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ON THE ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES

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1. Let Σa_n be a given infinite series with s_n as its n-th partial sum. We denote by $\{\sigma_n^\alpha\}$ and $\{t_n^\alpha\}$ the n-th (C, α), $(\alpha > -1)$ means of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. A series Σa_n is said to be summable $|C, \alpha|$ if $\Sigma|\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty$ and summable $|C, \alpha|_k$, $k \geqq 1$, $\alpha > -1$ if

$$\sum n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty . \tag{1.1}$$

In view of the well known identity $t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$, the condition (1.1) can also be written as

$$\sum_1^\infty \frac{|t_n^\alpha|^k}{n} < \infty . \tag{1.2}$$

Let $\{p_n\}$ be a sequence of positive real constants such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$. A series Σa_n is said to be summable $|\bar{N}, p_n|$ if $t_n^* \in BV$,

where
$$t_n^* = \frac{1}{P_n} \sum_{k=0}^n p_k s_k .$$

For $p_n = \dfrac{1}{n+1}$ the summability $|\bar{N}, p_n|$ is equivalent to the well known summability $|R, \log n, 1|$.

For any real α and integers $n \geqq 0$, we define $\overset{\alpha}{\triangle} U_n = \sum_{v=n}^\infty A_{v-n}^{-\alpha-1} U_v$, whenever the series is convergent.

2. It is known that summability $|\bar{N}, p_n|$ and the summability $|C, \alpha|_k$ are, in general, independent of each other. It is, therefore, natural to find out suitable summability factors $\{\varepsilon_n\}$ so that $\Sigma a_n \varepsilon_n$ may be summable $|C, \alpha|_k$, $\alpha > -1$, $k \geqq 1$, whenever Σa_n is summable $|\bar{N}, p_n|$, and conversely, if Σa_n is summable $|C, \alpha|_k$ then $\Sigma a_n \varepsilon_n$ may be summable $|\bar{N}, p_n|$. In a recent paper [5] the author has examined the summability factor problem of the first type. We propose to study the converse problem in the present note. In what follows we shall prove the following:

THEOREM. *The necessary and sufficient conditions for the series* $\Sigma a_n \varepsilon_n$ *to be summable* $|\bar{N}, p_n|$ *whenever* Σa_n *is summable* $|C, \alpha|_k$, $\alpha \geqq 0$, $k \geqq 1$, *are*

(i) $\left\{ n^{\alpha+1-\frac{1}{k'}} \overset{\alpha}{\triangle} \left(\frac{\varepsilon_n}{n} \right) \right\} \in l^{k'}, \frac{1}{k} + \frac{1}{k'} = 1,$

(ii)(a) $\left\{ n^{-\frac{1}{k'}} \varepsilon_n \right\} \in l^{k'}, \quad 0 \leqq \alpha \leqq 1,$

(ii)(b) $\left\{ n^{\alpha-\frac{1}{k'}} \left(\frac{p_n}{P_n} \right) \varepsilon_n \right\} \in l^{k'}, \quad \alpha > 1,$

where (a) $p_n = O(p_{n+1})$, (b) $(n+1)p_n = O(P_n)$ *and* (c) $P_n = O(n^\alpha p_n)$ $(\alpha > 1)$.

It may be remarked that our theorem includes, as a special case for $k = 1$, the following theorem of Mohapatra [8].

THEOREM A. *Let the sequence* $\{p_n\}$ *satisfy the following*:

(2.1) $$p_n = O(p_{n+1}),$$

(2.2) $$(n+1)p_n = O(P_n),$$

(2.3) $$P_n = O(p_n n^\alpha), \ \alpha > 1.$$

The necessary and sufficient conditions to be satisfied by a sequence $\{\varepsilon_n\}$ *such that* $\Sigma a_n \varepsilon_n$ *is summable* $|\bar{N}, p_n|$, *whenever* Σa_n *is summable* $|C, \alpha|$, $\alpha \geqq 0$ *are*

(2.4) $$\varepsilon_n = \begin{cases} O(1), \ 0 \leqq \alpha \leqq 1, \\ O\left(\frac{P_n n^{-\alpha}}{p_n} \right), \alpha > 1, \end{cases}$$

(2.5) $$\overset{\alpha}{\triangle} \left(\frac{\varepsilon_n}{n} \right) = O(n^{-\alpha-1}).$$

On the other hand if we take $p_n = 1$, we get the following result of Mehdi [6].

THEOREM B. *Let* $\alpha \geqq 0$, $k > 1$. *The necessary and sufficient conditions for* $\Sigma a_n \varepsilon_n$ *to be summable* $|C, 1|$ *whenever* Σa_n *is summable* $|C, \alpha|_k$ *are* (i) *and* (ii)′ (a) $\sum_{1}^{\infty} \frac{|\varepsilon_n|^{k'}}{n} < \infty, \quad \alpha \leqq 1,$

(ii)′ (b) $\sum_{1}^{\infty} n^{-1+\alpha k'-k'} |\varepsilon_n|^{k'} < \infty \quad \alpha \geqq 1.$

Similarly on taking $p_n = 1/n+1$, we deduce the following result concerning $|R, \log n, 1|$ summability factor of infinite series.

COROLLARY. *Let $\alpha \geqq 0$. The necessary and sufficient conditions for $\Sigma a_n \varepsilon_n$ to be summable $|R, \log n, 1|$ whenever Σa_n is summable $|C, \alpha|_{k'}$ $k \geqq 1$, are*

I $\quad \left\{ n^{\alpha+1-\frac{1}{k'}} \overset{\alpha}{\triangle} \left(\frac{\varepsilon_n}{n} \right) \right\} \in l^{k'}, \qquad \frac{1}{k} + \frac{1}{k'} = 1,$

II (a) $\quad \{ n^{-\frac{1}{k'}} \varepsilon_n \} \in l^{k'}, \qquad 0 \leqq \alpha \leqq 1,$

II (b) $\quad \{ n^{\alpha - \frac{1}{k'} - 1} (\log n)^{-1} \varepsilon_n \} \in l^{k'}, \qquad \alpha > 1.$

3. We require the following lemmas for the proof of our theorem.

LEMMA 1 [7]. *Let $p \geqq 1$, $k \geqq 1$ and suppose that x, y, u and v are related as:*

$$y_n = \sum_{m=0}^{\infty} C_{n,m} x_m, \qquad n \geqq 0,$$

$$v_m = \sum_{n=0}^{\infty} C_{n,m} u_n, \qquad m \geqq 0.$$

The necessary and sufficient condition for

(3.1) $\qquad y \in l^p$ *whenever* $x \in l^k$ *is*

(3.2) $\qquad v \in l^{k'}$ *whenever* $u \in l^{p'}$,

where k' and p' are the conjugate indices of k and p respectively.

LEMMA 2 [7]. *If $k > 1$ and $y_n = \sum_{m=0}^{n} C_{n,m} x_m$ and $\sum_{n=0}^{\infty} |y_n| < \infty$ whenever $\sum_{m=0}^{\infty} |x_m|^k < \infty$ then*

(3.3) $$\sum_{n=0}^{\infty} |C_{n,n}|^{k'} < \infty.$$

LEMMA 3[7]*). *If* $1 < k \leqq \infty$, *the necessary and sufficient condition for* $y_n = O(1)$ *whenever* $x_n \in l^k$ *is*

$$\sum_{m=0}^{\infty} |C_{n,m}|^{k'} < \infty ,$$

where x_n *and* y_n *are related as in Lemma* 1.

LEMMA 4. *Let* $k \geqq 1$. *If* $\Sigma a_n \varepsilon_n$ *is bounded* $(\overline{N}, p_n)$ *whenever* Σa_n *is summable* $|C, 0|_k$, *then*

$$\varepsilon_n = O(n^{1-\frac{1}{k}}).$$

PROOF. We write

$$y_n = \frac{1}{P_n} \sum_{r=0}^{n} (P_n - P_{r-1}) a_r \varepsilon_r$$

and

$$x_r = r^{1-\frac{1}{k}} a_r,\ r \geqq 1,\ x_0 = a_0 = 0.$$

Then

$$y_n = \frac{1}{P_n} \sum_{r=1}^{n} (P_n - P_{r-1}) x_r r^{\frac{1}{k}-1} \varepsilon_r = \sum_{r=1}^{\infty} b_{n,r} x_r ,$$

where

$$b_{n,r} = \frac{1}{P_n} \cdot (P_n - P_{r-1}) \varepsilon_r r^{\frac{1}{k}-1}, \qquad r \leqq n,$$
$$= 0, \qquad r > n.$$

By hypothesis $y_n = O(1)$ whenever $\Sigma |x_n|^k < \infty$. Then appealing to Lemma 3, a necessary and sufficient condition for the above is

$$\sum_{r=1}^{\infty} |b_{n,r}|^{k'} < \infty . \tag{3.4}$$

Now

*) This is given in Cooke's "Infinite Matrices and Sequence Spaces" with a superfluous hypothesis.

$$|y_n| \leqq \left(\sum_{r=1}^{\infty} |b_{n,r}|^{k'}\right)^{\frac{1}{k'}} \left(\sum_{r=1}^{\infty} |x_r|^k\right)^{\frac{1}{k}} \leqq C^{*)} \left(\sum_{r=1}^{\infty} |x_r|^k\right)^{\frac{1}{k}}.$$

Choose any $m \geqq 1$ and let $a_m = 1$, $a_n = 0$, $n \neq m$.

Then
$$x_m = m^{1-\frac{1}{k}}, \quad x_n = 0, \quad n \neq m$$

and
$$\begin{aligned} y_n &= \frac{1}{P_n}(P_n - P_{m-1})\varepsilon_m \qquad & m \leqq n, \\ &= 0 & m > n. \end{aligned}$$

Thus for $n \geqq m$

$$|y_n| = \frac{1}{P_n}(P_n - P_{m-1})|\varepsilon_m| \leqq C \ m^{1-\frac{1}{k}}.$$

Making $n \to \infty$ we get

$$|\varepsilon_m| \leqq C \ m^{1-\frac{1}{k}}$$

which is the required result.

LEMMA 5 [1, 2]. *If* $\varepsilon_n = O(1)$, *then* $\overset{\beta}{\triangle}(\overset{\alpha}{\triangle}\varepsilon_n) = \overset{\alpha+\beta}{\triangle}\varepsilon_n$ *if* $\alpha \geqq 0$, $\beta > -1$, $\alpha+\beta > 0$. *If* $\varepsilon_n = o(1)$ *then the equality holds for* $\alpha \geqq 0$, $\beta \geqq -1$, $\alpha+\beta \geqq 0$.

LEMMA 6 [8]. *If* $1 < \alpha < 2$, $\varepsilon_n = O(n)$, *then*

$$\begin{aligned} \sum_{v=r}^{n} \triangle\left(\frac{\varepsilon_v}{v}\right) A_{v-r}^{-\alpha} P_{v-1} = &- \sum_{m=r}^{n} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right) \sum_{v=r}^{m} p_v \sum_{j=r}^{v} A_{j-r}^{-\alpha} A_{m-j}^{\alpha-2} \\ &+ \sum_{m=r}^{n} P_m \overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right) A_{m-r}^{-1} \\ &- \sum_{m=n+1}^{\infty} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right) \sum_{v=r}^{n} p_v \sum_{j=r}^{v} A_{j-r}^{-\alpha} A_{m-j}^{\alpha-2} \\ &+ P_n \sum_{m=n+1}^{\infty} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right) \sum_{j=r}^{n} A_{j-r}^{-\alpha} A_{m-j}^{\alpha-2}. \end{aligned}$$

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*) Where C is a constant not necessarily the same at each occurrence.

LEMMA 7 [2]. *For* $0<\beta\leqq\alpha<1$, $0\leqq r\leqq v\leqq n$,

$$\left|\sum_{m=r}^{v} A_{n-m}^{\beta-1}A_{m-r}^{-\alpha-1}\right|\leqq CA_{n-r}^{\beta-1}A_{v-r}^{-\alpha}.$$

LEMMA 8. *If* $(n+1)p_n=O(P_n)$, $P_n\to\infty$, *then*

$$\lim_{n\to\infty}\frac{P_{n\pm v}}{P_n}=1$$

for finite v.

The proof is quite easy.

LEMMA 9 [7]. *Let* $1<k\leqq+\infty$, $\alpha>0$, $\theta_n=O(1)$,

and

$$\sum_{1}^{\infty} n^{(\alpha+1)k'-1}|\overset{\alpha}{\triangle}\theta_n|^{k'}\quad<\infty.$$

Then
$$\sum_{1}^{\infty} n^{(\gamma+1)k'-1}|\overset{\gamma}{\triangle}\theta_n|^{k'}<\infty,\qquad 0<\gamma\leqq\alpha.$$

LEMMA 10 [3]. *If* Σa_n *is summable* $|C,\alpha|_k$, $k\geqq1$, $\alpha\geqq0$, *then* $\Sigma n^{-1-k\delta}|t_n^{\alpha-\delta}|^k<\infty$, *where* $0\leqq\delta\leqq\alpha$.

LEMMA 11 [9]*). *For* $\alpha\geqq1$, $k-1<\alpha-1\leqq k$, *where* k *is an integer,*

$$\sum_{v=r}^{n} A_{n-v}^{\alpha-1-1}A_{v-r}^{-(\alpha-1)-1}\frac{1}{v+1}=\sum_{\rho=0}^{k-1}C_\rho(-1)^\rho\overset{\rho}{\triangle}\frac{1}{(r+1)}A_{n-\rho-r}^{\rho-1}$$
$$+A_{n-k-r}^{k-1}O\left(\frac{1}{(r+1)^{k+1}}\right).$$

LEMMA 12. *If* Σa_n *is summable* $|C,\alpha|_k$, $\alpha\geqq1$, *Then* $\Sigma\, t_n^1/n$ *is summable* $|C,\alpha-1|_k$.

For $k=1$ it is a special case of a general theorem due to Kogbetliantz [4].

*) This is a special case: $\beta=\alpha$, $\gamma=1$ and α replaced by $\alpha-1$ in[9].

PROOF. Let $T_n^{\alpha-1}$ denote $(C, \alpha-1)$ means of $\{t_n^1\}$.

Then
$$T_n^{\alpha-1} = \frac{1}{A_n^{\alpha-1}} \sum_{v=1}^{n} A_{n-v}^{\alpha-2} t_v^1 = \frac{1}{A_n^{\alpha-1}} \sum_{v=1}^{n} A_{n-v}^{\alpha-2} \frac{1}{v+1} \sum_{r=1}^{v} A_{v-r}^{-\alpha} A_r^{\alpha} t_r^{\alpha}$$
$$= \frac{1}{A_n^{\alpha-1}} \sum_{r=1}^{n} t_r^{\alpha} A_r^{\alpha} \sum_{v=r}^{n} \frac{A_{n-v}^{\alpha-2} A_{v-r}^{-\alpha}}{v+1}$$
$$\leqq \frac{1}{A_n^{\alpha-1}} \sum_{r=1}^{n} |t_r^{\alpha}| A_r^{\alpha} \left\{ \left| \sum_{\rho=0}^{q-1} C_\rho (-1)^\rho \overset{\rho}{\triangle} \frac{1}{(r+1)} A_{n-\rho-r}^{\rho n-1} \right| \right.$$
$$\left. + A_{n-q-r}^{q-1} O\left(\frac{1}{(r+1)^{q+1}} \right) \right\}, \quad q-1 < \alpha-1 \leqq q,$$

by virtue of Lemma 11.

It is, therefore, sufficient to prove that

$$I = \sum_{1}^{\infty} \frac{1}{n^{1+(\alpha-1)k}} \left(\sum_{r=1}^{n} |t_r^{\alpha}| A_r^{\alpha} \cdot \frac{1}{(r+1)^{1+\rho}} A_{n-\rho-r}^{\rho-1} \right)^k < \infty, \text{ for } 0 \leqq \rho \leqq q.$$

If $\rho = 0$, then

$$I = \sum_{1}^{\infty} \frac{1}{n^{1+(\alpha-1)k}} |t_n^{\alpha}|^k \cdot n^{(\alpha-1)k} < \infty,$$

by the hypothesis.

If $\rho > 0$, then $\alpha > q \geqq \rho$ so that

$$I = \sum_{1}^{\infty} \frac{1}{n^{1+(\alpha-1)k}} \left(\sum_{r=1}^{n} |t_r^{\alpha}| \cdot r^{\alpha-\rho-1} A_{n-\rho-r}^{\rho n-1} \right)^k$$
$$\leqq \sum_{1}^{\infty} \frac{1}{n^{1+(\alpha-1)k}} \left(\sum_{r=1}^{n-\rho} |t_r^{\alpha}|^k r^{\alpha-\rho-1} A_{n-\rho-r}^{\rho-1} \right) \left(\sum_{r=1}^{n-\rho} A_r^{\alpha-\rho-1} A_{n-\rho-r}^{\rho-1} \right)^{k/k'}$$
$$\leqq C \sum_{1}^{\infty} \frac{1}{n^{\alpha}} \sum_{r=1}^{n-\rho} |t_r^{\alpha}|^k r^{\alpha-\rho-1} A_{n-\rho-r}^{\rho-1}$$
$$= C \sum_{r=1}^{\infty} r^{\alpha-\rho-1} |t_r^{\alpha}|^k \sum_{n=r+\rho}^{\infty} \frac{A_{n-\rho-r}^{\rho-1}}{n \cdot A_n^{\alpha-1}}$$
$$= C \sum_{r=1}^{\infty} r^{-1} |t_r^{\alpha}|^k = O(1).$$

4. Proof of the Theorem. The result being known for $k=1$, we proceed to prove the same for $k>1$. We write $x_0 = a_0$ and

$$x_n = \frac{1}{n^{1/k}A_n^\alpha}\sum_{r=1}^{n} A_{n-r}^{\alpha-1} r a_r, \qquad n \geqq 1,$$

and
$$t_n = \frac{1}{P_n}\sum_{r=0}^{n}(P_n - P_{r-1})a_r\varepsilon_r, \qquad P_{-1} = 0,$$

so that

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{r=1}^{n} a_r\varepsilon_r P_{r-1}.$$

Putting

$$y_n = t_n - t_{n-1}, \qquad n \geqq 1,\ y_0 = a_0\varepsilon_0$$

we have

$$\begin{aligned} y_n &= \frac{p_n}{P_n P_{n-1}}\sum_{r=1}^{n} a_r\varepsilon_r P_{r-1} \\ &= \frac{p_n}{P_n P_{n-1}}\sum_{r=1}^{n}\frac{\varepsilon_r}{r}P_{r-1}\sum_{m=1}^{r} A_{r-m}^{-\alpha-1}x_m m^{1/k}A_m^\alpha \\ &= \frac{p_n}{P_n P_{n-1}}\sum_{m=1}^{n} m^{1/k}A_m^\alpha x_m \sum_{r=m}^{n} A_{r-m}^{-\alpha-1}\frac{\varepsilon_r}{r}P_{r-1} \\ &= \sum_{m=1}^{\infty} C_{n,m}x_m, \end{aligned}$$

where

$$\begin{aligned} C_{n,m} &= \frac{p_n}{P_n P_{n-1}}A_m^\alpha m^{1/k}\cdot\sum_{r=m}^{n} A_{r-m}^{-\alpha-1}\frac{\varepsilon_r}{r}P_{r-1}, \quad m \leqq n, \\ &= 0. \qquad\qquad m > n. \end{aligned}$$

Now $\Sigma a_n\varepsilon_n$ is summable $|\bar{N}, p_n|$ whenever Σa_n is summable $|C,\alpha|_k$ $\alpha \geqq 0$, $k > 1$ if and only if

(4.1) $$\Sigma|y_n| < \infty \text{ whenever } \Sigma|x_n|^k < \infty.$$

Using Lemma 1, the necessary and sufficient conditions for the same are:

(4.2) $$\sum_{n=m}^{\infty} C_{n,m}u_n \text{ be convergent for every } u_n = O(1),\ m \geqq 1,$$

and

(4.3) $$\sum_{m=1}^{\infty}\left|\sum_{n=m}^{\infty} C_{n,m}u_n\right|^{k'} < +\infty \text{ whenever } u_n = O(1).$$

Now

$$\begin{aligned}\sum_{n=m}^{\infty} C_{n,m}u_n &= m^{1/k}A_m^{\alpha}\sum_{n=m}^{\infty}\frac{p_n}{P_nP_{n-1}}u_n\sum_{r=m}^{n}A_{r-m}^{-\alpha-1}\frac{\varepsilon_r}{r}P_{r-1}\\ &= m^{1/k}A_m^{\alpha}\sum_{r=m}^{\infty}A_{r-m}^{-\alpha-1}\frac{\varepsilon_r}{r}P_{r-1}\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}u_n\\ &= m^{1/k}A_m^{\alpha}\sum_{r=m}^{\infty}A_{r-m}^{-\alpha-1}P_{r-1}\frac{\varepsilon_r}{r}\delta_r\\ &= m^{1/k}A_m^{\alpha}\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}P_{m-1}\delta_m\right),\end{aligned}$$

where $$\delta_m = \sum_{n=m}^{\infty}\frac{p_n}{P_nP_{n-1}}u_n.$$

Now

$$\begin{aligned}&\sum_{n=m}^{\infty}\frac{p_n}{P_nP_{n-1}}|u_n|\sum_{r=m}^{n}|A_{r-m}^{-\alpha-1}|\frac{|\varepsilon_r|}{r}P_{r-1}\\ &\leqq C\sum_{r=m}^{\infty}|A_{r-m}^{-\alpha-1}|\frac{|\varepsilon_r|}{r}P_{r-1}\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\\ &= C\sum_{r=m}^{\infty}\frac{|\varepsilon_r|}{r}\cdot|A_{r-m}^{-\alpha-1}| = O(1) \text{ if } \varepsilon_n = O(n).\end{aligned}$$

Thus if $\varepsilon_n = O(n)$ then the above series is absolutely convergent for every $u_n = O(1)$ and hence change of order of summation is justified.

Thus, if $\varepsilon_n = O(n)$ condition (4.2) is satisfied and hence a necessary and sufficient condition for (4.1) is

$$\sum_{m=1}^{\infty} m^{\alpha k'+k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}P_{m-1}\delta_m\right)\right|^{k'} < +\infty \tag{4.4}$$

whenever $\varepsilon_n = O(n)$ and $u_n = O(1)$.

NECESSITY: We are given that $\Sigma a_n \varepsilon_n$ is summable $|\overline{N}, p_n|$ whenever Σa_n is summable $|C, \alpha|_k$. Then applying Lemma 4 we have $\varepsilon_n = O(n^{1-1/k}) = O(n)$. Thus (4.4) is a necessary condition whenever $u_n = O(1)$.

NECESSITY OF (i). Let $u_n = 1$, then $\delta_n = \frac{1}{P_{n-1}}$. From (4.4) we obtain

$$\sum_{m=1}^{\infty} m^{k'\alpha+k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|^{k'} < +\infty,\ \alpha \geqq 0.$$

Thus (i) is necessary.

NECESSITY OF (ii) (b). From Lemma 2 we have

$$\sum_{n=1}^{\infty} |C_{n,n}|^{k'} < +\infty,$$

that is to say,

$$\sum_{n=1}^{\infty} n^{\alpha k'-1}\left(\frac{p_n}{P_n}\right)^{k'} |\varepsilon_n|^{k'} < +\infty.$$

This proves the necessity of (ii) (b).

NECESSITY OF (ii) (a). It follows from the case $\alpha=0$ and the fact that (i) is a necessary condition.

SUFFICIENCY: For $0 \leqq \alpha \leqq 1$ condition (ii) (a) implies that $\varepsilon_n = O(n)$. Also from (ii) (b) for $\alpha > 1$

$$\varepsilon_n = O\left(n^{-\alpha+1/k'}\frac{P_n}{p_n}\right) = O(n^{1/k'}) = O(n)$$

since
$$\frac{P_n}{p_n} = O(n^{\alpha}),\ \alpha > 1.$$

Thus (4.4) is also sufficient condition for the validity of (4.1).

Case (i): Suppose $\alpha = 0$. Then

$$\Sigma n^{-1}|\varepsilon_n|^{k'} < \infty .$$

Using Hölder's inequality we observe that

$$\Sigma|a_n\varepsilon_n| = \Sigma n^{1-1/k}|a_n|n^{-1/k'}|\varepsilon_n| \leqq (\Sigma n^{k-1}|a_n|^k)^{1/k}(\Sigma n^{-1}|\varepsilon_n|^{k'})^{1/k'} < \infty .$$

Hence on account of absolute regularity, the series $\Sigma a_n\varepsilon_n$ is summable $|\bar{N}, p_n|$.
Case (ii): $1 < \alpha \leqq 2$. We shall prove that

$$\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty} C_{n,r}u_n\right|^{k'} < \infty \text{ whenever } u_n = O(1).$$

We have

$$\begin{aligned}
C_{n,r} &= \frac{p_n}{P_nP_{n-1}} A_r^{\alpha} r^{1/k} \sum_{v=r}^{n} A_{v-r}^{-\alpha-1} \frac{\varepsilon_v}{v} P_{v-1} \\
&= \frac{p_n}{P_nP_{n-1}} A_r^{\alpha} r^{1/k} \left\{\sum_{v=r}^{n-1} \triangle\left(\frac{\varepsilon_v}{v} P_{v-1}\right) A_{v-r}^{-\alpha} + \frac{\varepsilon_n}{n} P_{n-1} A_{n-r}^{-\alpha}\right\} \\
&= \frac{p_n}{P_nP_{n-1}} A_r^{\alpha} r^{1/k} \left\{\sum_{v=r}^{n} P_{v-1} \triangle\left(\frac{\varepsilon_v}{v}\right) A_{v-r}^{-\alpha} - \sum_{v=r}^{n} p_v \frac{\varepsilon_{v+1}}{v+1} A_{v-r}^{-\alpha}\right. \\
&\quad \left. + P_n \frac{\varepsilon_{n+1}}{n+1} A_{n-r}^{-\alpha}\right\} = L_1^{(n)} + L_2^{(n)} + L_3^{(n)}, \text{ say}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty} C_{n,r}u_n\right|^{k'} &\leqq C\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty} L_1^{(n)}u_n\right|^{k'} + C\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty} L_2^{(n)}u_n\right|^{k'} + C\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty} L_3^{(n)}u_n\right|^{k'} \\
&= M_1 + M_2 + M_3, \text{ say}.
\end{aligned}$$

It is, therefore, sufficient to prove that

$$M_p = O(1),\ p = 1, 2, 3 \text{ whenever } u_n = O(1).$$

Let us first suppose that $1 < \alpha < 2$. Then applying Lemma 6 we get

$$
\begin{aligned}
M_1 &= C\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty}\frac{u_n\cdot p_n}{P_nP_{n-1}}A_r^{\alpha}r^{1/k}\sum_{v=r}^{n}P_{v-1}\triangle\left(\frac{\varepsilon_v}{v}\right)A_{v-r}^{-\alpha}\right|^{k'}\\
&= C\sum_{r=1}^{\infty}r^{(\alpha+1)k'-1}\left|\sum_{n=r}^{\infty}\frac{u_np_n}{P_nP_{n-1}}\sum_{v=r}^{n}P_{v-1}\triangle\left(\frac{\varepsilon_v}{v}\right)A_{v-r}^{-\alpha}\right|^{k'}\\
&\leqq C\sum_{r=1}^{\infty}r^{(\alpha+1)k'-1}\left|\sum_{n=r}^{\infty}\frac{u_np_n}{P_nP_{n-1}}\sum_{m=r}^{n}\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\sum_{v=r}^{m}p_v\sum_{j=r}^{v}A_{j-r}^{-\alpha}A_{m-j}^{\alpha-2}\right|^{k'}\\
&\quad+C\sum_{r=1}^{\infty}r^{(\alpha+1)k'-1}\left|\sum_{n=r}^{\infty}\frac{u_np_n}{P_nP_{n-1}}\sum_{m=r}^{n}\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)P_mA_{m-r}^{-1}\right|^{k'}\\
&\quad+C\sum_{r=1}^{\infty}r^{(\alpha+1)k'-1}\left|\sum_{n=r}^{\infty}\frac{u_np_n}{P_nP_{n-1}}\sum_{m=n+1}^{\infty}\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\sum_{v=r}^{n}p_v\sum_{j=r}^{v}A_{j-r}^{-\alpha}A_{m-j}^{\alpha-2}\right|^{k'}\\
&\quad+C\sum_{r=1}^{\infty}r^{(\alpha+1)k'-1}\left|\sum_{n=r}^{\infty}\frac{u_np_n}{P_{n-1}}\sum_{m=n+1}^{\infty}\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\sum_{j=r}^{n}A_{j-r}^{-\alpha}A_{m-j}^{\alpha-2}\right|^{k'}\\
&= M_{11}+M_{12}+M_{13}+M_{14},\ \text{say}.
\end{aligned}
$$

Now using Lemmas 7, 8 the hypotheses (b) and (i),

$$
\begin{aligned}
M_{11} &= C\sum_{r=1}^{\infty}r^{(\alpha+1)k'-1}\left|\sum_{n=r}^{\infty}\frac{u_np_n}{P_nP_{n-1}}\sum_{m=r}^{n}\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\sum_{v=r}^{m}p_v\sum_{j=r}^{v}A_{j-r}^{-\alpha}A_{m-j}^{\alpha-2}\right|^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{m=r}^{n}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|\sum_{v=r}^{m}p_vA_{m-r}^{\alpha-2}A_{v-r}^{1-\alpha}\right)^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{\alpha k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{m=r}^{n}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|P_mA_{m-r}^{\alpha-2}A_{m-r}^{2-\alpha}\right)^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{\alpha k'-1}\left(\sum_{m=r}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|\right)^{k'}, \qquad (4.5)\\
&= O(1)\sum_{r=1}^{\infty}r^{\alpha k'-1}\left(\sum_{m=r}^{\infty}m^{\alpha k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|^{k'}\right)\left(\sum_{m=r}^{\infty}m^{-\alpha k+k-1}\right)^{k'/k}\\
&= O(1)\sum_{r=1}^{\infty}r^{k'-1}\sum_{m=r}^{\infty}m^{\alpha k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|^{k'}\\
&= O(1)\sum_{m=1}^{\infty}m^{\alpha k'+k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|^{k'}=O(1).
\end{aligned}
$$

Similarly,

$$M_{12} = O(1)\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{m=r}^{n}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|P_mA_{m-r}^{-1}\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_r}{r}\right)\right|. P_r\right)^{k'}$$

$$= O\left(\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_r}{r}\right)\right|^{k'}\right) = O(1).$$

$$M_{13} = O(1)\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{m=n+1}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|\sum_{v=r}^{n}p_v A_{m-r}^{\alpha-2}A_{v-r}^{1-\alpha}\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{m=n+1}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|A_{m-r}^{\alpha-2}\frac{P_n}{r+1}A_{n-r}^{2-\alpha}\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{-k'+\alpha k'-1}\left(\sum_{m=r}^{\infty} m\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{-k'+\alpha k'-1}\left(\sum_{m=r}^{\infty} m^{\delta k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|^{k'}\right)\left(\sum_{m=r}^{\infty} m^{-\delta k+k+k-1}\right)^{k'/k} \quad 2<\delta<\alpha+1$$

$$= O(1)\sum_{r=1}^{\infty} r^{-k'+\alpha k'-1} r^{-\delta k'+2k'}\sum_{m=r}^{\infty} m^{\delta k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|^{k'}$$

$$= O(1)\sum_{m=1}^{\infty} m^{(\alpha+1)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|^{k'} = O(1).$$

Next using Lemma 7.

$$M_{14} = O(1)\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_{n-1}}\sum_{m=n+1}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|A_{n-r}^{1-\alpha}A_{m-r}^{\alpha-2}\right)^{k}$$

$$= O(1)\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{m=r}^{\infty} A_{m-r}^{\alpha-2}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|\sum_{n=r}^{m}\frac{p_n}{P_{n-1}}A_{n-r}^{1-\alpha}\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{\alpha k'-1}\left(\sum_{m=r}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_m}{m}\right)\right|\right)^{k'} = O(1)$$

as shown in (4.5).

Thus $M_1 = O(1)$, for $1<\alpha<2$.
Now let $\alpha = 2$. Then

$$\begin{aligned}
M_1 &= C\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty}\frac{u_n p_n}{P_n P_{n-1}}A_r^2 r^{1/k}\left(P_{r-1}\triangle\left(\frac{\varepsilon_r}{r}\right)-P_r\triangle\left(\frac{\varepsilon_{r+1}}{r+1}\right)\right)\right|^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{3k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_n P_{n-1}}\left|P_{r-1}\overset{2}{\triangle}\left(\frac{\varepsilon_r}{r}\right)-p_r\triangle\left(\frac{\varepsilon_{r+1}}{r+1}\right)\right|\right)^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{3k'-1}\left|\overset{2}{\triangle}\left(\frac{\varepsilon_r}{r}\right)\right|^{k'}\left(\sum_{n=r}^{\infty}\frac{p_n P_{r-1}}{P_n P_{n-1}}\right)^{k'}\\
&\quad + O(1)\sum_{r=1}^{\infty}r^{3k'-1}\left|\triangle\left(\frac{\varepsilon_{r+1}}{r+1}\right)\right|^{k'}\left(\frac{p_r}{P_{r-1}}\right)^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{3k'-1}\left|\overset{2}{\triangle}\left(\frac{\varepsilon_r}{r}\right)\right|^{k'} + O(1)\sum_{r=1}^{\infty}(r+1)^{2k'-1}\left|\triangle\left(\frac{\varepsilon_{r+1}}{r+1}\right)\right|^{k'}\\
&= O(1),
\end{aligned}$$

by virtue of Lemma 9.

Hence, $M_1 = O(1)$ for $1<\alpha\leqq 2$. We shall now consider M_2. We have

$$\begin{aligned}
M_2 &= C\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty}\frac{u_n\cdot p_n}{P_n P_{n-1}}A_r^{\alpha} r^{1/k}\sum_{v=r}^{n}p_v\frac{\varepsilon_{v+1}}{v+1}A_{v-r}^{-\alpha}\right|^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{\alpha k'+k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_n P_{n-1}}\sum_{v=r}^{n}p_v\frac{|\varepsilon_{v+1}|}{v+1}\left|A_{v-r}^{-\alpha}\right|\right)^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{\alpha k'-1}\left(\sum_{v=r}^{\infty}\left(\frac{p_v}{P_{v-1}}\right)^{k'}|\varepsilon_{v+1}|^{k'}|A_{v-r}^{-\alpha}|\right)\\
&= O(1)\sum_{v=1}^{\infty}v^{\alpha k'-1}\left(\frac{p_v}{P_{v-1}}\right)^{k'}|\varepsilon_{v+1}|^{k'} = O(1),
\end{aligned}$$

by virtue of (ii)(b) and (a). Also

$$\begin{aligned}
M_3 &= C\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty}\frac{u_n p_n}{P_n P_{n-1}}A_r^{\alpha} r^{1/k}P_n\frac{\varepsilon_{n+1}}{n+1}A_{n-r}^{-\alpha}\right|^{k'}\\
&= O(1)\sum_{r=1}^{\infty}r^{\alpha k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_{n-1}}|\varepsilon_{n+1}|\,|A_{n-r}^{-\alpha}|\right)^{k'} = O(1),
\end{aligned}$$

as shown in the proof of $M_2 = O(1)$.

This proves the theorem for the case: $1<\alpha\leqq 2$.

Case (iii): $\alpha>2$. Choose a positive integer r such that $1\leqq r<\alpha\leqq r+1$. By case (ii) the result is true when $r=1$. Suppose the result is true for $s<\alpha\leqq s+1, s\geqq 1$. We shall show that it is also true for $s+1<\alpha\leqq s+2$.

Now we have on applying Abel's transformation

$$\sum_{v=1}^{n} a_v \varepsilon_v = \sum_{1}^{n} \varepsilon_v \frac{t_v^1}{v} + \sum_{v=1}^{n} (v \triangle \varepsilon_v) \frac{t_v^1}{v} + \varepsilon_{n+1} t_n^1$$
$$= J_1(n) + J_2(n) + J_3(n), \text{ say}.$$

The series $\Sigma a_n \varepsilon_n$ will be summable $|\bar{N}, p_n|$ if each of the sequences $\{J_p(n)\}$, $p=1,2,3$, is summable $|\bar{N}, p_n|$. By virtue of Lemma 12 and the hypothesis $\Sigma t_n^1/n$ is summable $|C, \alpha-1|_k$. Hence to prove that $\{J_1\}$ and $\{J_2\}$ are summable $|\bar{N}, p_n|$ it is sufficient to show that $\{\varepsilon_v\}$ and $\{v \triangle \varepsilon_v\}$ satisfy the conditions of the theorem with $\alpha-1$ in place of α. Since in the case of $1<\alpha\leqq 2$ we require for the proof (i), (ii)(b), (a), (b) and $\Sigma|\varepsilon_n|^{k'}/n<\infty$ we assume the *same* set of conditions for $\alpha>2$.

Since $\sum_{1}^{\infty} n^{-1}|\varepsilon_n|^{k'}<\infty$ implies that $\varepsilon_n=O(n)$, it follows from Lemma 9 that

(4.5)(i) $$\sum_{1}^{\infty} n^{\alpha k'-1}\left|\overset{\alpha-1}{\triangle}\left(\frac{\varepsilon_n}{n}\right)\right|^{k'}<\infty.$$

Also it is obvious that

(4.5)(ii) $$\sum_{1}^{\infty} n^{(\alpha-1)k'-1}\left(\frac{p_n}{P_n}\right)^{k'}|\varepsilon_n|^{k'}<\infty.$$

Also since $\alpha>2$ (i) implies that

$$\sum_{1}^{\infty} n^{2k'-1}\left|\triangle\left(\frac{\varepsilon_n}{n}\right)\right|^{k'}<\infty,$$

and from this it follows that

(4.6)(i) $$\sum_{1}^{\infty} n^{k'-1}|\triangle \varepsilon_n|^{k'}<\infty.$$

Now

$$\overset{\alpha}{\triangle}\varepsilon_n = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1}\varepsilon_v$$
$$= \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1}\{(n+\alpha)+(-\alpha+v-n)\}\frac{\varepsilon_v}{v}$$
$$= (n+\alpha)\overset{\alpha}{\triangle}\left(\frac{\varepsilon_n}{n}\right)-\alpha\overset{\alpha-1}{\triangle}\left(\frac{\varepsilon_n}{n}\right).$$

Therefore

$$\sum_1^\infty n^{\alpha k'-1}\left|\overset{\alpha-1}{\triangle}\left(\frac{n\triangle\varepsilon_n}{n}\right)\right|^{k'}$$
$$=\sum_1^\infty n^{\alpha k'-1}\,|\overset{\alpha}{\triangle}\varepsilon_n|^{k'}$$
$$\leqq C\sum_1^\infty n^{(\alpha+1)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_n}{n}\right)\right|^{k'}+C\sum_1^\infty n^{\alpha k'-1}\left|\overset{\alpha-1}{\triangle}\left(\frac{\varepsilon_n}{n}\right)\right|^{k'}$$
$$=O(1).$$

Thus

(4. 6)(ii) $$\sum_1^\infty n^{\alpha k'-1}\left|\overset{\alpha-1}{\triangle}\left(\frac{n\triangle\varepsilon_n}{n}\right)\right|^{k'}<\infty.$$

Again

(4. 6)(iii) $$\sum_1^\infty n^{(\alpha-1)k'-1}\left(\frac{p_n}{P_n}\right)^{k'}|n\triangle\varepsilon_n|^{k'}=O(1).$$

Thus from (4. 5)−(4. 6) it is clear that $\{\varepsilon_n\}$ and $\{n\triangle\varepsilon_n\}$ satisfy the conditions (i), (ii)(b) and $\Sigma|\varepsilon_n|^{k'}/n<\infty$ with $(\alpha-1)$ in place of α.
Hence $\{J_1(n)\}$ and $\{J_2(n)\}$ are summable $|\overline{N},p_n|$.

We shall now consider $J_3(n)$. We will show that $\{\varepsilon_{n+1}t_n^1\}$ is summable $|\overline{N},p_n|$.

$$\sum_1^\infty\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v|\varepsilon_{v+1}|\,|t_v^1|+\sum_1^\infty\frac{p_n}{P_n}|\varepsilon_{n+1}|\,|t_n^1|$$
$$=2\sum_{v=1}^\infty\frac{p_v}{P_v}|\varepsilon_{v+1}|\,|t_v^1|$$
$$\leqq C\left(\sum_{v=1}^\infty\left(\frac{p_v}{P_v}\right)^{k'}|\varepsilon_{v+1}|^{k'}\cdot v^{\alpha k'-1}\right)^{1/k'}\left(\sum_{v=1}^\infty v^{-\alpha k+k-1}|t_v^1|^k\right)^{1/k}$$
$$=O(1),$$

by (ii)(b), condition (a) and Lemma 10.

Hence $\{J_3(n)\}$ is summable $|\overline{N},p_n|$.

Therefore the theorem is proved for $s+1<\alpha\leqq s+2$ $(s\geqq 1)$ and consequently theorem holds for $\alpha>2$.

Case (iv): $0<\alpha\leqq 1$. We have

$$
\begin{aligned}
C_{n,r} &= \frac{p_n}{P_nP_{n-1}} A_r^\alpha r^{1/k} \sum_{v=r}^{n} P_{v-1} \frac{\varepsilon_v}{v} A_{v-r}^{-\alpha-1} \\
&= \frac{p_n}{P_nP_{n-1}} A_r^\alpha r^{1/k} \sum_{v=r}^{n} P_{v-1} A_{v-r}^{-\alpha-1} \overset{-\alpha}{\triangle}\left(\overset{\alpha}{\triangle} \frac{\varepsilon_v}{v}\right) \\
&= \frac{p_n}{P_nP_{n-1}} A_r^\alpha r^{1/k} \sum_{v=r}^{n} P_{v-1} A_{v-r}^{-\alpha-1} \sum_{q=v}^{\infty} A_{q-v}^{\alpha-1} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right) \\
&= \frac{p_n}{P_nP_{n-1}} A_r^\alpha r^{1/k} \left[\sum_{v=r}^{n} P_{v-1} A_{v-r}^{-\alpha-1} \sum_{q=v}^{n} A_{q-v}^{\alpha-1} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right. \\
&\quad \left. + \sum_{v=r}^{n} P_{v-1} A_{v-r}^{-\alpha-1} \sum_{q=n+1}^{\infty} A_{q-v}^{\alpha-1} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right] \\
&= \frac{p_n}{P_nP_{n-1}} A_r^\alpha r^{1/k} \sum_{q=r}^{n} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right) \sum_{v=r}^{q} P_{v-1} A_{v-r}^{-\alpha-1} A_{q-v}^{\alpha-1} \\
&\quad + \frac{p_n}{P_nP_{n-1}} A_r^\alpha r^{1/k} \sum_{q=n+1}^{\infty} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right) \sum_{v=r}^{n} P_{v-1} A_{v-r}^{-\alpha-1} A_{q-v}^{\alpha-1} \\
&= Q_1 + Q_2, \quad \text{say.}
\end{aligned}
$$

It is, therefore, sufficient to prove that

$$
\sum_{r=1}^{\infty} \left| \sum_{n=r}^{\infty} Q_1 u_n \right|^{k'} < \infty, \tag{4.7}
$$

$$
\sum_{r=1}^{\infty} \left| \sum_{n=r}^{\infty} Q_2 u_n \right|^{k'} < \infty, \text{ whenever } u_n = O(1). \tag{4.8}
$$

PROOF OF (4.7). We have for $0<\alpha<1$

$$
\begin{aligned}
&\sum_{r=1}^{\infty} \left| \sum_{n=r}^{\infty} \frac{p_n}{P_nP_{n-1}} A_r^\alpha r^{1/k} u_n \sum_{q=r}^{n} \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right) \sum_{v=r}^{q} P_{v-1} A_{v-r}^{-\alpha-1} A_{q-v}^{\alpha-1} \right|^{k'} \\
&\leqq \sum_{r=1}^{\infty} r^{(\alpha+1)k'-1} \left(\sum_{n=r}^{\infty} \frac{p_n}{P_nP_{n-1}} \sum_{q=r}^{n} \left| \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right| \left| \sum_{v=r}^{q-1} - p_v \sum_{m=r}^{v} A_{m-r}^{-\alpha-1} A_{q-m}^{\alpha-1} \right.\right. \\
&\quad \left.\left. + P_{q-1} \sum_{m=r}^{q} A_{m-r}^{-\alpha-1} A_{q-m}^{\alpha-1} \right| \right)^{k'} \\
&\leqq C \sum_{r=1}^{\infty} r^{(\alpha+1)k'-1} \left(\sum_{n=r}^{\infty} \frac{p_n}{P_nP_{n-1}} \sum_{q=r}^{n} \left| \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right| \left| \sum_{v=r}^{q-1} \frac{P_v}{v} A_{q-r}^{\alpha-1} A_{v-r}^{-\alpha} \right)^{k'} \\
&\quad + C \sum_{r=1}^{\infty} r^{(\alpha+1)k'-1} \left(\sum_{n=r}^{\infty} \frac{p_n}{P_nP_{n-1}} \sum_{q=r}^{n} \left| \overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right| \left| P_{q-1} A_{q-r}^{-1} \right)^{k'}
\end{aligned}
$$

$$\leqq C\sum_{r=1}^{\infty} r^{\alpha k'-1}\left(\sum_{q=r}^{\infty} P_{q-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|\sum_{n=q}^{\infty}\frac{p_n}{P_nP_{n-1}}\right)^{k'}$$

$$+ C\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_r}{r}\right)\right|^{k'} = C\sum_{r=1}^{\infty} r^{\alpha k'-1}\left(\sum_{q=r}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|\right)^{k'} + C$$

$$= O(1)\sum_{r=1}^{\infty} r^{\alpha k'-1}\sum_{q=r}^{\infty} q^{(\delta+\alpha)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|^{k'}\cdot\left(\sum_{q=r}^{\infty} q^{-\delta k-\alpha k+k-1}\right)^{k'/k}$$

$$+ O(1),\ 1-\alpha<\delta<1,$$

$$= O(1)\sum_{r=1}^{\infty} r^{\alpha k'-1-\alpha k'-\delta k'+k'}\sum_{q=r}^{\infty} q^{(\alpha+\delta)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|^{k'} + O(1)$$

$$= O(1)\sum_{q=1}^{\infty} q^{(\alpha+1)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|^{k'} + O(1) = O(1),\ \text{by (i)}.$$

If $\alpha=1$, then the proof is easy.
This proves (4.7) for $0<\alpha\leqq 1$.

PROOF OF (4.8). For $0<\alpha<1$ we have as in (4.7)

$$\sum_{r=1}^{\infty}\left|\sum_{n=r}^{\infty} Q_2 u_n\right|^{k'}$$

$$= \sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left|\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}u_n\sum_{q=n+1}^{\infty}\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\sum_{v=r}^{n} P_{v-1}A_{v-r}^{-\alpha-1}A_{q-v}^{\alpha-1}\right|^{k'}$$

$$\leqq \sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{q=n+1}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|\left|\sum_{v=r}^{n-1} -p_v\sum_{m=r}^{v} A_{m-r}^{-\alpha-1}A_{q-m}^{\alpha-1}\right.\right.$$

$$\left.\left. + P_{n-1}\sum_{m=r}^{n} A_{m-r}^{-\alpha-1}A_{q-m}^{\alpha-1}\right|\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{\alpha k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_n}\sum_{q=n+1}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|A_{q-r}^{\alpha-1}A_{n-r}^{1-\alpha}\right)^{k'}$$

$$+ O(1)\sum_{r=1}^{\infty} r^{(\alpha+1)k'-1}\left(\sum_{n=r}^{\infty}\frac{p_n}{P_n}\sum_{q=n+1}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|A_{q-r}^{\alpha-1}A_{n-r}^{-\alpha}\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{\alpha k'-1}\left(\sum_{q=r}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|\log\frac{2q}{r}\right)^{k'}$$

$$+ O(1)\sum_{r=1}^{\infty} r^{\alpha k'-1}\left(\sum_{q=r}^{\infty}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|\right)^{k'}$$

$$= O(1)\sum_{r=1}^{\infty} r^{\alpha k'-1}\left(\sum_{q=r}^{\infty} q^{k'-1+(\alpha/4)k'}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|^{k'}\right)\left(\sum_{q=r}^{\infty} q^{-1-k\alpha/4}\log^k\frac{2q}{r}\right)^{k'/k}$$

$$+ O(1) = O(1)\sum_{q=1}^{\infty} q^{(\alpha+1)k'-1}\left|\overset{\alpha}{\triangle}\left(\frac{\varepsilon_q}{q}\right)\right|^{k'} + O(1) = O(1).$$

The case $\alpha = 1$ can be easily disposed of.

This completes the proof of the theorem.

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