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ON THE ALGEBRA OF MEASURABLE OPERATORS FOR A GENERAL AW^*-ALGEBRA II

KAZUYUKI SAITÔ

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1. Introduction. Let us consider the following problem: Let ΠM_i be the c^*-sum of a family (M_i) of AW^*-algebras, that is, the algebra of all bounded sequences $\{a_i\}$, $a_i \in M_i$, with natural norm and $*$-operations (note that $M_\infty = \prod_i M_i$ is an AW^*-algebra ([5])) and C_∞(resp. C_i) be the algebra of "measurable operators" affiliated with M_∞(resp. M_i)([8]), then is it true that C_∞ is the complete direct sum of C_i(the set of all families $x=(x_i)$ with $x_i \in C_i$ for each i, with the coordinatewise operations)? S.K.Berberian [2] showed that it is true when M_∞ is of finite type.

However, we shall show that the answer to this problem is negative in general. Then we shall define a "locally measurable operator" affiliated with the given AW^*-algebra M(LMO(M))(when M is of finite class or a factor, then any LMO(M) is a "measurable operator" affiliated with M(MO(M))([2], [8])) and show that with suitable operations, then set $\mathcal{M}(M)$ of all LMO(M)is a $*$-algebra in which the set $C(M)$of all MO(M) is naturally imbedded as a $*$-subalgebra. Moreover we shall prove: Suppose M_∞ is the c^*-sum of a family (M_i) of AW^*-algebras and $\mathcal{M}_\infty$(resp. $\mathcal{M}_i$) is the algebra of all LMO(M_∞)(resp. LMO(M_i)), then $\mathcal{M}_\infty$ is the complete direct sum of the $\mathcal{M}_i$. In the course of the proof, we shall show, along the same lines with [8], for any given AW^*-algebra M,(1) $\mathcal{M}(M)$ is a Baer*-ring in the sense of [7], (2) every element x of $\mathcal{M}(M)$ has a polar decomposition $x=w(x^*x)^{1/2}$, where w^*w and ww^* are the right and left projections of x(in particular $RP(x)\sim LP(x)$, where $RP(x)$ (resp. $LP(x)$) is the right (resp. left) projection of x) and (3) if $\mathcal{M}(M)$ is regular([10], Definition 2. 2), then M is of finite class (an alternative proof of ([6]) Theorem) and $\mathcal{M}(M) = C(M)$ ([2] Lemma). Finally we remark that if M is a semi-finite von Neumann algebra, then any integrable element of $\mathcal{M}(M)$ with respect to a normal trace is necessarily measurable. All other notations and definitions are referred to [8].

2. A characterization of the algebra of measurable operators. We shall start with the following characterization of $C(M)$ for any given AW^*-algebra M. Suppose N is an algebra over the complex numbers with involution $*$, the unit element 1 and containing the given AW^*-algebra M as a $*$-subalgebra. Assume further that (1)$x, y \in N$ and $x^*x+y^*y=1$, then $x, y \in M$(2) for any $x \in N$, $1+x^*x$

is invertible and $(1+x^*x)^{-1} \in M$ and (3) let u be a unitary element in M, write $\{u\}'' = C(\Omega)$ (where $\{u\}''$ is the double commutant of $\{u\}$ in M and $C(\Omega)$ is the algebra of all complex-valued continuous functions on a Stone space Ω [4]) and let Ω_0 be the open set $\{\omega, \omega \in \Omega, u(\omega) \neq 1\}$, then $1-u$ is invertible in N if and only if Ω_0 is dense in Ω and there exist clopen subsets Ω_n such that $\bigcup_{n=1}^{\infty} \Omega_n = \Omega_0$ and $\{\chi\Omega_n\}$ (where $\chi\Omega_n$ is the characteristic function of Ω_n for each n) is an SDD, that is, $\chi\Omega_n \uparrow (n)$ and $1-\chi\Omega_n$ is a finite projection in M for each n([8], Definition 3.1). Then we have

PROPOSITION 2.1. *For any AW*-algebra M, N is *-isomorphic to $C(M)$.*

PROOF. Note first by([3], Theorem 2.3) that N is a Baer*-ring in the sense of [7]. By axiom (2), for any $x \in N$ with $x = x^*$, $x+i1$ and $x-i1$ are invertible in N and $(x+i1)^{-1}, (x-i1)^{-1} \in \{x\}''$(the double commutant of $\{x\}$ in N). Let $u = (x-i1)(x+i1)^{-1}$, then u is a unitary element in M and $1-u$ is invertible in N. Therefore by axiom (3), there is an SDD$\{e_n\}$ in $\{u\}''$ such that $ue_n - e_n$ is invertible in e_nMe_n for each n. Observe that $\{x\}'' \cap M = \{u\}''$ and $x = i(1+u)(1-u)^{-1}$, $xe_n = i(1+u)(1-u)^{-1}e_n \in M$ for each n. Therefore for any $x \in N$ with $x = y_1 + iy_2(y_1, y_2 \in N_{sa}$ the self-adjoint part of N)let u_1(resp. u_2) be the Cayley transform of y_1(resp. y_2), then there exist SDD's$\{e_n(1)\}(\subset \{u_1\}'')$ and $\{e_n(2)\}(\subset \{u_2\}'')$ such that $y_1e_n(1)$, $y_2e_n(2) \in M$ for all n. Now put $e_n(1) \wedge e_n(2)$, $x_n = e_nxe_n$ and $f_n = (xe_n)^{-1}[e_n] \wedge [x^*e_n)^{-1}[e_n]$ where $x^{-1}[e]$ is the largest projection right-annihilating $(1-e)x$ in N(note that N is a Baer*-ring and by axiom (1) it has no new projections), then by the definition and ([8] Lemma 3.1), $\{e_n\}$ and $\{f_n\}$ are SDD's and $\{x_n, f_n\}$ is an EMO(according to ([8], Definition 3.1), a pair of sequences$\{x_n, f_n\}$ with $x_n \in M$ and $f_n \in M_p$ (the set of all projections in M) is an EMO if $\{f_n\}$ is an SDD and $m < n$ implies $x_nf_m = x_mf_m$ and $x_n^*f_m = x_m^*f_m$). Let us consider the Cayley transforms of the real and imaginary parts of the MO$[x_n, f_n]$ ([8], Definition 3.4 and Theorem 3.1. Two EMO's$\{x_n, e_n\}$ and $\{y_n, f_n\}$ are equivalent if there exists an SDD$\{g_n\}$ such that $x_ng_n = y_ng_n$, $x_n^*g_n = y_n^*g_n$ for all n. $[x_n, e_n]$ is its equivalence class and is said a "measurable operator" (MO)). Then, an easy calculation shows that $[1/2(x_n + x_n^*), f_n] = [y_1e_n, e_n]$, $[1/2i(x_n - x_n^*), f_n] = [y_2e_n, e_n]$,$(y_1+i1)^{-1}e_n \in M$ for each n and $(i1 + [y_1e_n, e_n])^{-1} = [(y_1 + i1)^{-1}e_n, e_n]$. Therefore the Cayley transform of $[1/2(x_n + x_n^*), f_n]$ (resp. $[1/2i(x_n - x_n^*), f_n]$) is u_1(resp. u_2). Thus for any x, $[x_n, f_n]$ is uniquely determined. Set $\Phi(x) = [x_n, f_n]$, then by a direct calculation Φ is one to one, linear and *-preserving map of N into $C(M)$. For any x, y and $xy \in N$, *let* $\{e_n(x)\}$, $\{f_n(x)\}$, $\{e_n(y)\}$, $\{f_n(y)\}$, $\{e_n(xy)\}$ and $\{f_n(xy)\}$ be SDD'*s* such that $\Phi(x) = [e_n(x)xe_n(x), f_n(x)]$, $\Phi(y) = [e_n(y)ye_n(y), f_n(y)]$ and $\Phi(xy) = [e_n(xy)xye_n(xy), f_n(xy)]$, then since $\Phi(x)\Phi(y) = [e_n(x)xe_n(x)e_n(y)ye_n(y), g_n]$ for some SDD$\{g_n\}$ by ([8], Theorem 3.1), the SDD$\{h_n\}$(where $h_n = f_n(x) \wedge f_n(y) \wedge f_n(xy) \wedge ((ye_n(y))^{-1}[e_n(x)]) \wedge ((x_ne_n(x))^{-1}[e_n(y)]) \wedge (xye_n(y))^{-1}[e_n(x)]) \wedge (y^*x^*e_n(x))^{-1}[e_n(y)]))$ implements the equivalence $\{e_n(x)xe_n(x)e_n(y)ye_n(y),\ g_n\} \equiv \{e_n(xy)xye_n(xy), f_n(xy)\}$. Thus Φ is a *-isomorphism of N into $C(M)$. Using axiom (3) and Cayley transform, it is easy to see that this map Φ is onto. This completes the proof.

Now let M_∞ be the c^*-sum of a family (M_i) of AW^*-algebras, C_∞(resp. C_i) be the algebra of MO(M_∞) (resp. M_i) and let $\mathcal{D}$ be the set of all families $A=(a_i)$ with $a_i \in C_i$ and $a_i=[x_n(i), e_n(i)]$ such that $\|x_n(i)e_n(i)\| \leqq k_n$ for all i for each n where k_n is a positive real number which is independent on i. The operations in $\mathcal{D}$ are coordinatewise, then one knows that $\mathcal{D}$ is a *-algebra over the complex numbers field with unit $1=(1)$ which contains M as the *-subalgebra.

PROPOSITION 2.2. *Let $\mathcal{D}$ be defined as above. Then $\mathcal{D}$ is *-isomorphic to C_∞.*

PROOF. We have only to show that $\mathcal{D}$ satisfies axioms (1)−(3) mentioned in the first paragraph of this section. Since for any $x \in C_\infty$, $1+x^*x$ is invertible in C_∞ and $0 \leqq (1+x^*x)^{-1} \leqq 1$, it is easy to verify that $\mathcal{D}$ satisfies axioms (1) and (2). For any $A=(a_i) \in \mathcal{D}$ with $A=A^*$, let u be the Cayley transform of A and $\{u\}''=C(\Omega)$ (where $C(\Omega)$ is the algebra of continuous complex-valued functions on a Stone space Ω). Since $1-u$ is invertible in $\mathcal{D}$, $\Omega_0=\{\omega;\ u(\omega)\neq 1\}$ is dense in Ω. Taking an increasing sequence $\{r_n\}$ of positive numbers satisfying $r_n > k_n$ for each n and we define clopen subset $\Omega_n=\{\omega;\ |u(\omega)-1| > 2/((r_n)^2+1)^{1/2}\}^-$. Note that if $e_i=(x_j^i)$where $x_j^i=0 (i\neq j)$ and $x_i^i=1$ (the unit of M_i), then e_i is a central projection of M_∞ and $M_\infty e_i \cong M_i$, we can easily show that $(1-\chi\Omega_n)e_i$ is finite for each i, which implies $\{\chi\Omega_n\}$ is an SDD. Conversely if for given unitary element u of M_∞, there is an SDD$\{e_n\}$ which satisfies the condition of axiom (3), we can take a self-adjoint element x of C_∞ such that u is the Cayley transform of (xe_i) ($\in \mathcal{D}$). Thus $\mathcal{D}$ satisfies axiom (3), which implies $\mathcal{D} \cong C_\infty$. The proposition follows.

Note (1). In Proposition 2.1, the axiom (3)cannot be dropped. In fact, let H be an infinite dimensional Hilbert space and $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. Then an easy calculation shows that the algebra of MO affiliated with $\mathcal{B}(H)$ is exactly $\mathcal{B}(H)$. Now suppose given a countable set of infinite dimensional Hilbert spaces $H_1, H_2, \cdots$, let K be tht direct sum of the spaces $H_1, H_2, \cdots$. We may suppose that for each i, H_i becomes a subspace of K. For any family (T_i) with $T_i \in \mathcal{B}(H_i)$ (for each i) set $D(T_{(T_i)})=\{\xi;\ \Sigma_i\|P_iT_iP_i\xi\|^2<\infty$ where P_i is the orthogonal projection on $H_i\}$and define a linear operator $T_{(T_i)}$ on $D(T_{(T_i)})$ by $T_{(T_i)}\xi=\Sigma_iP_iT_iP_i\xi$, $\xi \in D(T_{(T_i)})$, then $T_{(T_i)}$ is a densely defined closed linear operator such that $T_{(T_i)}^*=T_{(T_i^*)}$. Let $\mathcal{D}'=\{T_{(T_i)};\ T_i \in \mathcal{B}(H_i)\}$, then for $T_{(T_i)}, T_{(S_i)} \in \mathcal{D}'$ and a complex number λ, $T_{(T_i+S_i)}=T_{(T_i)} \dot{+} T_{(S_i)}$ (where $T_{(T_i)} \dot{+} T_{(S_i)}$ is the minimal closed extension of $T_{(T_i)}+T_{(S_i)}$), $\lambda T_{(T_i)}=T_{(\lambda T_i)}$ and $T_{(T_iS_i)}=T_{(T_i)}\cdot T_{(S_i)}$

(where $T_{(T_i)}\cdot T_{(S_i)}$ is the minimal closed extension of $T_{(T_i)}T_{(S_i)}$). Thus $\mathcal{D}'$ is a *-algebra with unit relative to the sum $\dot{+}$ and product $\cdot$ which contains the c^*-sum $\prod_{i=1}^{\infty}\mathcal{B}(H_i)$ of $(\mathcal{B}(H_i))$ as the *-subalgebra and it is easy to verify that $\mathcal{D}'$ satisfies axioms (1) and (2). Let us consider an element $T_{(n1_n)}$ in $\mathcal{D}'$ where 1_n is the identity operator on H_n for each n and suppose $\mathcal{D}'$ is *-isomorphic to $C\left(\prod_{n=1}^{\infty}\mathcal{B}(H_n)\right)$, then there is a *-isomorphism ϕ of $\mathcal{D}'$ onto $C\left(\prod_{n=1}^{\infty}\mathcal{B}(H_n)\right)$ such that $\phi(T_{(n1_n)}) = [s_m, f_m]$ ([8]) $\left([s_m, f_m]\in C\left(\prod_{n=1}^{\infty}(\mathcal{B}(H_n))\right)\right)$. Thus $\phi(T_{(n1_n)})\phi(P_n) = n\phi(P_n)$ and $\phi(T_{(n1_n)})\phi(P_n)f_m = n\phi(P_n)f_m = s_m f_m$ for each m and n. Since $\phi(P_n)f_m \neq 0$, $n = \|s_m f_m\|$ for all m and n. This is a contradiction and $\mathcal{D}'$ is not *-isomorphic to $C\left(\prod_{n=1}^{\infty}\mathcal{B}H_n)\right)$. We also note that the above example is a negative one to the problem mentioned in the introduction.

3. The algebra of "locally measurable operators" affiliated with an AW^*-algebra.

Let M be an arbitrary AW^*-algebra and let $C(M)$ be the algebra of MO(M).

DEFINITION 3. 1. An essentially locally measurable operator affiliated with M(ELMO(M)) is an indexed family of ordered pairs $\{x_\alpha, e_\alpha\}$ where $x_\alpha \in C(M)$ and $\{e_\alpha\}$ is an orthogonal family of central projections such that $\Sigma_\alpha e_\alpha = 1$.

DEFINITION 3. 2. Two ELMO(M)'s $\{x_\alpha, e_\alpha\}$ and $\{y_\beta, f_\beta\}$ are equivalent, denoted by $\{x_\alpha, e_\alpha\} \equiv \{y_\beta, f_\beta\}$, if $e_\alpha f_\beta x_\alpha = e_\alpha f_\beta y_\beta$ for all α and β.

Since $C(M)$ is a Baer*-ring, it is immediate that the relation just defined is an equivalence relation.

DEFINITION 3. 3. An equivalence class (x_α, e_α) of an ELMO(M) $\{x_\alpha, e_\alpha\}$ is called a "locally measurable operator affiliated with M"(LMO(M)); the set of all LMO(M) is denoted by $\mathcal{M}(M)$ and we use letters $x, y, z, \cdots$ for the elements of $\mathcal{M}(M)$.

We shall define the algebraic operations in $\mathcal{M}(M)$. If (x_α, e_α) and (y_β, f_β) are LMO(M)'s and λ is a complex number, we define $\lambda(x_\alpha, e_\alpha) = (\lambda x_\alpha, e_\alpha)$, $(x_\alpha, e_\alpha) + (y_\beta, f_\beta) = (x_\alpha + y_\beta, e_\alpha f_\beta)$, $(x_\alpha, e_\alpha)^* = ((x_\alpha)^*, e_\alpha)$ and $(x_\alpha, e_\alpha)(y_\beta, f_\beta) = (x_\alpha y_\beta, e_\alpha f_\beta)$.
Since $C(M)$ is a Baer*-ring, the above definitions are unambiguous. With these

definitions, $\mathcal{M}(M)$ becomes a *-algebra over complex numbers. By the Baer*-ring property of $C(M)$, it is easy to see that $x(\in C(M)) \longrightarrow (x,1)((x,1)$is an equivalence class of an ELMO(M) $\{x,1\})$is a *-isomorphism of $C(M)$ into $\mathcal{M}(M)$ and $(1,1)$ is the unit element ot $\mathcal{M}(M)$. To simplify the notations, we shall denote $(x,1)$ by x; and thus M is a *-subalgebra of $\mathcal{M}(M)$.

LEMMA 3.1. *If* $x=(x_\alpha,e_\alpha)\in\mathcal{M}(M)$ *and all the* x_α *are invertible in* $C(M)$, *then* x *is invertible in* $\mathcal{M}(M)$ *and* $x^{-1}=((x_\alpha)^{-1},e_\alpha)$.

PROOF. $x((x_\alpha)^{-1},e_\alpha)=(x_{\alpha'},e_{\alpha'})((x_\alpha)^{-1},e_\alpha)=(x_{\alpha'}(x_\alpha)^{-1},e_{\alpha'}e_\alpha)=(1,e_\alpha)=(1,1)=1$. The lemma follows.

LEMMA 3.2. *If* $x\in\mathcal{M}(M)$ *with* $x=x^*$, *then we can write* $x=(x_\alpha,e_\alpha)$ *with* $(x_\alpha)^*=x_\alpha$ *for each* α.

Now let $x\in C(M)$ and $\{e_\alpha\}$ be an orthogonal family of central projections such that $xe_\alpha\geqq 0$ for all α and $\Sigma_\alpha e_\alpha=1$, then $x\geqq 0$. In fact, since $C(M)$ is a Baer*-ring, x is automatically self-adjoint and by [8] we can write $x=[x_n,e_n]$ with $x_n=x_n{}^*$ for each n. By the assumption $e_nx_ne_ne_\alpha\geqq 0$ for all n and α. Noting that $e_\alpha\in\{e_nx_ne_n\}''$ for all n, if we assume that there are a non-zero projection $g\in\{e_nx_ne_n\}''$ and a real number $\delta<0$ such that $ge_nx_ne_n\leqq\delta g$, then $ge_\alpha=0$ for all α, which implies $g=0$, contradicting the above result $g\neq 0$. Thus $e_nx_ne_n\geqq 0$ for all n and by ([8], Theorem 5.5), $x\geqq 0$ follows.

THEOREM 3.1. $\mathcal{M}(M)$ *satisfies axioms* (1) *and* (2) *mentioned in the first paragraph of this section and* $\mathcal{M}(M)$*is a Baer*-ring.*

PROOF. Suppose $x=(x_\alpha,e_\alpha)$, $y=(y_\beta,f_\beta)\in\mathcal{M}(M)$, and $x^*x+y^*y=1$. Then $(x_\alpha{}^*x_\alpha+y_\beta{}^*y_\beta)e_\alpha f_\beta=e_\alpha f_\beta$ for all α and β, $x_\alpha{}^*x_\alpha e_\alpha f_\beta\leqq f_\beta$ for all β, which implies by the considerations following Lemma 3.2, $x_\alpha{}^*x_\alpha e_\alpha\leqq 1$. Therefore $x_\alpha e_\alpha\in M$ and $\|x_\alpha e_\alpha\|\leqq 1$. By ([5] Lemma 2.5), there is a unique element $x\in M$ such that $xe_\alpha=x_\alpha e_\alpha$. This implies $x=(x,1)=x\in M$. By the same way $y\in M$. Now for any $x=(x_\alpha,e_\alpha)\in\mathcal{M}(M)$, $1+x^*x=(1+x_\alpha{}^*x_\alpha,e_\alpha)$. Therefore by Lemma 3.1, $1+x^*x$ is invertible in $\mathcal{M}(M)$ and $(1+x^*x)^{-1}=((1+x_\alpha^*x_\alpha)^{-1},e_\alpha)$. Since $(1+x_\alpha{}^*x_\alpha)^{-1}\in M$ and $(1+x_\alpha{}^*x_\alpha)^{-1}\leqq 1$ for each α, by ([5] Lemma 2.5), $(1+x^*x)^{-1}\in M$. Thus $\mathcal{M}(M)$ satisfies axioms (1) and (2). By ([3] Theorem 2.3) $\mathcal{M}(M)$ is a Baer*-ring. This completes the proof.

Now we discuss the spectral theory for $\mathcal{M}(M)$. Let $x\in\mathcal{M}(M)$ with $x=x^*$, then $x+i1$ is invertible in $\mathcal{M}(M)$. Put $u=(x-i1)(x+i1)^{-1}$, u is unitary in M and $\{x\}''\cap M=\{u\}''$. Write $\{u\}''=C(\Omega)$ (where $C(\Omega)$ is the algebra of complex-valued continuous functions on a Stone space Ω). We may write $x=(x_\alpha,e_\alpha)$ with

$x_\alpha = x_\alpha^*$ and $(1-u)e_\alpha = 2ie_\alpha(x_\alpha + i1)^{-1}$ for all α. Since $e_\alpha \in \{u\}''$, let Ω_α be the clopen subset of Ω corresponding to e_α, then $\Omega_0 = \{\omega;\ u(\omega) \neq 1\}$ is dense in Ω and there exist clopen subsets $\Omega_n(\alpha)$ such that $\bigcup_{n=1}^{\infty} \Omega_n(\alpha) = \Omega_0 \cap \Omega_\alpha$ and $\chi\Omega_\alpha - \chi\Omega_n(\alpha)$ is a finite projection for each n and α. Conversely for given unitary element $u \in M$, if $\Omega_0 = \{\omega;\ u(\omega) \neq 1\}$ is dense in Ω and there are clopen subsets Ω_α and $\Omega_n(\alpha)$ such that $\bigcup_{n=1}^{\infty} \Omega_n(\alpha) = \Omega_0 \cap \Omega_\alpha$, $\chi\Omega_\alpha$ is a central projection for each α and $\{\chi\Omega_n(\alpha)\}$ is an SDD in Me_α, then ue_α is a Cayley transform of some self-adjoint element x_α of $C(M)e_\alpha$. A direct calculation shows that u is the Cayley transform of $(x_\alpha, e_\alpha) \in \mathcal{M}(M)$.

PROPOSITION 3.1. *Let $u \in M_u$, write $\{u\}'' = C(\Omega)$ with Ω a Stone space ([4]) and let $\Omega_0 = \{\omega;\ u(\omega) \neq 1\}$. Then $1-u$ is invertible in $\mathcal{M}(M)$ if and only if Ω_0 is dense in Ω and there exist families $\{\Omega_\alpha\}$ and $\{\Omega_n(\alpha)\}$ of clopen subsets such that $\overline{\cup_\alpha \Omega_\alpha} = \Omega$, $\Omega_\alpha \cap \Omega_0 = \bigcup_{n=1}^{\infty} \Omega_n(\alpha)$, $\chi\Omega_\alpha$ is a central projection for each α and $\{\chi\Omega_n(\alpha)\}$ is an* SDD *in $M_{\chi\Omega_\alpha}$ for each α. In this case, u is the Cayley transform of some self-adjoint element of $\mathcal{M}(M)$.*

COROLLARY. *Let x be a self-adjoint element of $\mathcal{M}(M)$, and u be its Cayley transform. Then we can write $x = (x_\alpha, e_\alpha)$ with $x_\alpha = [x_n(\alpha), e_n(\alpha)]$ such that $x_n(\alpha)$, $e_n(\alpha) \in \{u\}''$, $x_n(\alpha)e_n(\alpha)e_\alpha = x_n(\alpha) = x_n(\alpha)^*$ and $x_n(\alpha)^2 \uparrow (n)$ (that is, $x_n(\alpha)^2 \leqq x_{n+1}(\alpha)^2$).*

Next we introduce a partial order in the self-adjoint part of $\mathcal{M}(M)$.

DEFINITION 3.4. An element $x \in \mathcal{M}(M)$ is non-negative ($x \geqq 0$) if $x = y^*y$ for some $y \in \mathcal{M}(M)$. If $x, y \in \mathcal{M}(M)$ are self-adjoint, write $x \leqq y$ if $y - x \geqq 0$.

To show that $\mathcal{M}(M)_{s.a.}$ (the self-adjoint part of $\mathcal{M}(M)$) form a partially ordered real linear space with respect to this order, we have only to prove the following:

PROPOSITION 3.2. *Let x be a self-adjoint element of $\mathcal{M}(M)$, u be its Cayley transform. Then the following four conditions are equivalent:*

(1) $x \geqq 0$,

(2) *we can write $x = (y_\beta, f_\beta)$ with $y_\beta \geqq 0$ for each β;*

(3) *$\sigma(u)$ (the spectrum of u) $\subset \{e^{i\theta}:\ -\pi \leqq \theta \leqq 0\}$;*

(4) *we may write $x = (x_\alpha, e_\alpha)$ with $x_\alpha = [x_n(\alpha), e_n(\alpha)]$ such that $x_n(\alpha), e_n(a) \in \{u\}''$, $x_n(\alpha)e_n(\alpha)e_\alpha = x_n(\alpha) \geqq 0$ and $x_n(\alpha) \uparrow (n)$ for each α.*

Observe that for any family $\{y_\beta\}$ of self-adjoint elements in $C(M)$ and a negative number λ, $(y_\beta+i1)(y_\beta-\lambda1)^{-1}\in M$ and $\|(y_\beta+i1)(y_\beta-\lambda1)^{-1}\|\leqq k$ for some positive number k which is independent on β, by ([5] Lemma 2.5), (2)$\longrightarrow$(3) follows. The proofs of (1)$\longrightarrow$(2), (3)$\longrightarrow$(4)$\longrightarrow$(1) are the same as those of ([8], Theorem 5.5), so we omit them.

COROLLARY. *If $x\geqq 0$ $(x\in \mathcal{M}(M))$, then there exists a unique $y\geqq 0$ in $\mathcal{M}(M)$ such that $x=y^2$ and $y\in\{x\}''$, that is to say, the Baer*-ring $\mathcal{M}(M)$ satisfies the* (SR)*-axiom in the sense of* ([7] p.37).

Proof is the same as that of ([8] Corollary 5.2).

DEFINITION 3.5. Let $x\in\mathcal{M}(M)$ with $x\geqq 0$, write $y=x^{1/2}$ for the unique $y\geqq 0$ in $\{x\}''$ such that $x=y^2$. For $x\in\mathcal{M}(M)$, write $|x|=(x^*x)^{1/2}$.

THEOREM 3.2. *For any $x\in\mathcal{M}(M)$, let u(resp. v) be the Cayley transform of x^*x(resp. xx^*), $e=\mathrm{LP}(1+u)$ and $f=\mathrm{LP}(1+v)$. Then we may write $x=w|x|$, where w is a partial isometry such that $w^*w=e$, $ww^*=f$. In particular $e\sim f$ and $e=\mathrm{RP}(x)$, $f=\mathrm{LP}(x)$.*

PROOF. By [4], we may write $\{u\}''$(resp. $\{v\}''$) as the algebra $C(\Omega)$ (resp. $C(\Gamma)$) of continuous complex-valued functions on a Stone space Ω (resp. Γ). A direct calculation shows by ([3], Lemma 2.1) that the characteristic function of $\{\omega;\ u(\omega)+1\neq 0\}^-$ (resp. $\{\gamma;\ v(\gamma)+1\neq 0\}^-$) is $e=\mathrm{RP}(x)$ (resp. $f=\mathrm{LP}(x)$). Now let $x^*x=(y_\alpha, e_\alpha)$ such that $y_\alpha=[y_n(\alpha), e_n(\alpha)]$, $y_n(\alpha)$, $e_n(\alpha)\in\{u\}''$, $y_n(\alpha)\geqq 0$, $y_n(\alpha)\uparrow(n)$ and $y_n(\alpha)e_n(\alpha)e_\alpha=y_n(\alpha)$. For each $y_n(\alpha)$, we can choose families $\{c_m^n(\alpha)\}$, $\{e_m^n(\alpha)\}$ satisfying the conditions (1)$-$(5) in the proof of ([8], Theorem 6.3). Now let $c_m^n=(c_m^n(\alpha), e_\alpha)$ $(\in\mathcal{M}(M))$, $e_m^n=\Sigma_\alpha e_m^n(\alpha)e_\alpha$ and $e_n=\Sigma_\alpha e_n(\alpha)e_\alpha$, then we have $x^*x(c_m^n)^2=e_m^n$ and $e_m^n e_n\uparrow e(m,n)$. In fact, $x^*x(c_m^n)^2=(y_\alpha, e_\alpha)((c_m^n(\alpha))^2, e_\alpha)=(y_\alpha(c_m^n(\alpha))^2, e_\alpha)=(y_n(\alpha)(c_m^n(\alpha))^2, e_\alpha)=(e_m^n(\alpha), e_\alpha)\Sigma_\alpha e_m^n(\alpha)e=e_m^n$. The last statement is proved by the same method as that of ([8], Theorem 6.3). Thus $\mathcal{M}(M)$ satisfies the (EP)-axiom in the sense of ([7], p.37). Therefore by ([7] Appendix II, Theorem 2), $x=w|x|$ with $w^*w=e$, $ww^*=f$. It remains to prove the unicity of such decomposition. Let $x=w_1y$ with $y\geqq 0$, $w_1{}^*w_1=e, ey=y$, then $x^*x=yey=y^2$ and by the unicity of the square root of x^*x, $y=|x|$ and $w_1|x|^2=w|x|^2$ implies $w|x|^2(c_m^n)^2=w|x|^2(c_m^n)^2$, $w_1e_m^n=we_m^n$ for all m, n. $w_1e=we$, that is, $w_1=w$. This completes the proof.

4. Algebraic structure of $\mathcal{M}(M)$.

S.K.Berberian showed in([2], Lemma): Let M be a finite AW^*-algebra, and $\{e_\alpha\}$ a set of orthogonal central projections with $\Sigma_\alpha e_\alpha=1$. Then $C(M)$ is isomorphic to the complete direct sum of $\{C(M)e_\alpha\}$. As we showed in Note in section 2, we cannot drop the condition that M is of finite class, but we have

THEOREM 4. 1. *Suppose M_∞ is a c*-sum of a family (M_ι) of AW*-algebras, $\mathcal{M}_\infty$(resp. $\mathcal{M}_\iota$) is the algebra of* LMO's *affiliated with M_∞(resp. M_ι). Then $\mathcal{M}_\infty$ is the complete direct sum of the $\mathcal{M}_\iota$.*

PROOF. Let $\mathcal{S}$ be the complete direct sum of the $\mathcal{M}_\iota$, that is the set of all families $x=(x_\iota)$ with $x_\iota \in \mathcal{M}_\iota$ with coordinatewise operations and let $e_j=(x_\iota)$ with $x_\iota=0$ if $\iota \neq j$ and $x_j=1_j$ the unit of $\mathcal{M}_j$. Then for any $x \in \mathcal{M}_\infty$ with $x=(x_\alpha, e_\alpha)(x_\alpha \in C(M_\infty))$, observe that $C(M_\infty)e_\iota \cong C(M_\iota)$ (the algebra of MO's of M_ι) if we put $x_\iota=(x_\alpha e_\iota, e_\iota e_\alpha)$, then $x_\iota \in \mathcal{M}_\iota$ and$(x_\iota) \in \mathcal{S}$ is uniquely determined by $x \in \mathcal{M}_\infty$. By an easy calculation $x \longrightarrow (x_\iota)$ is a *-isomorphism of $\mathcal{M}_\infty$ into $\mathcal{S}$. By the nature of the construction, it is easy to show that this map is onto. This completes the proof.

Because of the inherent nature of the construction of $\mathcal{M}(M)$, we have:

THEOREM 4. 2. *Let M and N be AW*-algebras and $\mathcal{M}(M)$* (resp. $\mathcal{M}(N)$) *be the algebra of* LMO(M)'s *(resp.* LMO(N)'s*). There exists a one to one correspondence between the *-isomorphisms Φ: $\mathcal{M}(M) \longrightarrow \mathcal{M}(N)$ and the *-isomorphisms ϕ: $M \longrightarrow N$ and the correspondence $\Phi \longrightarrow \phi$ is obtained by restricting Φ to M. Moreover for any $e \in M_p$, the algebra of all* LMO's *of eMe is *-isomorphic to $e\,\mathcal{M}(M)e$.*

THEOREM 4. 3. *$\mathcal{M}(M)$ is regular in the sense of* [10] *if and only if M is finite.*

PROOF. Suppose M is finite, then $\mathcal{M}(M)=C(M)$ and $C(M)$ is regular by ([1], Corollary 7. 1). Conversely if $\mathcal{M}(M)$ is regular, and suppose M is not finite, then there exists a family of increasing projections $\{e_\iota\}$ in M such that $1-e_\iota$ is not finite and $e_\iota \uparrow 1$. By the same way as in the proof of ([8], Theorem 6. 2), we can find a non-negative invertivle elements in M. Thus there is $y \in \mathcal{M}(M)$ with $y=(y_\alpha, e_\alpha)$ and $ys=sy=1$. Since $C(M)e_\alpha \cong C(Me_\alpha)$, we can show that $(1-e_n)e_\alpha$ is finite for each n and α, and contradicting the choice of $\{e_\iota\}$, that is, M is of finite class. This completes the proof.

5. Complements. We first remark the following: (1) If M has the monotone convergence property, that is, it satisfies that if $\{a_n\}$ is a monotone increasing sequence of self-adjoint elements in M such that $a_n \leqq b$ $(n=1, 2, 3, \cdots)$ for some self-adjoint element b of M, then Sup a_n exists with respect to the ordering defined in Definition 3. 4, then so does $\mathcal{M}(M)$. (2) If M has the monotone convergence property, and if $x, y \in \mathcal{M}(M)$ satisfy $0 \leqq x \leqq y$, then $x^{1/2} \leqq y^{1/2}$. The proofs are the same as those of [3], so we omit them. The rest of this section is devoted to the non-commutative integration theory.

DEFINITION 5.1. We call that a sequence $\{x(n)\}$ of $\mathcal{M}(M)$ converges nearly everywhere (or converges n.e.) to an element x in $\mathcal{M}(M)$ if for any positive ε, there exist a positive integer $n_0(\varepsilon)$ and an SDD$\{e_n(\varepsilon)\}$ such that $|(x(n)-x)e_n(\varepsilon)| \leqq \varepsilon \cdot 1$ for all $n \geqq n_0(\varepsilon)$.

REMARK. By the same way as that in $C(M)$, we can show that a limit n.e is unique.

Now let M be a von Neumann algebra and τ be a faithful normal semi-finite trace on M, then we have

PROPOSITION 5.1. *If $x \in \mathcal{M}(M)$ is integrable with respect to τ, that is, there exists a sequence $\{x(n)\}$ of elements of τ-finite rank such that $x(n) \longrightarrow x$ n.e. and $\tau(|x(n)-x(m)|) \longrightarrow 0 (n, m \longrightarrow \infty)$, then $x \in C(M)$.*

PROOF. Let $x=w|x|$ be the polar decomposition of x, then $w^*x(n) \longrightarrow |x|$ n.e. and $\tau(|w^*x(n)-w^*x(m)|) \longrightarrow 0 (m, n \longrightarrow \infty)$, that is, $|x|$ is integrable with respect to τ. Let u be the Cayley transform of $|x|$, then $1+u=(1/i)(-u+1)|x|$ implies $1+u$ is integrable with respect to τ. Write$\{u\}''=\mathrm{C}(\Omega)$ with Ω a Stone space ([4]), and let $\Omega_0 = \{\omega;\ u(\omega) \neq 1\}$. Set $\Omega_n = \{\omega;\ |u(\omega)-1| > 2/((n+1)^2+1)^{1/2}\}^-$, then $\Omega_0 = \bigcup_{n=1}^{\infty} \Omega_n$ and $\Omega_n^c \subset \{\omega;\ |u(\omega)+1| > 2n/(n^2+1)^{1/2}\}^-$. Let f_n be the projection corresponding to the clopen set $\{\omega;\ |u(\omega)+1| > 2n/(n^2+1)^{1/2}\}^-$, then write $w_n =(1+u(\omega))^{-1}f_n$ and observe $w_n \in M$, and we have $w_n(1+u)=f_n$ is also τ-integrable, that is, f_n is τ-finite. Thus $\chi\Omega_n^c$ is a finite projection and $\{\chi\Omega_n\}$ is an SDD. Therefore by([8], Theorem 5.1), $|x| \in C(M)$ and $x=w|x| \in C(M)$. The proposition follows.

The result of this paper should prove to be useful for attacking the Widom's problem concerning to the "Embedding as a double commutator in algebras of type 1" for the semi-finite case. We propose to investigate this in a subsequent paper.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN