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CONTINUOUS DEPENDENCE FOR SOME FUNCTIONAL DIFFERENTIAL EQUATIONS

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Several authors have discussed global behaviors of trajectories of functional differential equations with the phase space considered by Hale ([2], [3], [4], [5]). The purpose of this paper is to discuss the continuity of solutions on initial values.

Let x be any vector in R^n and let $|x|$ be any norm of x. Let $B=B((-\infty,0],R^n)$ be a Banach space of functions mapping $(-\infty,0]$ into R^n with norm $\|\cdot\|$. For any φ in B and any σ in $[0,\infty)$, let φ^σ be the restriction of φ to the interval $(-\infty,-\sigma]$. This is a function mapping $(-\infty,-\sigma]$ into R^n. We shall denote by B^σ the space of such functions φ^σ. For any $\eta\in B^\sigma$, we define the semi-norm $\|\eta\|_{B^\sigma}$ of η by

$$\|\eta\|_{B^\sigma}=\inf_{\varphi}\{\|\varphi\|:\varphi^\sigma=\eta\}\ .$$

Then we can regard the space B^σ as a Banach space with norm $\|\cdot\|_{B^\sigma}$. If x is a function defined on $(-\infty,a)$, then for each t in $(-\infty,a)$ we define the function x_t by the relation $x_t(s)=x(t+s)$, $-\infty<s\leqq 0$. For numbers a and τ, $a>\tau$, we denote by $A_\tau{}^a$ the class of functions x mapping $(-\infty,a)$ into R^n such that x is a continuous function on $[\tau,a)$ and $x_\tau\in B$. The space B is assumed to have the following properties:

(I) If x is in $A_\tau{}^a$, then x_t is in B for all t in $[\tau,a)$ and x_t is a continuous function of t, where a and τ are constants such that $\tau<a\leqq\infty$.

(II) All bounded continuous functions mapping $(-\infty,0]$ into R^n are in B.

(III) If a sequence $\{\varphi_k\}$, $\varphi_k\in B$, is uniformly bounded on $(-\infty,0]$ with respect to norm $|\cdot|$ and converges to φ uniformly on any compact subset of $(-\infty,0]$, then $\varphi\in B$ and $\|\varphi_k-\varphi\|\to 0$ as $k\to\infty$.

(IV) There are continuous, nondecreasing and nonnegative functions $b(r)$, $c(r)$ defined on $[0,\infty)$, $b(0)=c(0)=0$, such that

$$\|\varphi\|\leqq b(\sup_{-\sigma\leqq s\leqq 0}|\varphi(s)|)+c(\|\varphi^\sigma\|_{B^\sigma})$$

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for any φ in B and any $\sigma \geqq 0$.

(V) If σ is a nonnegative number and φ is an element in B, then $T_\sigma\varphi$ defined by $T_\sigma\varphi(s)=\varphi(s+\sigma)$, $s \in (-\infty, -\sigma]$, is an element in B^σ.

(VI) $|\varphi(0)| \leqq M_1\|\varphi\|$ for some constant $M_1>0$.

(VII) $\|T_t\varphi\|_{B^t} \leqq M_2\|\varphi\|$ for all $t \geqq 0$ and for some constant $M_2>0$.

REMARK 1. When we discussed the global behaviors of trajectories in the phase space, the property of the fading memory, that is, $\|T_\sigma\varphi\|_{B^\sigma} \to 0$ as $\sigma \to \infty$, played an important role, but in this paper, this property is not required.

REMARK 2. The class of phase spaces considered by Coleman and Mizel [1] has the properties (I)$\sim$(VII), and hence the result in this paper holds good for this class of phase spaces.

Consider the functional differential equations

$$\dot{x}(t) = f(t, x_t)\,. \tag{1}$$

The superposed dot denotes the right-hand derivative and $f(t, \varphi)$ is a continuous function of (t, φ) which is defined on $I \times B^*$ and takes values in R^n, where I and B^* are open subsets of $[0, \infty)$ and B, respectively. We shall denote by $x(t_0, \varphi)$ a solution of (1) such that $x_{t_0}(t_0, \varphi)=\varphi$ and denote by $x(t, t_0, \varphi)$ the value at t of $x(t_0, \varphi)$.

THEOREM. *Suppose that a solution* $u(t)=u(t, t_0, \varphi^0)$, $(t_0, \varphi^0) \in I \times B^*$, *of* (1) *defined on* $[t_0, t_0+a]$ *for some* $a>0$ *is unique for initial value problem. Then for any* $\varepsilon>0$, *there exists a* $\delta(\varepsilon)>0$ *such that if* $(s, \psi) \in I \times B^*$, $|s-t_0|<\delta(\varepsilon)$ *and* $\|\psi-\varphi^0\|<\delta(\varepsilon)$, *then* $\|x_t(s, \psi)-u_t(t_0, \varphi^0)\|<\varepsilon$ *for all* $t \in [\max\{t_0, s\}, t_0+a]$, *where* $x(s, \psi)$ *is a solution of* (1) *through* (s, ψ).

PROOF. The set $\{u_t : t \in [t_0, t_0+a]\}$ is a compact subset of B, and hence there exists a positive number d such that if $\|\varphi-u_t\| \leqq d$ and $|s-t| \leqq d$, then $(s, \varphi) \in I \times B^*$ for all $t \in [t_0, t_0+a]$, because $I \times B^*$ is an open subset. Since f is continuous in (t, φ), we can assume that if $|t-s| \leqq d$ and $\|\varphi-u_t\| \leqq d$, then $|f(s, \varphi)-f(t, u_t)| \leqq 1$ for a $t \in [t_0, t_0+a]$. Thus it follows that

$$|f(s, \varphi)| \leqq 1 + \max\{|f(t, u_t)| : t \in [t_0, t_0+a]\}\,,$$

and hence there exists an $M>0$ such that $|f(s, \varphi)|<M$ on the set $D=\{(s, \varphi) : |s-t| \leqq d, \|\varphi-u_t\| \leqq d, t \in [t_0, t_0+a]\}$. Moreover, there is a continuous function $g(t, \varphi)$ defined on $[t_0-d, t_0+a+2d] \times B$ such that $|g|<M$ and

(2) $$g(t,\varphi)=f(t,\varphi) \quad \text{for } (t,\varphi)\in D.$$

Clearly the solutions of

(3) $$\dot{y}(t) = g(t, y_t)$$

are continuable to t_0+a+2d.

Suppose that the conclusion of this theorem is false. Then there exists a positive number $\varepsilon_0, \varepsilon_0<d$, and sequences $\{\varphi^m\}$, $\{t_m\}$ and $\{\tau_m\}$ such that $\|\varphi^m-\varphi^0\|\to 0$, $t_m\to t_0$, $\max\{t_0,t_m\}<\tau_m\leqq t_0+a$, and $\tau_m\to\tau_0$ as $m\to\infty$ and that

(4) $$\|x_{\tau_m}(t_m,\varphi^m) - u_{\tau_m}(t_0,\varphi^0)\| = \varepsilon_0$$

and

(5) $$\|x_t(t_m,\varphi^m) - u_t(t_0,\varphi^0)\| < \varepsilon_0 \qquad \text{for} \quad \max\{t_m,t_0\}\leqq t<\tau_m.$$

For all sufficiently large m, the function $g^m(t,\varphi)$ given by

$$g^m(t,\varphi) = g(t+t_m-t_0,\varphi)$$

is defined on $[t_0,\tau_0+d]\times B$. Let $y(t,t_m,\varphi^m)$, $t_m\leqq t\leqq t_0+a+d$, be a solution of (3) through (t_m,φ^m). Then $y^m(t)$ given by

$$y^m(t) = \begin{cases} y(t+t_m-t_0,t_m,\varphi^m) & \text{for} \quad t\in[t_0,\tau_0+d] \\ \varphi^m(t-t_0) & \text{for} \quad t\in(-\infty,t_0) \end{cases}$$

is a solution through (t_0,φ^m) of the functional differential equation

(6) $$\dot{y}(t) = g^m(t,y_t).$$

We shall show that the sequence $\{y^m(t)\}$ is uniformly bounded and equi-continuous on the interval $[t_0,\tau_0+d]$ for all large m. For all large m we have $|\varphi^m(0)-\varphi^0(0)|\leqq M_1\|\varphi^m-\varphi^0\|\leqq K$ for some constant $K>0$ by (VI), and hence

$$\begin{aligned} |y^m(t)| &\leqq |\varphi^m(0)| + \int_{t_0}^{t} |g^m(s,y^m{}_s)|\,ds \\ &\leqq |\varphi^0(0)| + K + M(\tau_0+d-t_0). \end{aligned}$$

Therefore $\{y^m(t)\}$ is uniformly bounded on $[t_0,\tau_0+d]$. For any t_1,t_2, $t_0\leqq t_2<t_1\leqq\tau_0+d$, we have

$$|y^m(t_1)-y^m(t_2)| \leqq \int_{t_2}^{t_1} |g^m(s, y^m{}_s)|\, ds \leqq M(t_1-t_2),$$

and hence $\{y^m(t)\}$ is equicontinuous on $[t_0, \tau_0+d]$. By Ascoli-Arzelà's Theorem, there exists a subsequence of $\{y^m(t)\}$ which converges to a function $y^*(t)$ uniformly on $[t_0, \tau_0+d]$. We shall denote it by $\{y^m(t)\}$ again. The limit function $y^*(t)$ is continuous and bounded on $[t_0, \tau_0+d]$.

Define $y(t)$ by

$$y(t) = \begin{cases} y^*(t) & \text{for } t \in [t_0, \tau_0+d] \\ \varphi^0(t-t_0) & \text{for } t \in (-\infty, t_0). \end{cases}$$

Then y_t belongs to B for all $t \in [t_0, \tau_0+d]$, because $y(t_0)=\varphi^0(0)$ and $y_t \in A_{t_0}{}^{\tau_0+d}$. We shall show that $y(t)$ is a solution of (3) through (t_0, φ^0).

First of all, we shall see that the set $S=\{y^m{}_s : s \in [t_0, \tau_0+d],\ m\,;$ sufficiently large$\}$ is a relative compact subset of B. Take any sequence $\{\psi^m\}$, $\psi^m \in S$. Then, corresponding to each m, there are k_m and s_m such that $s_m \in [t_0, \tau_0+d]$, and $\psi^m = y^{k_m}{}_{s_m}$. If the set $\{k_m\,; m=1, 2, \cdots\}$ is finite, we can assume that $\psi^m = y^k{}_{s_m}$ for a specified k. In this case, it is clear that there is a subsequence of $\{\psi^m\}$ which converges in S. In the case where the set $\{k_m\}$ is infinite, we can set $\psi^m = y^m{}_{s_m}$. We can also assume that the sequence $\{y^m(t)\}$ converges to the function $y(t)$ uniformly on $[t_0, \tau_0+d]$. There exists an s_0 such that $s_m \to s_0 \in [t_0, \tau_0+d]$ as $m \to \infty$. Define $z^m(t)$, $\xi^m(t)$, $z(t)$ and $\xi(t)$ by

$$z^m(t) = \begin{cases} y^m(t) & \text{for } t \in [t_0, \tau_0+d] \\ \varphi^m(0) & \text{for } t \in (-\infty, t_0), \end{cases}$$

$$\xi^m(t) = \begin{cases} 0 & \text{for } t \in [t_0, \tau_0+d] \\ \varphi^m(t-t_0)-\varphi^m(0) & \text{for } t \in (-\infty, t_0], \end{cases}$$

$$z(t) = \begin{cases} y(t) & \text{for } t \in [t_0, \tau_0+d] \\ \varphi^0(0) & \text{for } t \in (-\infty, t_0) \end{cases}$$

and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [t_0, \tau_0+d] \\ \varphi^0(t-t_0)-\varphi^0(0) & \text{for } t \in (-\infty, t_0), \end{cases}$$

respectively. For any $a \in R^n$, the symbol $<a>$ will denote the constant function α such that $\alpha(s)=a$ for all $s \in (-\infty, 0]$. Since $y^m{}_{s_m} = z^m{}_{s_m} + \xi^m{}_{s_m}$ and $y_{s_0} = z_{s_0} + \xi_{s_0}$, we have

$$(7)\qquad \begin{aligned}\|y^m{}_{s_m}-y_{s_0}\| &= \|z^m{}_{s_m}+\xi^m{}_{s_m}-z_{s_0}-\xi_{s_0}\| \\ &\leqq \|z^m{}_{s_m}-z_{s_0}\| + \|\xi^m{}_{s_m}-\xi_{s_0}\| \\ &\leqq \|z^m{}_{s_m}-z_{s_m}\| + \|z_{s_m}-z_{s_0}\| + \|\xi^m{}_{s_m}-\xi_{s_m}\| + \|\xi_{s_m}-\xi_{s_0}\| \\ &\leqq b(\sup_{-(s_m-t_0)\leqq s\leqq 0}|y^m(s_m+s)-y(s_m+s)|) \\ &\quad + c(\|<\varphi^m(0)>^{s_m-t_0} - <\varphi^0(0)>^{s_m-t_0}\|_{B^{s_m-t_0}}) + \|z_{s_m}-z_{s_0}\| \\ &\quad + b(\sup_{-(s_m-t_0)\leqq s\leqq 0}|\xi^m(s_m+s)-\xi(s_m+s)|) \\ &\quad + c(\|T_{s_m-t_0}\xi^m{}_{t_0} - T_{s_m-t_0}\xi_{t_0}\|_{B^{s_m-t_0}}) + \|\xi_{s_m}-\xi_{s_0}\|\,.\end{aligned}$$

And hence, we have

$$(8)\qquad \begin{aligned}\|y^m{}_{s_m}-y_{s_0}\| &\leqq b(\sup_{-(\tau_0+d-t_0)\leqq s\leqq 0}|y^m(\tau_0+d+s)-y(\tau_0+d+s)|) \\ &\quad + c(M_2\|<\varphi^m(0)> - <\varphi^0(0)>\|)+\|z_{s_m}-z_{s_0}\| \\ &\quad + c(M_2\|\xi^m{}_{t_0}-\xi_{t_0}\|) + \|\xi_{s_m}-\xi_{s_0}\|\,,\end{aligned}$$

because we have $0\leqq s_m-t_0\leqq\tau_0+d-t_0$ and because $\xi^m(t)$ and $\xi(t)$ are identically zero on the interval $[t_0, s_m]$. Since $y^m(t)$ converges to the function $y(t)$ uniformly on $[t_0, \tau_0+d]$ as $m\to\infty$, the first term on the right-hand side of (8) tends to zero as $m\to\infty$. By (III), the second term also tends to zero, since $|\varphi^m(0)-\varphi^0(0)|\leqq M_1\|\varphi^m-\varphi^0\|$ by (IV). We have $z,\ \xi\in A_{t_0}^{\tau_0+d}$, and therefore the third term and the fifth term tend to zero as $m\to\infty$ by (I). The fourth term also tends to zero as $m\to\infty$ by (III), because

$$\|\xi^m{}_{t_0}-\xi_{t_0}\| \leqq \|\varphi^m-\varphi^0\| + \|<\varphi^m(0)> - <\varphi^0(0)>\|\,.$$

Thus we have

$$\|y^m{}_{s_m}-y_{s_0}\|\to 0 \qquad \text{as} \quad m\to\infty\,,$$

which shows that $\bar{S}$ is a compact subset of B, where $\bar{S}$ is the closure of the set S.

Therefore $g^m(t,\varphi)$ is a uniformly continuous function on $[t^0, \tau_0+d]\times\bar{S}$. Since $y^m(t)$ is a solution of (6) through (t_0, φ^m), we have

$$(9)\qquad y^m(t) = \varphi^m(0) + \int_{t_0}^t g^m(s, y^m{}_s)\,ds$$

for all $t\in[t_0, \tau_0+d]$. The left-hand side of (9) tends to $y(t)$ as $m\to\infty$. The first term on the right-hand side of (9) tends to $\varphi^0(0)$ as $m\to\infty$. Noting the uniform continuity of $g^m(t,\varphi)$ on $[t_0, \tau_0+d]\times\overline{S}$, we have

$$\begin{aligned}
\lim_{m\to\infty}\left|\int_{t_0}^{t} g^m(s, y^m{}_s)\,ds - \int_{t_0}^{t} g(s, y_s)\,ds\right| \\
&\leqq \int_{t_0}^{t} \lim_{m\to\infty}|g^m(s, y^m{}_s) - g(s, y_s)|\,ds \\
&\leqq \int_{t_0}^{t} \lim_{m\to\infty}|g^m(s, y^m{}_s) - g(s, y^m{}_s)|\,ds \\
&\quad + \int_{t_0}^{t} \lim_{m\to\infty}|g(s, y^m{}_s) - g(s, y_s)|\,ds \\
&= 0\,,
\end{aligned}$$

and hence the second term on the right-hand side of (9) tends to $\int_{t_0}^{t} g(s, y_s)\,ds$ as $m\to\infty$. Since $y^m{}_{t_0}\to\varphi^0$ as $m\to\infty$, $y(t)$ is a solution of (3) through (t_0, φ^0) which is defined on $t_0\leqq t\leqq\tau_0+d$, and hence $y(t)$ can be expressed by $y(t, t_0, \varphi^0)$.

$y(t, t_m, \varphi^m)$ is clearly a solution of (1) through (t_m, φ^m) until $(t, y_t(t_m, \varphi^m))$ leaves the domain D by (2). Since $(t, x_t(t_m, \varphi^m))$ belongs to D on $[t_m, \tau_m]$ by (5), we can assume that

$$y(t, t_m, \varphi^m) = x(t, t^m, \varphi^m)$$

for $t\in[t_m, \tau_m]$. Thus clearly $y(t, t_0, \varphi^0)$ is a solution of (1) through (t_0, φ^0) defined on $[t_0, \tau_0]$. On the other hand, we have

$$\begin{aligned}
(10)\qquad \|x_{\tau_m}(t_m, \varphi^m) - u_{\tau_m}(t_0, \varphi^0)\| \\
&= \|y_{\tau_m}(t_m, \varphi^m) - u_{\tau_m}(t_0, \varphi^0)\| \\
&= \|y^m{}_{\tau_m+t_0-t_m} - u_{\tau_m}(t_0, \varphi^0)\| \\
&\leqq \|y^m{}_{\tau_m+t_0-t_m} - y_{\tau_0}(t_0, \varphi^0)\| \\
&\quad + \|y_{\tau_0}(t_0, \varphi^0) - u_{\tau_0}(t_0, \varphi^0)\| \\
&\quad + \|u_{\tau_0}(t_0, \varphi^0) - u_{\tau_m}(t_0, \varphi^0)\|\,,
\end{aligned}$$

and hence it follows from (4) and (10) that

(11) $$\varepsilon_0 \leqq \|y^m{}_{\tau_m+t_0-t_m} - y_{\tau_0}(t_0, \varphi^0)\| + \|y_{\tau_0}(t_0, \varphi^0) - u_{\tau_0}(t_0, \varphi^0)\| + \|u_{\tau_0}(t_0, \varphi^0) - u_{\tau_m}(t_0, \varphi^0)\| .$$

Taking sufficiently large m, we have $s_m = \tau_m + t_0 - t_m \in [t_0, \tau_0 + d]$ and $s_m \to \tau_0$ as $m \to \infty$, and hence the first term on the right-hand side of (11) tends to zero as $m \to \infty$, as in the calculation of (8). The third term on the right-hand side of (11) also tends to zero by (I). Thus we have

$$\varepsilon_0 \leqq \|y_{\tau_0}(t_0, \varphi^0) - u_{\tau_0}(t_0, \varphi^0)\| .$$

It follows from (III) that

$$\varepsilon_0 \leqq \|y_{\tau_0}(t_0, \varphi^0) - u_{\tau_0}(t_0, \varphi^0)\| \leqq b(\sup_{-(\tau_0 - t_0) \leqq s \leqq 0} |y(\tau_0 + s, t_0, \varphi^0) - u(\tau_0 + s, t_0, \varphi^0)|) ,$$

and hence there exists an $s^* \in [-(\tau_0 - t_0), 0]$ such that

$$|y(\tau_0 + t^*, t_0, \varphi^0) - u(\tau_0 + s^*, t_0, \varphi^0)| \neq 0 ,$$

which contradicts the uniqueness of the solution $u(t)$. This proves Theorem.

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