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ON MIXING AND THE CENTRAL LIMIT THEOREM

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In Memory of Professor Alfred Rényi and in honour of Professor Shigeru Takahashi

1. Introduction. Let $[\Omega, \mathcal{A}, P]$ be a probability space and $\{X_j\}$ a sequence of random variables. Write $S_n = \sum_{j=1}^{n} X_j$. We shall say that the central limit theorem (CLT) holds if for all x,

$$P[S_n/\sqrt{n} \leqq x] \to \Phi(x) \tag{1}$$

where Φ is the normal distribution function with mean 0 and variance 1.

Sufficient conditions for (1) to hold are of course well known.

We shall say that a sequence $\{Y_n\}$ of random variables is mixing with density $F(x)$ if for all B in $\mathcal{A}$, and x a continuity point of F,

$$P[Y_n \leqq x, B] \to F(x)P(B) .$$

For example if the CLT holds and if the X_j are independent identically distributed with mean 0 and variance 1 then it is known ([8]) that $\{Y_n\} = \{S_n/\sqrt{n}\}$ is mixing.

Now let $\{\nu_n\}$ be a sequence of random variables on Ω which only take on positive integer values, and suppose that $\{Y_n\}$ is a sequence of random variables such that $P[Y_n \leqq x] \to F(x)$ at continuity points of F, then we want to know under what conditions one has also that $P[Y_{\nu_n} \leqq x] \to F(x)$.

The following result is known ([5]).

THEOREM A. *$P[Y_{\nu_n} \leqq x] \to F(x)$ if*

(i) *$\{Y_n\}$ is mixing with density $F(x)$*

(ii) *$\forall \varepsilon, \delta > 0, \exists\ n_0$ and $c > 0 \ni P[\max_{|m-n|<nc} |Y_n - Y_m| > \varepsilon] < \delta$ whenever $n \geqq n_0$*

(iii) *$\exists$ a strictly positive random variable ν and a sequence of integers $\{f(n)\}$ increasing with $n \ni \nu_n/f(n) \to \nu$ in probability.*

As a special case we have that if the CLT holds with $\{X_j\}$ independent and if ν_n satisfies condition (iii) *then*

$$P[S_{\nu_n}/\sqrt{\nu_n} \leqq x] \to \Phi(x) . \tag{2}$$

This is the so called random-sum central limit theorem (RSCLT) (see eg. [9], [7]).

In this paper we shall give a proof of a result, of the type of Theorem A, which gives conditions under which the $\{Y_{\nu_n}\}$ are themselves mixing. We will then combine this result with another theorem concerning mixing sequences of random variables to obtain some quite surprising results in the CLT and RSCLT cases.

2. Mixing of the randomized sequences.

THEOREM 1. *Suppose that conditions* (i) *and* (ii) *of Theorem A hold and that condition* (iii) *holds with the added proviso that* ν *be discrete, then* $\{Y_{\nu_n}\}$ *is also mixing with density* $F(x)$.

PROOF. Let B in $\mathcal{A}$ be any event with $P(B)>0$. We want to verify

$$P\{Y_{\nu_n}\leqq x\,|\,B\}\to F(x), \quad \text{as } n\to\infty\,.$$

First we assume that $\nu_n=[f(n)\nu]$, where ν is a positive random variable having a discrete distribution. Let $a_k, k=1,2,\cdots,(0<a_1<a_2<\cdots)$ be the possible values taken up by ν with positive probability. Let A_k be the events $A_k=\{\omega:\ \nu(\omega)=a_k\}$, $k=1,2,\cdots$.
Then

$$(3)\qquad P\{Y_{\nu_n}\leqq x, B\}=\textstyle\sum_{k=1}^{\infty}P\{Y_{[f(n)a_k]}\leqq x\,|\,B\cap A_k\}P(B\cap A_k)\,,$$

with $P(\cdot\,|B\cap A_k)=0$ if $P(B\cap A_k)=0$. $\{Y_n\}$ mixing implies

$$P\{Y_{[f(n)a_k]}\leqq x\,|\,B\cap A_k\}\to F(x) \text{ as } n\to\infty\,,$$

for every continuity point x of F and for all k for which $P(B\cap A_k)\neq 0$. It follows then from (3)

$$(4)\qquad P\{Y_{\nu}\leqq x, B\}\to F(x)P(B)\,,$$

that is to say, the theorem is true if $\nu_n=[f(n)\nu]$ with ν-discrete. Next we write

$$(5)\qquad Y_{\nu_n}=Y_{[f(n)\nu]}+\sqrt{\frac{[f(n)\nu]}{\nu_n}}\left(\frac{\sqrt{\nu_n}\,Y_{\nu_n}-\sqrt{[f(n)\nu]}\,Y_{[f(n)\nu]}}{\sqrt{[f(n)\nu]}}\right)$$
$$+Y_{[f(n)\nu]}\left(\sqrt{\frac{[f(n)\nu]}{\nu_n}}-1\right).$$

Now, condition (iii) implies that $\frac{\nu_n}{[f(n)\nu]}\to 1$ in probability as $n\to\infty$ and we have just shown that $\{Y_{[f(n)\nu]}\}$ given B converges in distribution to F. Therefore the third term of (5) converges in probability to 0, given B. Thus, to prove the theorem it suffices by (4) to show that in the second term of (5)

$$(6)\qquad \left(Y_{\nu_n}-Y_{[f(n)\nu]}\sqrt{\frac{[f(n)\nu]}{\nu_n}}\right)\to 0 \quad \text{in probability}$$

as $n\to\infty$. First we show that $(Y_{\nu_n}-Y_{[f(n)\nu]})\to 0$ in probability as $n\to\infty$. Let $B_n(c)$, $c>0$, be the event $|\nu_n-[f(n)\nu]|\leqq f(n)c$. Put $n_k=[f(n)a_k]$. We choose an arbitrary $\varepsilon>0$ and let C_{n_k} denote the event $|Y_{\nu_n}-Y_{n_k}|>\varepsilon$. Then we have

$$(7)\qquad P\{|Y_{\nu_n}-Y_{[f(n)\nu]}|>\varepsilon\}\leqq\sum_{k=1}^{\infty}P(A_k\cap C_{n_k}\cap B_n(c))+P(\overline{B_n(c)}).$$

Let D_M denote the event $\{\nu>a_M\}$. Then

$$(8)\qquad \sum_{k=1}^{\infty}P(A_k\cap C_{n_k}\cap B_n(c))\leqq\sum_{k=1}^{M-1}P(A_k\cap C_{n_k}\cap B_n(c))+P(D_M)$$

and

$$(9)\qquad \sum_{k=1}^{M-1}P(A_k\cap C_{n_k}\cap B_n(c))\leqq\sum_{k=1}^{M-1}P\{\max_{|l-n_k|\leqq f(n)c}|Y_l-Y_{n_k}|>\varepsilon\}$$

Let $0<\delta<1$ and such that $\delta/(1-\delta)<\eta/3$, where $\eta>0$ is arbitrarily small. It follows from condition (ii) that for the given $\varepsilon>0$ and $\delta^k>0$ there exist positive numbers $c_k(\varepsilon,\delta)$ and integers $m_k(\varepsilon,\delta)$ such that for $f(n)\geqq m_k$,

$$(10)\qquad P\{\max_{|l-n_k|\leqq f(n)c_k}|Y_l-Y_{n_k}|>\varepsilon\}<\delta^k,\ k=1,2,\cdots$$

For the above η, choose M so large that $P(D_M)<\frac{\delta}{3}$. Fixing M this way, we choose m_M such that for $n\leqq m_M$

$$f(n)\geqq\max_{1\leqq k\leqq M-1}m_k(\varepsilon,\delta)$$

is satisfied. Then it follows from (10) that for $n\geqq m_M$

$$\sum_{k=1}^{M-1}P\{\max_{|l-n_k|\leqq f(n)c_k}|Y_l-Y_{n_k}|>\varepsilon\}<\sum_{k=1}^{M-1}\delta^k<\eta/3\,,$$

and if we now take $c=\min_{1\leqq k\leqq M-1}c_k(\varepsilon,\delta)$ for c of (9), we also have that the right hand side expression of (9) is also less than $\eta/3$. Having fixed c of $B_n(c)$ this way we now choose $m_c \geqq m_M$ so large that for $n \geqq m_c$ we have $P\{\overline{B_n(c)}\} < \eta/3$. It now follows form (7)and (8) that $(Y_{\nu_n} - Y_{[f(n)\nu]}) \to 0$ in probability as $n \to \infty$. This, together with the fact that $\frac{\nu_n}{[f(n)\nu]} \to 1$ in probability as $n \to \infty$, implies that every subsequence of $\left\{\left(Y_{\nu_n} - Y_{[f(n)\nu]}\sqrt{\frac{[f(n)\nu]}{\nu_n}}\right)\right\}$ contains a subsequence which converges almost everywhere to zero and the statement of (6) follows. This completes the proof.

This result was originally proven by Richter [12] (see also [11]) using Prohorov's theory of convergence in metric spaces. Our proof has the advantage of being straight forward and only using the usual probabalistic method of reasoning. A proof in the case where ν is not assumed discrete will be given elsewhere.

3. Some extensions of the central limit theorem. We shall need the following result which is a special case of some results proved in [3] and [4]. (See also [2], [6]).

We first need to introduce the following notations and definitions. If F is a distribution function we designate by μ_F the probability measure on the Borel sets of the real line which is determined by $\mu_F((a, b]) = F(b) - F(a)$. If Z is any random variable then a random measure ν_F^Z is defined on the Borel sets of the plane by $\nu_F^Z(\omega, E) = \mu_F\{y : (y, Z(\omega)) \in E\}$. The measure P_Z induced by P and Z is defined for all Borel sets by $P_Z(B) = P[Z \in B]$. Finally if g is a Borel function then the distribution function F^g induced by F and g is defined by $F^g(x) = \mu_F(g^{-1}(-\infty, x])$.

THEOREM B. *Let $\{Y_n\}$ be mixing of density $F(x)$ and let Z be an arbitrary random variable. Suppose that $h(\cdot, \cdot)$ is a function of two real variables which is almost everywhere continuous with respect to the random measure $\nu_F^Z(\omega, \cdot)$ for almost all (with respect to P) ω. Then for any sequence $\{Z_n\}$ of random variables which converges in probability to Z we have that*

$$P[h(Y_n, Z_n) \leqq x] \to \int_\Omega \nu_F^Z(\omega, h^{-1}(-\infty, x])P(d\omega)$$

$$= \int_\Omega \mu_F(\{y : h(y, Z(\omega)) \leqq x\})P(d\omega)$$

$$= \int_R \mu_F\{y : h(y, z) \leqq x\} P_Z(dz)\,.$$

To put the continuity condition in a more tractable form we reason as follows. Let D be the discontinuity set of h. The hypothesis implies that

$$0=\int_{\Omega}\nu_F^Z(\omega, D)\,P(d\omega)=\int_{\Omega}\mu_F(y\ :(y, Z(\omega))\in D)P(d\omega)=\int_{R}\mu_F(D_z)P_Z(dz)$$

(where D_z is the section of D by $z)=\mu_F\times P_Z(D)$. The last equality being just one of the forms of Fubini's theorem. Using the fact that the product measure of a set is zero if and only if almost every section has zero measure we have

COROLLARY 2. *The result of Theorem B holds if either* (i) $\mu_F\times P_Z(D)=0$, (ii) $\mu_F(D_z)=0$ *for almost every* z *section of* D *or* (iii) $P_Z(D_y)=0$ *for almost every* y *section of* D, *where* D *is the discontinuity set of* h.

We now give some applications to limit theorems.

THEOREM 2. *Suppose that* $\{X_j\}$ *is a sequence of independent identically distributed random variables with mean* 0 *and variance* 1. *Let* $\{\nu_n\}$ *be a sequence of integer valued random variables such that* ν_n/n *converges in probability to a positive random variable* ν. *Let* $\{Z_n\}$ *be an arbitrary sequence of random variables which converge in probability to a random variable* Z. *Let* $g(\cdot)$ *and* $f(\cdot)$ *be functions of a single real variable which are almost everywhere continuous with respect to Lebesgue measure and* P_Z *respectively. Then*:

$$P[g(S_{\nu_n}/\sqrt{\nu_n})\leqq f(Z_n)]\rightarrow\int_R\Phi^g(f(z))P_Z(dz)$$

PROOF. We take $h(y, z)=g(y)-f(z)$. Since Φ is absolutely continuous with respect to Lebesgue measure the hypotheses on f and g above imply that condition (i) of Corollary 2 is satisfied.

As mentioned in the introduction $S_n/\sqrt{n}$ is mixing with density Φ. Condition (ii) of Theorem A is satisfied by Kolmogorov's inequality. Thus by Theorem 1 and the remark following it, $S_{\nu_n}/\sqrt{n}$ is also mixing. Putting $x=0$ in Theorem B and realizing that $\mu_\Phi\{y: g(y)-f(z)\leqq 0\}$ is the same thing as $\Phi^g(f(z))$ we have by Corollary 2, that the desired result holds.

Results of this type were proved for $g(t)=t, Z_n=Z$ (with the exception that $\sqrt{\nu_n}$ was replaced by $\sqrt{n}$ and ν_n satisfied an additional requirement) in the Rademacher case by J. C. Smith [13]; for independent random variables by S. Takahashi [15] and for a different example of mixing by S. Takahashi [14].

In fact even the following, first proved in [15], which is the most special case seems to be generally unknown.

COROLLARY 3. *Let $\{X_j\}$ be a sequence of independent identically distributed random variables with mean* 0 *and variance* 1 *and let Z be an arbitrary random variable then*:

$$P[S_n/\sqrt{n}\leqq Z]\to\int_{-\infty}^{\infty}\Phi(t)dP[Z\leqq t].$$

The result is of course what we would obtain if S_n and Z were independent.

It would be of interest to know if the Z_n in Theorem 3 can also be randomized. This would follow if Z_n converged to Z W. P. 1, but since convergence of Z_n in probability does not imply convergence in probability of Z_{ν_n} (see [1]) the answer is not immediately forthcoming by the methods used above.

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