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ON THE UNIQUENESS OF THE CAUCHY PROBLEM FOR CERTAIN ELLIPTIC EQUATIONS WITH TRIPLE CHARACTERISTICS

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1. Introduction. The uniqueness theorem of the Cauchy problem for elliptic operators with double characteristics have been studied by many authors, [3], [4], [6], etc. On the other hand, in [5], A.Plis has constructed an elliptic equation of fouth order with real-valued C^∞ coefficients, which has non-trivial solutions with compact support and P.Cohen [1] has constructed an equation of order m with constant leading coefficients, with a complex characteristics of multiplicity r and with a lower order term of order $m-1$ with C^k coefficients, $k<r-2$, for which uniqueness fails. It is the point worthy of remark that there exists an equation whose principal part is equal to $\triangle^3$ and for which uniqueness fails, where $\triangle$ is the Laplacian operator. The purpose of this paper is to show the uniqueness and the unique continuation theorem for some class of elliptic equations which include every equation with principal part $\triangle^3$ and with a lower order term with Lipschitz continuous coefficients. At first, we shall study the solutions of a differential inequality

$$|P(D)u+Q(x;D)u| \leqq C \sum_{|\alpha|\leqq m-q} |D^\alpha u| \tag{1.1}$$

where $P(D)$ is a linear partial differential operator of order m with constant coefficients and the principal part $P_m(D)$ of $P(D)$ is of real (resp. complex) characteristics of multiplicity at most s (resp. r) and where $Q(x;D)$ is a homogeneous operator of order $m-q+1$ with Lipschitz continuous coefficients defined in a neighbourhood of the origin 0. When $Q\equiv 0$, $q=[(r+1)/2]$ and $P(D)$ is homogeneous, Goorjian [2] proved that each solution $u\in C^m$ for the differential inequality (1.1), which vanishes for

$$x_1 < x_2^2+\cdots+x_n^2 \tag{1.2}$$

when $x=(x_1,\cdots,x_n)$ is in a neighbourhood of 0, also vanishes in a full neighbourhood of 0. We are interested in the case of $r=3$, $q=2$. Next, applying the results for operators with constant leading coefficients, we shall also consider the solutions of a differential inequality

$$|A(x; D)^3u + B(x; D)u| \leqq C \sum_{|\alpha| \leqq 3m-2} |D^\alpha u| \tag{1.3}$$

where $A(x; D)$ is a homogeneous operater of order m with C^{2m+1} coefficients such that $A(0; D)$ is an elliptic operator of order m with simple characteristics and where $B(x; D)$ is a homogeneous operator of order $3m-1$ with Lipschitz continuous coefficients. The coefficients of $A(x; D)$ and $B(x; D)$ are defined in a neighbourhood of 0. The purpose is to prove the following.

THEOREM 1. *If r is an odd integer $\geqq 3$ and $[r/2] \geqq s$, then each solution $u \in C^m$ for the differential inequality* (1.1) *with $q = [(r+1)/2]$, which vanishes for* (1.2) *when x is in a neighbourhood of* 0, *vanishes also in a full neighbourhood of* 0.

Applying this Theorem to elliptic operators, we have

COROLLARY 1. *Suppose that $P(D)$ is a homogeneous elliptic operator and that $r=3$. Then the same conclusion as in Theorem* 1 *holds with $q=2$ in the differential inequality* (1.1).

When operators are of variable coefficients, we have

THEOREM 2. *Let $u \in C^{3m}$ be a solution of the differential inequality* (1.3), *which vanishes for* (1.2) *when x is in a neighbourhood of* 0. *Then u must vanish in a full neighbourhood of* 0.

REMARK. If $m=2$ and $n \geqq 3$, or if $m=2$ and $A(x; D)$ has real-valued coefficients, then Theorem 2 implies the uniqueness of solutions for (1.3) across arbitrary surface Γ of the class C^{3m}, since, for each point $\bar{x}$ on Γ, by a change of coordinates, the surface Γ can be made to coincide with the paraboloid $x_1 = x_2^2 + \cdots + x_n^2$ in a neighbourhood of $\bar{x}$ with $\bar{x}$ at the origin of the new coordinates, and the class of differential inequalities considered in (1.3) is invariant under the change of coordinates. Hence the unique continuation theorem holds for such operators. Explaining in detail, for a second order elliptic operator $A(x; D)$ in a domain $\Omega \subset R^n$ with $C^5(\Omega)$ coefficients such that if $n=2$ the coefficients are real-valued and for a homogeneous operator $B(x; D)$ of fifth order whose coefficients are locally Lipschitz continuous in Ω, we have the following.

COROLLARY 2. *Let $u \in C^6(\Omega)$ be a solution for the differential inequality in Ω*

$$|A(x; D)^3u + B(x; D)u| \leqq C(x) \sum_{|\alpha| \leqq 4} |D^\alpha u| ,$$

with a locally bounded function $C(x)$ in Ω. Then u must vanish identically in Ω if u vanishes in some non-empty open subset of Ω.

The proofs of Theorems 1, 2 and Corollary 2 will be given in §3 by using Carleman type estimates. This estimates will be established in §2, when operators are of constant leading coefficients and in §3, in the case of variable coefficients.

2. Carleman type estimates for operators with constant leading coefficients. Notation and Definitions. Let $x=(x_1,\cdots,x_n)$ be a point in the real n-dimensional Euclidean space R^n, $n\geqq 2$, $\alpha=(\alpha_1,\cdots,\alpha_n)$ is a multi-index where the α_k is non-negative integer, $|\alpha|=\sum_{k=1}^{n}\alpha_k$. k is always a single index, $k=0,1,\cdots$, $[a]$ is the largest integer less than or equal to a. For the brevity we use the following notation; $\xi=(\xi_1,\cdots,\xi_n)\in R^n$, $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, $D_k=-i\,\partial/\partial x_k$, $(i)^2=-1$, $D=(D_1,\cdots,D_n)$, $D^\alpha=D_1^{\alpha_1},\cdots,D_n^{\alpha_n}$.

We denote by $P(D)$ a linear partial differential operator of order m with constant coefficients and by $P_m(D)$ the principal part of $P(D)$. For a multi-index α, set $P^{(\alpha)}(\xi)=\partial^{|\alpha|}P(\xi)/\partial\xi_1^{\alpha_1}\cdots\partial\xi_n^{\alpha_n}$; in particular, for a single index k, $P^{(k)}(\xi)=\partial^k P(\xi)/\partial\xi_1^k$.

We say that a homogeneous operator $P(D)$ with constant coefficients is of real (resp. complex) characteristics of multiplicity at most s (resp. r) if the hyperplane $x_1=0$ is not characteristic for $P(D)$ and if a family of polynomianls of ζ_1,

$$\{P^{(k)}(\zeta_1,\xi_2,\cdots,\xi_n);\ 0\leqq k\leqq s\}$$

$$(\text{resp. } \{P^{(k)}(\zeta_1,\xi_2,\cdots,\xi_n);\ 0\leqq k\leqq r\})$$

has no common real (resp. complex) zero whenever $(\xi_2,\cdots,\xi_n)$ is non-zero real vector. If $r=1$, we say $P(D)$ is of simple characteristics. Without loss of generality, we always assume that $s\leqq r\leqq$ the order of $P(D)$.

Using the methods of Hörmander [3], we shall choose as a weight function in the exponents

$$\varphi_\delta(x)=(x_1-\delta)^2+\delta(x_2^2+\cdots+x_n^2), \tag{2.1}$$

where $\delta>0$ will be taken sufficiently small. Then we shall now prove the basic Carleman type estimates, which was proven by Goorjian [2], when the operator is homogeneous, by using the weight function in the exponents

$$\varphi_\delta=(x_1-\delta)^2+\delta^2(x_2^2+\cdots+x_n^2)$$

instead of (2. 1).

THEOREM 3. *If, for an operator $P(D)$ of order m with constant coefficients, the principal part $P_m(D)$ is of real (resp. complex) characteristics of multiplicity at most s (resp. r), then there exist positive constants $\varepsilon_0, \delta_0, M_0$ and C_0 such that the inequality*

$$(2.2) \qquad \int \delta^p(\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|}|D^\alpha u|^2\exp(2\tau\varphi_\delta)dx \leqq C_0\int |P(D)u|^2 \exp(2\tau\varphi_\delta)dx$$

holds for $u\in C_c^\infty(U_\delta)$, α, δ and τ with $|\alpha|\leqq m-s$, $0<\delta<\delta_0$, $M_0<\tau\delta^{3+s}$ where $U_\delta=\{x\in R^n;\ |x|<\varepsilon_0\delta\}$ and $p=s$ if $s=r$, or $p=0$ if $s<r$.

PROOF. At first, we use algebraic properties of the characteristic polynominal $P_m(\xi)$ to prove a key inequality, that is : there is an open cone V in R^n, containing the point $N_0=(-1, 0, \cdots, 0)$, with the vertex at the origin such that the inequality

$$(2.3) \qquad C_1|\xi+i\tau N|^{2(m-s)} \leqq \sum_{|\alpha|=s}|P^{(\alpha)}(\xi+i\tau N)|^2 + \sum_{k=s+1}^{r}(\tau|N|)^{2(k-s)}|P^{(k)}(\xi+i\tau N)|^2$$

holds for $(\xi, \tau)\in R^{n+1}$ and $N\in V$ satisfying

$$(2.4) \qquad M_1\leqq|\tau N|$$

with some positive constants C_1 and M_1. If $s=r$, then the second sum does not appear in (2. 3). It is sufficient to prove the inequality (2. 3) for P_m instead of P. In virtue of the continuity and homogeneity of $P_m(\xi)$, it is easy to show that (2. 3) holds for $P=P_m$, $N\in U$ and $|\xi+i\tau N|=1$, where U is an open neighbourhood on the unit sphere $|x|=1$ and contains N_0. For $N\in U$ and a non-zero real vector (ξ, τ), by setting

$$\xi'+i\tau' N=(\xi+i\tau N)/|\xi+i\tau N|$$

we have (2. 3) holds for $P=P_m$, $N\in U$ and (ξ, τ). Now set $V=\{\varepsilon N;\ \varepsilon>0,\ N\in U\}$. This completes the proof of (2. 3). Using the properties of this cone V, we define neighbourhood U_δ, $0<\delta\leqq 1$, in the following. Set $W_\delta=\{x\in R^n;\ \mathrm{grad}\varphi_\delta(x)\in V,\ |\mathrm{grad}\varphi_\delta(x)-\mathrm{grad}\varphi_\delta(0)|<\delta\}$. Since V is open and contains the points $\mathrm{grad}\varphi_\delta(0)=2\delta N_0$, $\delta>0$, the set W_δ is a neighbourhood of 0. Notice that $(x_1, x_2, \cdots, x_n)\in W_1$, if and only if $(\delta x_1, x_2, \cdots, x_n)\in W_\delta$. Hence W_δ contains $\{x;\ |x|<2\varepsilon_0\delta\}$ if W_1 contains $\{x; |x|<2\varepsilon_0\}$. Set $U_\delta=\{x; |x|<\varepsilon_0\delta\}$.

Now we shall use the partition of unity given by the functions

$$\Theta(x_1(\tau)^{1/2}-g_1, x_2(\tau\delta)^{1/2}-g_2, \cdots, x_n(\tau\delta)^{1/2}-g_n)$$

with $\delta, \tau > 0$, where Θ is a C^∞ function with the support contained in the cube $\max|x_k| < 1$, such that for each $x \in R^n$,

$$\sum_g \Theta(x-g) = 1$$

where $g = (g_1, g_2, \cdots, g_n)$ runs over all the points with integer coordinates. Hence, for $u \in C_c^\infty(U_\delta)$, we have $u = \sum_g u_g$ where

$$u_g(x) = u(x) \cdot \Theta(x_1(\tau)^{1/2}-g_1, x_2(\tau\delta)^{1/2}-g_2, \cdots, x_n(\tau\delta)^{1/2}-g_n).$$

When we set

$$N_g = \operatorname{grad} \varphi_\delta(x_g),\ x_g = (g_1/(\tau)^{1/2}, g_2/(\tau\delta)^{1/2}, \cdots, g_n/(\tau\delta)^{1/2}),$$

we have, if $u_g(x) \neq 0$, then $x_g \in W_\delta$, $N_g \in V$ and

$$\delta \leqq |N_g| \leqq 3\delta$$

whenever $n\varepsilon_0^{-2} < \tau\delta^3$, $0 < \delta < 1$. Let $\tau\delta^3$ (resp. δ) be a sufficiently large (resp. small) positive number. Integrating the inequality which is obtained by multiplying (2. 3) by $|\hat{u}_g(\xi + i\tau N_g)|^2$, we have

$$\begin{aligned} C_1 \int |\xi + i\tau N_g|^{2(m-s)} |\hat{u}_g(\xi + i\tau N_g)|^2 d\xi \\ \leqq \int \Bigg\{ \sum_{|\alpha|=s} |P^{(\alpha)}(\xi + i\tau N_g)|^2 + \sum_{k=s+1}^{r} (\tau|N_g|)^{2(k-s)} \\ \times |P^{(k)}(\xi + i\tau N_g)|^2 \Bigg\} |\hat{u}_g(\xi + i\tau N_g)|^2 d\xi, \end{aligned} \tag{2. 5}$$

where $\hat{u}_g(\xi + i\tau N_g)$ is the Fourier transform of $u_g(x)\exp(< x, \tau N_g >)$. Using Parseval's formula, from (2. 5) we get

$$\begin{aligned} C \int |D^{m-s} u_g|^2 \exp(2\tau < x, N_g >) dx \\ \leqq \int \Bigg\{ \sum_{|\alpha|=s} |P^{(\alpha)}(D) u_g|^2 + \sum_{k=s+1}^{r} (\tau|N_g|)^{2(k-s)} \\ \times |P^{(k)}(D) u_g|^2 \Bigg\} \exp(2\tau < x, N_g >) dx, \end{aligned} \tag{2. 6}$$

where we use the notation

$$|D^{m-s}u_g|^2 = \sum_{|\beta|=m-s} |D^\beta u_g|^2$$

and where, henceforth, C is a generic positive constant depending on n, m, P and Θ, but independent on τ, δ and u. Multiplying (2. 6) by $\exp(2\tau\{\varphi_\delta(x_g) - <x_g, N_g>\})$, we have

$$C\int |D^{m-s}u_g|^2 \exp(2\tau\psi)dx \tag{2.7}$$
$$\leqq \int\left\{\sum_{|\alpha|=s} |P^{(\alpha)}(D)u_g|^2 + \sum_{k=s+1}^{r} (\tau|N_g|)^{2(k-s)} \times |P^{(k)}(D)u_g|^2\right\}\exp(2\tau\psi)dx,$$

where $\psi = \varphi_\delta(x_g) + <x - x_g, N_g>$. By Taylor's formula, we see

$$\psi = \varphi_\delta(x) - (x_1 - x_{g,1})^2 - \delta\sum_{k=2}^{n} (x_k - x_{g,k})^2 .$$

Since, in the support of u_g, $|x_1 - x_{g,1}|^2 < \tau^{-1}$, $|x_k - x_{g,k}|^2 < (\tau\delta)^{-1}$, $k \geqq 2$ and $|N_g| < 3\delta$, we obtain

$$C\int |D^{m-s}u_g|^2 \exp(2\tau\varphi_\delta)\, dx \tag{2.8}$$
$$\leqq \int\left\{\sum_{|\alpha|=s} |P^{(\alpha)}(D)u_g|^2 + \sum_{k=s+1}^{r} (\tau\delta)^{2(k-s)} |P^{(k)}(D)u_g|^2\right\}\exp(2\tau\varphi_\delta)dx .$$

On the other hand, from Cauchy's inequality, we have

$$|D^{m-s}u|^2 \leqq 2^n \sum_g |D^{m-s}u_g|^2 . \tag{2.9}$$

Let α^* denote the multi-index obtained by setting the first components of α equal to zero, that is, $\alpha^* = (0, \alpha_2, \cdots, \alpha_n)$, $k^* = (0, 0, \cdots, 0)$ if $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and k is a single index. By Leibniz' formula, we see

$$P^{(\alpha)}(D)u_g = \sum_\beta \tau^{|\beta|/2}\delta^{|\beta^*|/2} \frac{D^\beta\Theta}{\beta!} P^{(\alpha+\beta)}(D)u . \tag{2.10}$$

Consequently we have

$$C\int |D^{m-s}u|^2\exp(2\tau\varphi_\delta)dx \tag{2.11}$$

$$\leqq \int \sum_\beta \tau^{|\beta|}\delta^{|\beta^*|}\Big\{\sum_{|\alpha|=s} |P^{(\alpha+\beta)}(D)u|^2$$

$$+\sum_{k=s+1}^{r} (\tau\delta)^{2(k-s)}|P^{(k+\beta)}(D)u|^2\Big\}\exp(2\tau\varphi_\delta)dx\,.$$

Next, we need the following fundamental inequality proven by Trèves [7]

$$C(n,m)\int \tau^{|\alpha|}\delta^{|\alpha^*|}|Q^{(\alpha)}(D)v|^2\exp(2\tau\varphi_\delta)dx \tag{2.12}$$

$$\leqq \int |Q(D)v|^2\exp(2\tau\varphi_\delta)dx$$

for $v\in C_c^\infty$, α with $|\alpha|\leqq m$ and for the linear differential operator $Q(D)$ of order m with constant coefficients, where $C(n,m)$ is a positive constant depending only on n and m.

Applying this inequality to (2.11), we have if $s=r$,

$$C\int |D^{m-s}u|^2\exp(2\tau\varphi_\delta)dx \leqq (\tau\delta)^{-s}\int |P(D)u|^2\exp(2\tau\varphi_\delta)dx$$

and, if $s<r$,

$$C\int |D^{m-s}u|^2\exp(2\tau\varphi_\delta)dx \leqq \tau^{-s}\{\delta^{-s}+(\tau\delta^2)^{r-s}\}\int |P(D)u|^2\exp(2\tau\varphi_\delta)dx\,.$$

Without loss of generality, we assume that $0<\varepsilon_0<1/2$. Then we can use the inequality

$$(\tau\delta)^2\int |v|^2\exp(2\tau\varphi_\delta)dx \leqq \int |D_1v|^2\exp(2\tau\varphi_\delta)dx \tag{2.13}$$

for $v\in C_c^\infty(U_\delta)$, since, by setting $w=v\cdot\exp(\tau\varphi_\delta)$,

$$\int |\partial v/\partial x_1|^2 \exp(2\tau\varphi_\delta)dx = \int |\partial w/\partial x_1 - 2\tau(x_1-\delta)w|^2 \, dx$$

$$\geqq 4\tau^2(x_1-\delta)^2 \int |w|^2 dx \geqq (\tau\delta)^2 \int |v|^2 \exp(2\tau\varphi_\delta)dx .$$

Hence we have the inequality (2.2) for a sufficiently large $\tau\delta^{3+s}>0$ and a small $\delta>0$. This completes the proof.

Next we shall prove the similar estimates to (2.2) for the operator $P(D)+Q(x;D)$.

THEOREM 4. *Let $P(D)$, $Q(x;D)$ be the operators given in Theorem 1. Assume that r is an odd integer $\geqq 3$ and that $[r/2]\geqq s$. Then there exist positive constants $\varepsilon_1, \delta_1, M_1$ and C_1 such that the inequality*

$$\int (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} |D^\alpha u|^2 \exp(2\tau\varphi_\delta)dx \tag{2.14}$$
$$\leqq C_1 \int |P(D)u + Q(x;D)u|^2 \exp(2\tau\varphi_\delta)dx$$

holds for $u\in C_c^\infty(U_\delta)$, α, τ, δ with $|\alpha|\leqq m-[r/2]$, $0<\delta<\delta_1$, $M_1<\tau\delta^3$ where $U_\delta=\{x;\ |x|<\varepsilon_1\delta\}$.

PROOF. Let $r=2r_0+1$. By the continuity of the coefficients of $Q(x;D)$, Theorem 3 and the inequality (2.12) for operator $P(D)+Q(y;D)$ may be applied and we have

$$\int \sum_{|\alpha|\leqq m-r_0} (\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|} |D^\alpha u|^2 \exp(2\tau\varphi_\delta)dx \tag{2.15}$$
$$+\int \sum_{1\leqq|\beta|} \tau^{|\beta|}\delta^{|\beta^*|} |P^{(\beta)}(D)u + Q^{(\beta)}(y;D)u|^2 \exp(2\tau\varphi_\delta)dx$$
$$\leqq C\int |P(D)u+Q(y;D)u|^2 \exp(2\tau\varphi_\delta)dx$$

for $u\in C_c^\infty(U_\delta)$, a sufficiently large $\tau\delta^3>1$, a small $\delta>0$ and for y belonging to a neighbourhood of 0, with some positive constants ε_1 and C. Now we use the partition of unity given by the functions $\Theta((\tau\delta^2)^{1/2}x-g)$, where Θ is the function given in the proof of Theorem 3. For $u\in C_c^\infty(U_\delta)$ and g, put

$$u_g(x) = u(x)\cdot\Theta((\tau\delta^2)^{1/2}x-g),\ x_g = g/(\tau\delta^2)^{1/2} .$$

Since, in the support of u_g, $|x-x_g|^2<(\tau\delta^2)^{-1}$ and the coefficients of $Q(x;D)$ are Lipschitz continuous, we see

$$\int |\{Q^{(\beta)}(x_g;D)-Q^{(\beta)}(x;D)\}u_g|^2\exp(2\tau\varphi_\delta)dx$$
$$\leqq C\int(\tau\delta^2)^{-1}|D^{m-r_0-|\beta|}u_g|^2\exp(2\tau\varphi_\delta)dx.$$

Hence, replacing y by x_g and u by u_g in (2.15), we have

$$\int\sum_{|\alpha|\leqq m-r_0}(\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|}|D^\gamma u_g|^2\exp(2\tau\varphi_\delta)dx \tag{2.16}$$
$$+\int\sum_{1\leqq|\beta|}\tau^{|\beta|}\delta^{|\beta^*|}|P^{(\beta)}(D)u_g+Q^{(\beta)}(x;D)u_g|^2\exp(2\tau\varphi_\delta)dx$$
$$\leqq C\int|P(D)u_g+Q(x;D)u_g|^2\exp(2\tau\varphi_\delta)dx$$

with some another constant C if we take

$$\tau^{r_0}(\tau\delta^2)^{r_0-r}\gg(\tau\delta^2)^{-1}$$

and for $|\beta|\geqq 1$

$$\tau^{r_0+|\beta|}(\tau\delta^2)^{r_0+|\beta|-r}\gg\tau^{|\beta|-1}\delta^{-2}$$

thus, if we take a sufficiently small $\delta>0$ and a large $\tau\delta^3>0$. Summing these inequalities (2.16) over g, we obtain

$$\int\sum_{|\alpha|\leqq m-r_0}(\tau\delta^2)^{m-|\alpha|-r}\tau^{m-|\alpha|}|D^\alpha u|^2\exp(2\tau\varphi_\delta)dx$$
$$+\int\sum_{1\leqq|\beta|}\tau^{|\beta|}\delta^{|\beta^*|}|P^{(\beta)}(D)u+Q^{(\beta)}(x;D)u|^2\exp(2\tau\varphi_\delta)dx$$
$$\leqq C\int\sum_\beta(\tau\delta^2)^{|\beta|}|P^{(\beta)}(D)u+Q^{(\beta)}(x;D)u|^2\exp(2\tau\varphi_\delta)dx.$$

Thus we can get the inequality (2.14) for a sufficiently large $\tau\delta^3>0$ and a small $\delta>0$. This completes the proof.

Now we shall prove the following lemma, in order to obtain estimates of operators with variable coefficients.

LEMMA 1. *Let $A(D)$ be a homogeneous elliptic operator of order m with constant coefficients and with simple characteristics and let $B(D)$ be an operator of order $3m-1$ with constant coefficients. Set $P(D)=A(D)^3+B(D)$. Then the inequality*

$$\int (\tau\delta^2)^{-1}|D^m A(D)^2 u|^2 \exp(2\tau\varphi_\delta)\, dx \leqq C_2 \int |P(D)u|^2 \exp(2\tau\varphi_\delta)\, dx \tag{2.17}$$

holds for $u \in C_c^\infty(U_\delta)$, τ, δ with $0<\delta<\delta_2$, $M_2<\tau\delta^3$ where δ_2, M_2 and C_2 are positive constants and where $U_\delta=\{x;\, |x|<\varepsilon_2\delta\}$ with some $\varepsilon_2>0$,

PROOF. Let V be an open cone in R^n, containing the point $N_0=(-1, 0, \cdots, 0)$, with the vertex at 0 such that the inequalities

$$C|\xi+i\tau N|^{2(m-1)} \leqq \sum_{|\alpha|=1} |A^{(\alpha)}(\xi+i\tau N)|^2$$

and

$$C|\xi+i\tau N|^{2m} \leqq |A(\xi+i\tau N)|^2 + |\tau N|^2 |A^{(1)}(\xi+i\tau N)|^2$$

holds for $(\xi,\tau)\in R^{n+1}$ and $N\in V$, with some constant $C>0$. Then we have

$$\begin{aligned} C|\xi+i\tau N|^{2m}|A(\xi+i\tau N)|^4 &\leqq |P(\xi+i\tau N)|^2 + |\tau N|^2 |P^{(1)}(\xi+i\tau N)|^2 \\ &+ \sum_{|\alpha|=1} |A^{(\alpha)}(\xi+i\tau N)|^2 \{|A(\xi+i\tau N)|^2 + |\tau N|^2 |A^{(1)}(\xi+i\tau N)|^2\}^2, \end{aligned}$$

with another constant $C>0$. Using the same method as in the proof of Theorem 3, we have

$$\begin{aligned} &C\int |D^m A(D)^2 u|^2 \exp(2\tau\varphi_\delta) dx \\ &\leqq \int \Big\{ |P(D)u|^2 + (\tau\delta)^2 |P^{(1)}(D)u|^2 + \sum_{|\alpha|=1} \{|A^{(\alpha)}(D)A(D)^2 u|^2 \\ &\quad + (\tau\delta)^4 |A^{(\alpha)}(D)A^{(1)}(D)^2 u|^2\} \Big\} \exp(2\tau\varphi_\delta) dx. \end{aligned}$$

Consequently using the inequalities (2.12) and (2.13), we obtain the inequality (2.17). This completes the proof.

3. Carleman type estimates for certain elliptic operators with variable coefficients. In this section, we shall give the similar estimates to Theorem 3 for the operator $P(x;D)=A(x;D)^3+B(x;D)$, where $A(x;D)$ is a homogeneous elliptic operator of order m with C^{2m+1} coefficients such that $A(0;D)$ is of simple characteristics and where $B(x;D)$ is a homogeneous operator of order $3m-1$ with Lipschitz continuous coefficients. Then we shall give the proofs of Theorems 1, 2 and Corollary 2. The following lemma was proven by Hörmander [3].

LEMMA 2. *The inequality*

$$\int (\tau\delta^2)^{m-|\alpha|-1}\tau^{m-|\alpha|}|D^\alpha u|^2\exp(2\tau\varphi_\delta)dx \leqq C_3\int |A(x;D)u|^2\exp(2\tau\varphi_\delta)dx \tag{3.1}$$

holds for $u\in C_c^\infty(U_\delta)$, α, τ, δ *with* $|\alpha|\leqq m$, $0<\delta<\delta_3$, $M_3<\tau\delta^3$ *where* $U_\delta=\{x;\ |x|<\varepsilon_3\delta\}$ *with some* $\varepsilon_3>0$.

Then, using this Lemma and Lemma 1, we have

THEOREM 5. *There exist positive constants* $\varepsilon_4, \delta_4, M_4$ *and* C_4 *such that the inequality*

$$\int (\tau\delta^2)^{3m-|\alpha|-3}\tau^{3m-|\alpha|}|D^\alpha u|^2\exp(2\tau\varphi_\delta)dx \leqq C_4\int |P(x;D)u|^2\exp(2\tau\varphi_\delta)dx \tag{3.2}$$

holds for $u\in C_c^\infty(U_\delta)$, α, τ, δ *with* $|\alpha|\leqq 3m$, $0<\delta<\delta_4$, $M_4<\tau\delta^3$ *where* $U_\delta=\{x;\ |x|<\varepsilon_4\delta\}$.

PROOF. In virtue of the continuity of the coefficients of $A(x;D)$ and $B(x;D)$, applying Theorem 3, Lemma 1 and the inequality (2.13) to the operator $A(y;D)^3+B(y;D)$, we have

$$\begin{aligned}(3.3)\qquad &\int \sum_{|\alpha|\leqq 3m}(\tau\delta^2)^{3m-|\alpha|-3}\tau^{3m-|\alpha|}|D^\alpha u|^2\exp(2\tau\varphi_\delta)dx\\ &\quad+\int(\tau\delta^2)^{-1}\sum_{|\alpha|\leqq 1}\tau^{|\alpha|}|A^{(\alpha)}(y;D)A(y;D)D^m u|^2\exp(2\tau\varphi_\delta)dx\\ &\quad+\int\sum_{1\leqq|\alpha|}\tau^{|\alpha|}\delta^{|\alpha^*|}|A_3^{(\alpha)}(y;D)u+B^{(\alpha)}(y;D)u|^2\exp(2\tau\varphi_\delta)dx\\ &\leqq C\int |A(y;D)^3u+B(y;D)u|^2\exp(2\tau\varphi_\delta)dx\end{aligned}$$

for $u\in C_c^\infty(U_\delta)$, a sufficiently large $\tau\delta^3>1$, a small $\delta>0$ and for y belonging to a neighbourhood of 0 with some positive constants ε_4 and C where $A_3(y; D)=A(y; D)^3$. Now we use the partition of unity given by the functions $\Theta(w(\tau,\delta)^{1/2}x-g)$ where Θ is the function given in the proof of Theorem 3 and where $w(\tau,\delta)=\tau\delta^{3/2}$. If we set, for $u\in C_c^\infty(U_\delta)$ and g,

$$u_g(x)=u(x)\cdot\Theta(w(\tau,\delta)^{1/2}x-g),\ x_g=g/w(\tau,\delta)^{1/2}$$

then we have, for a sufficiently large $\tau\delta^3>0$ and a small $\delta>0$,

$$
\begin{aligned}
(3.4)\qquad \sum_{j=1}^{3}I_j &\equiv \int\sum_{|\alpha|\leqq 3m}(\tau\delta^2)^{3m-|\alpha|-3}\tau^{3m-|\alpha|}|D^\alpha u_g|^2\exp(2\tau\varphi_\delta)dx\\
&+\int(\tau\delta^2)^{-1}\sum_{|\alpha|\leqq 1}\tau^{|\alpha|}|A^{(\alpha)}(x; D)A(x; D)D^m u_g|^2\exp(2\tau\varphi_\delta)dx\\
&+\int\sum_{1\leqq|\alpha|}\tau^{|\alpha|}\delta^{|\alpha^*|}|P^{(\alpha)}(x; D)u_g|^2\exp(2\tau\varphi_\delta)dx\\
&\leqq C\int\Big\{|P(x; D)u_g|^2+|\{A(x; D)^3-A(x_g; D)^3\}u_g|^2\\
&+(\tau\delta^2)^{-1}|\{A(x; D)^2-A(x_g; D)^2\}D^m u_g|^2\\
&+\sum_{|\beta|=1}\tau\delta^{|\beta^*|}|\{P^{(\beta)}(x; D)-P^{(\beta)}(x_g; D)\}u_g|^2\Big\}\exp(2\tau\varphi_\delta)dx\\
&\equiv\sum_{j=4}^{7}I_j
\end{aligned}
$$

where, henceforth, C is a positive constant depending on n, m, $A(x; D)$, $B(x; D)$ and Θ, but independent on τ,δ and u. We shall give the proof of the inequality (3.4) from (3.3). Since, in the support of u_g, $|x-x_g|^2<w(\tau,\delta)^{-1}$ and the coefficients of $B(x; D)$ are Lipschitz continuous, we have

$$
\begin{aligned}
&\int|\{B(x; D)-B(x_g; D)\}u_g|^2\exp(2\tau\varphi_\delta)dx\\
&\qquad\leqq C\int w(\tau,\delta)^{-1}|D^{3m-1}u_g|^2\exp(2\tau\varphi_\delta)dx\,,
\end{aligned}
$$

and since, for $|\alpha|=1$,

$$\begin{aligned}
&A^{(\alpha)}(x_g; D)A(x_g; D) - A^{(\alpha)}(x; D)A(x; D) \\
&\quad = \{A^{(\alpha)}(x_g; D) - A^{(\alpha)}(x; D)\}\{A(x_g; D) - A(x; D)\} \\
&\qquad + \{A^{(\alpha)}(x_g; D) - A^{(\alpha)}(x; D)\}A(x; D) \\
&\qquad + \{A(x_g; D) - A(x; D)\}A^{(\alpha)}(x; D) \\
&\qquad + \text{lower order term of order} \leqq 2m-2,
\end{aligned}$$

we have

$$\begin{aligned}
&\int \delta^{-2}|\{A^{(\alpha)}(x_g; D)A(x_g; D) - A^{(\alpha)}(x; D)A(x; D)\}D^m u_g|^2 \exp(2\tau\varphi_\delta)dx \\
&\leqq C\delta^{-2}\int \Big\{ w(\tau,\delta)^{-2}|D^{3m-1}u_g|^2 + w(\tau,\delta)^{-1}\{|A(x; D)D^{2m-1}u_g|^2 \\
&\quad + |A^{(\alpha)}(x; D)D^{2m}u_g|^2\} + |D^{3m-2}u_g|^2 \Big\} \exp(2\tau\varphi_\delta)dx .
\end{aligned}$$

And since, for $|\alpha| \geqq 2$,

$$\begin{aligned}
A_3^{(\alpha)}(x_g; D) + B^{(\alpha)}(x_g; D) - P^{(\alpha)}(x; D) &= P^{(\alpha)}(x_g; D) - P^{(\alpha)}(x; D) \\
&\quad + \text{lower order term of order} \leqq 3m - |\alpha| - 1
\end{aligned}$$

and the coefficients of $P(x; D)$ are Lipschitz continuous, we have

$$\begin{aligned}
&\int \tau^{|\alpha|}\delta^{|\alpha^*|}|\{A_3^{(\alpha)}(x_g; D) + B^{(\alpha)}(x_g; D) - P^{(\alpha)}(x; D)\}u_g|^2 \exp(2\tau\varphi_\delta)dx \\
&\leqq C\int \tau^{|\alpha|}\delta^{|\alpha^*|}\{w(\tau,\delta)^{-1}|D^{3m-|\alpha|}u_g|^2 + |D^{3m-|\alpha|-1}u_g|^2\}\exp(2\tau\varphi_\delta)dx .
\end{aligned}$$

Hence, replacing y by x_g and u by u_g in (3. 3), and then using the inequality (3. 1), we have the inequality (3. 4), if we take

$$(\tau\delta^2)^{-2}\tau \gg (\delta^2 w(\tau,\delta))^{-1}, \qquad (\tau\delta^2)^{-1}\tau^2 \gg \delta^{-2},$$

and for $|\alpha| \geqq 2$,

$$(\tau\delta^2)^{|\alpha|-3}\tau^{|\alpha|} \gg \tau^{|\alpha|}w(\tau,\delta)^{-1}, \qquad (\tau\delta^2)^{|\alpha|-2}\tau^{|\alpha|+1} \gg \tau^{|\alpha|}$$

thus, if we take a sufficiently large $\tau\delta^3 > 0$ and small $\delta > 0$.

On the other hand, we can also write the last three terms on the right hand side of (3. 4) as follows.

$$A(x;D)^2-A(x_0;D)^2=-\{A(x;D)-A(x_0;D)\}^2+2\{A(x;D)-A(x_0;D)\}A(x;D)$$
$$+\text{ lower order term of order}\leqq 2m-1$$

and

$$\begin{aligned}A(x;D)^3-A(x_0;D)^3=&\{A(x;D)-A(x_0;D)\}^3-3\{A(x;D)-A(x_0;D)\}^2A(x;D)\\&+3\{A(x;D)-A(x_0;D)\}A(x;D)^2\\&+\{\text{a differential operator of order }2m-1\}A(x;D)\\&+\{A(x;D)-A(x_0;D)\}\{\text{a differential operator of order }2m-1\}\\&+\text{lower order term of order}\leqq 3m-2\end{aligned}$$

and for $|\beta|=1$,

$$\begin{aligned}P^{(\beta)}(x;D)-P^{(\beta)}(x_0;D)&=3\{A^{(\beta)}(x;D)A(x;D)^2-A^{(\beta)}(x_0;D)A(x_0;D)^2\}\\&\quad+\text{lower order term of order}\leqq 3m-2\\&=3\{A(x;D)-A(x_0;D)\}^2\{A^{(\beta)}(x;D)-A^{(\beta)}(x_0;D)\}\\&\quad-3\{A(x;D)-A(x_0;D)\}^2A^{(\beta)}(x;D)\\&\quad-6\{A(x;D)-A(x_0;D)\}\{A^{(\beta)}(x;D)-A^{(\beta)}(x_0;D)\}A(x;D)\\&\quad+6\{A(x;D)-A(x_0;D)\}A^{(\beta)}(x;D)A(x;D)\\&\quad+3\{A^{(\beta)}(x;D)-A^{(\beta)}(x_0;D)\}A(x;D)^2\\&\quad+\text{lower order term of order}\leqq 3m-2.\end{aligned}$$

Hence we have

$$\begin{aligned}&\int\sum_{|\alpha|\leqq 3m}(\tau\delta^2)^{3m-|\alpha|-3}\tau^{3m-|\alpha|}|D^\alpha u_0|^2\exp(2\tau\varphi_\delta)dx\\&\quad+\int\sum_{1\leqq|\alpha|}\tau^{|\alpha|}\delta^{|\alpha^*|}|P^{(\alpha)}(x;D)u_0|^2\exp(2\tau\varphi_\delta)dx\\&\leqq C\int|P(x;D)u_0|^2\exp(2\tau\varphi_\delta)dx\end{aligned}\tag{3.5}$$

for a sufficiently large $\tau\delta^3>0$ and a small $\delta>0$. Here we give the proof of the inequality (3.5) by (3.4). Using the above three identities, we have

$$I_5 \leqq C\int \Big\{ w(\tau,\delta)^{-3}|D^{3m}u_g|^2 + w(\tau,\delta)^{-2}|A(x;D)D^{2m}u_g|^2$$
$$+ w(\tau,\delta)^{-1}\{|A(x;D)^2 D^m u_g|^2 + |D^{3m-1}u_g|^2\}$$
$$+ |A(x;D)D^{2m-1}u_g|^2 + |D^{3m-2}u_g|^2 \Big\} \exp(2\tau\varphi_\delta)dx\,,$$

$$I_6 \leqq C(\tau\delta^2)^{-1}\int \{w(\tau,\delta)^{-2}|D^{3m}u_g|^2$$
$$+ w(\tau,\delta)^{-1}|A(x;D)D^{2m}u_g|^2 + |D^{3m-1}u_g|^2\}\exp(2\tau\varphi_\delta)dx$$

and

$$I_7 \leqq C\tau\int \{w(\tau,\delta)^{-3}|D^{3m-1}u_g|^2 + w(\tau,\delta)^{-2}|A(x;D)D^{2m-1}u_g|^2$$
$$+ w(\tau,\delta)^{-1}|A(x;D)^2 D^{m-1}u_g|^2 + |D^{3m-2}u_g|^2\}\exp(2\tau\varphi_\delta)dx$$
$$+ C\sum_{|\beta|=1} \tau\delta^{|\beta^*|}\int \{w(\tau,\delta)^{-2}|A^{(\beta)}(x;D)D^{2m}u_g|^2$$
$$+ w(\tau,\delta)^{-1}|A^{(\beta)}(x;D)A(x;D)D^m u_g|^2\}\exp(2\tau\varphi_\delta)dx\,.$$

Hence we have

$$I_5 + I_6 + I_7 \leqq (I_1 + I_2)/2\,,$$

for a sufficiently large $\tau\delta^3 > 0$ and a small $\delta > 0$. This completes the proof of the inequality (3. 5). Thus, summing these inequalities (3. 5) over g, we have

$$\int \sum_{|\alpha|\leqq 3m} (\tau\delta^2)^{3m-|\alpha|-3}\tau^{3m-|\alpha|}|D^\alpha u|^2 \exp(2\tau\varphi_\delta)dx$$
$$+ \int \sum_{1\leqq|\alpha|} \tau^{|\alpha|}\delta^{|\alpha^*|}|P^{(\alpha)}(x;D)u|^2\exp(2\tau\varphi_\delta)dx$$
$$\leqq C\int \sum_\beta w(\tau,\delta)^{|\beta|}|P^{(\beta)}(x;D)u|^2\exp(2\tau\varphi_\delta)dx\,.$$

Consequently we have the inequality (3. 2) for a sufficiently large $\tau\delta^3 > 0$ and a small $\delta > 0$. This completes the proof.

Now we are ready to prove Theorems 1, 2 and Corollary 2.

THE PROOF OF THEOREM 1. Let U_δ be a neighbourhood of such that (2. 14) holds and U_δ is contained in the neighbourhood initially given in this Theorem.

Take a function $\chi \in C_c^\infty(U_\delta)$ such that $\chi = 1$ in a neighbourhood $U_{\delta,1}$, of 0, and set $v = \chi u$. Then we have in $U_{\delta,1}$

$$|P(D)v + Q(x; D)v| \leqq C \sum_{|\alpha| \leqq m-q} |D^\alpha v| \tag{3.6}$$

where $q = [(r+1)/2]$. Since, in the support of v, $x_2^2 + \cdots + x_n^2 \leqq x_1$, and $x_1 < \delta$ in $U_{\delta,1}$, we have $\varphi_\delta(x) < \varphi_\delta(0)$ in the support of v except when $x=0$, Hence we have for some positive constant L_δ

$$\varphi_\delta(x) \leqq \varphi_\delta(0) - L_\delta\,, \tag{3.7}$$

when x is in the support of v and in the complement $CU_{\delta,1}$ of $U_{\delta,1}$. By approximation, we can apply (2.14) to v. This gives, by using (3.6),

$$\begin{aligned} \int \sum_{|\alpha| \leqq m-q} (\tau\delta^2)^{m-|\alpha|-r} \tau^{m-|\alpha|} |D^\alpha v|^2 \exp(2\tau\varphi_\delta) dx \\ \leqq C \int_{U_{\delta,1}} \sum_{|\alpha| \leqq m-q} |D^\alpha v|^2 \exp(2\tau\varphi_\delta) dx \\ + C \int_{CU_{\delta,1}} |P(D)v + Q(x; D)v|^2 \exp(2\tau\varphi_\delta) dx\,, \end{aligned}$$

or, by restricting the integration in the left hand side to $U_{\delta,1}$, it gives

$$\begin{aligned} \int_{U_{\delta,1}} \{\tau\delta^{-2} - C\} \sum_{|\alpha| \leqq m-q} |D^\alpha v|^2 \exp(2\tau\varphi_\delta) dx \\ \leqq C \int_{CU_{\delta,1}} |P(D)v + Q(x; D)v|^2 \exp(2\tau\varphi_\delta) dx\,. \end{aligned}$$

Let $U_{\delta,2} \subset U_{\delta,1}$ be a neighbourhood of 0 where $\varphi_\delta(x) \geqq \varphi_\delta(0) - L_\delta/2$. Choose $\tau\delta^{-2} > 2C$ and then fix $\delta > 0$. Then we have, using (3.7),

$$\int_{U_{\delta,2}} |v|^2 dx \leqq \exp(-\tau L_\delta) \int_{CU_{\delta,1}} |P(D)v + Q(x; D)v|^2 \, dx$$

and, letting $\tau \to +\infty$, we have $v = u = 0$ in $U_{\delta,2}$. This completes the proof.

The proof of Theorem 2 is exactly the same as the proof of Theorem 1. We have only to use the estimates (3.2) instead of (2.14).

THE PROOF OF COROLLARY 2. We denote by Ω_0 the set of points x in Ω such that u vanishes in some neighbourhood at x. By the assumption and the definition, Ω_0 is the non-empty open set. Give any points x_0 in Ω_0 and x_1 in Ω. Since Ω is connected, there is a continuous arc $\gamma = \{x(t);\ 0 \leqq t \leqq 1\}$ in Ω such that $x(0) = x_0$, $x(1) = x_1$.

Take a positive number r_0 such that $B(x_0, r_0) \subset \Omega_0$, $B(x_0, r_0)$ being the open ball at the centre x_0 with the radius r_0, and then set

$$4r_1 = \min\ (r_0,\ \mathrm{dis}(\gamma, \partial\Omega)) > 0\,,$$

and

$$t_0 = \sup\ \{t\,;\, x(s) \in \Omega_1 \ \text{ for } \ 0 \leqq s \leqq t\},$$

where $\partial\Omega$ is the boundary of Ω and $\mathrm{dis}(\gamma, \partial\Omega)$ is the distance from γ to $\partial\Omega$, and where Ω_1 is the subset of Ω_0, consisting of points x such that u vanishes in $\Omega \cap B(x, r_1)$. Since $B(x_0, 2r_1) \subset \Omega_0$, we have $t_0 > 0$, and since γ is a continuous arc, for any points x in $B(x(t_0), r_1)$

$$|x - x(t_0 - \varepsilon)| < |x - x(t_0)| + |x(t_0) - x(t_0 - \varepsilon)| < r_1$$

if we take a sufficiently small $\varepsilon > 0$. This means $B(x(t_0), r_1)$ is contained in Ω_0 and $x(t_0) \in \Omega_1$.

Suppose that $t_0 < 1$. Since the surface $S(x(t_0), r_1) = \{x, |x - x(t_0)| = r_1\}$ is smooth, after a change of coordinates, applying Theorem 2, we have that $S(x(t_0), r_1)$ is contained in Ω_0. Hence $B(x(t_0), \varepsilon_0)$ is contained in Ω_1 for some $\varepsilon_0 > 0$. In virtue of the continuity of γ, we can get $x(s) \in \Omega_1$ for $0 \leqq s \leqq t_0 + \varepsilon_1$ with some $\varepsilon_1 > 0$. This gives a contradiction. Consequently we have $x_1 = x(1) \in \Omega_1$. This completes the proof.

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