

Thus

$$M\leqq C\|h^\wedge\|_\infty\int_T |f^\wedge(\theta)|^2 d\theta \leqq C\,\|h^\wedge\|_\infty\int_0^1 |f^\wedge(\theta)|^2\theta^{-2\lambda}d\theta$$

when $\lambda > 0$. Now applying Lemma 2.1 we obtain

$$M \leqq C\|h^\wedge\|_\infty (N_{2,\lambda}[f])^2$$

where C is a constant depending on λ and α only.

THEOREM 3.5. *Suppose $h^\wedge$ satisfies the conditions of theorem 3.4. Then if $0<\lambda<\frac{\alpha}{2}$, there exists a constant C which depends on α and λ such that if H is the associated multiplier transformation such that $h(0)=0$, then*

$$(N_{2,\lambda}[H])^2 \leqq CAB.$$

PROOF. An application of Lemmas 3.2 and 3.3 together with Theorem 3.4 gives the result.

THEOREM 3.6. *Suppose $h^\wedge$ satisfies the condition of Theorem 3.1. Then H is a bounded linear transformation of $l^{p,\lambda}(Z)$ into itself, where $\frac{\alpha}{2} > |\lambda| > \alpha - \frac{1}{2}$ and*

$$\frac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|} < p < \frac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|-2\alpha}.$$

PROOF. Suppose $s^\wedge_k$ is defined in the proof of Theorem 3.1 and H_k the multiplier transformation defined there. Then since

$$\|s^\wedge_k\|_\infty \leqq AC(\alpha, \varepsilon) 2^{-k(\alpha-\varepsilon)}$$

and, as can be easily verified,

$$|h^\wedge_k(\theta) - h^\wedge_k(\theta+t)| \leqq AC(\alpha, \varepsilon) 2^{\varepsilon k} |t|^\alpha$$

we have by virtue of Theorem 3.5

$$N^2_{2,\lambda}[H_k] \leqq A 2^{-k(\alpha-\varepsilon)}$$

which implies that

$$(8) \qquad N_{2,\lambda}[H_k] \leqq A . 2^{-k/2(\alpha-\varepsilon)}$$

Now suppose $\frac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|}<p\leqq 2$. Then if $1/p=(1-\omega)/1+\omega/2$ we have $\omega>\frac{1-2\alpha+2|\lambda|}{1-\alpha+2|\lambda|}$. By the Riesz-Thorin convexity theorem (this is possible since $0<\omega<1$ under the condition that $|\lambda|>\alpha-1/2$) we obtain from (6) and (8)

$$N_{p,\lambda}[H_k]\leqq A\,2^{k[(1/2-\alpha+|\lambda|+\varepsilon)(1-\omega)-\omega(\alpha-\varepsilon)/2]}. \tag{9}$$

Now under the above condition on ω, it is possible to choose ε small enough such that the quantity in the exponent of the right hand side of (9) is negative and we obtain the result for $\frac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|}<p\leqq 2$. The result for $2\leqq p<\frac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|-2\alpha}$ follows by the conjugacy argument.

In theorems 3.1 and 3.6 we have assumed that $\alpha>\frac{1}{2}$. We have not asserted that they are the best possible. There are multiplier transformations for some p and λ even if $\alpha<\frac{1}{2}$ as can be seen from the following result.

THEOREM 3.7. *If $h^\wedge$ satisfies conditions of Theorem* 3.4, *then H is a bounded linear transformation of $l^{p,\lambda}(Z)$ into itself if $\frac{2}{1+2(\alpha-\lambda)}<p<2$ and λ is a nonnegative number such that $\alpha>\lambda>\alpha-\frac{1}{2}$.*

PROOF. With the notations as in the proof of Theorem 3.1 we have

$$N_{2,0}[H_k]\leqq AC(\alpha,\varepsilon)2^{-k(\alpha-\varepsilon)} \tag{10}$$

Let $\gamma=(2-p)/p$. Then $1/p=(1-\gamma)/2+\gamma/1$ and let $\lambda=(1-\gamma)0+\gamma\eta$. Applying Riesz Thorin theorem to (10) and to

$$N_{1,\eta}[H_k]\leqq AC(\alpha,\varepsilon)2^{k(1/2-\alpha+\eta+\varepsilon)}$$

we obtain Theorem 3.7.

REMARK. If $\alpha < 1/2$, then $\lambda > \alpha - 1/2$ is satisfied by any nonnegative λ. In particular when $\lambda = 0$ the range for p reduces to $2/(1+2\alpha) < p < 2$ and this is the result given by Hirschman [4, Th. 2a].

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