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THE EQUATION $\Delta u = Pu$ ON E^m WITH ALMOST ROTATION FREE $P \geqq 0$

MITSURU NAKAI

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Consider a connected C^∞ Riemannian m-manifold R ($m \geqq 2$) and a continuously differentiable function P ($\geqq 0$ and $\not\equiv 0$) on R. The space of solutions of $d{*}du = Pu{*}1$ or $\Delta u = Pu$ on R will be denoted by $P(R)$. Let $\mathcal{O}_{PX}$ be the set of pairs (R, P) such that the subspace $PX(R)$ of $P(R)$ consisting of functions with a certain property X reduces to $\{0\}$. Here we let X be B which stands for boundedness, D for the finiteness of the Dirichlet integral $D_R(u) = \int_R du \wedge {*}du$, and E for the finiteness of the energy integral $E_R^P(u) = D_R(u) + \int_R Pu^2{*}1$; we also consider nontrivial combinations of these properties. We denote by $\mathcal{O}_G$ the set of pairs (R, P) such that there exists no harmonic Green's function on R.

The purpose of this paper is to show that (E^m, P) will be an example for the strictness of each of the following inclusion relations

$$(1) \qquad \mathcal{O}_G \subset \mathcal{O}_{PB} \subset \mathcal{O}_{PD} \subset \mathcal{O}_{PE}$$

if P is properly chosen, where $E^m (m \geqq 3)$ is m-dimensional Euclidean space and P is a continuously differentiable function on $E^m (\geqq 0, \not\equiv 0)$.

More precisely let

$$(2) \qquad P(x) \sim |x|^{-\alpha}$$

as $|x| \to \infty$, i.e. there exists a constant $c > 1$ such that $c^{-1}|x|^{-\alpha} \leqq P(x) \leqq c|x|^{-\alpha}$ for large $|x|$. Then the following is true:

$$(3) \qquad \begin{cases} (E^m, P) \in \mathcal{O}_{PB} - \mathcal{O}_G \text{ if } \alpha \leqq 2; \\ (E^m, P) \in \mathcal{O}_{PD} - \mathcal{O}_{PB} \text{ if } 2 < \alpha \leqq (m+2)/2; \\ (E^m, P) \in \mathcal{O}_{PE} - \mathcal{O}_{PD} \text{ if } (m+2)/2 < \alpha \leqq m. \end{cases}$$

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By definition, $(E^m, P) \notin \mathcal{O}_G$ for every α, and $(E^m, P) \notin \mathcal{O}_{PE}$ for $\alpha > m$.

These relations will be proven first for a $P(x)$ which is invariant under every rotation of E^m with respect to the origin. To settle the general case (2) we will study the dependence of the linear space structure of $PX(R)$ on P for general Riemannian manifolds R, where $X=B$, BD, and BE. This problem also has interest in its own right.

Comparison theorems

1. Let (g_{ij}) be the metric tensor on R, $(g^{ij}) = (g_{ij})^{-1}$, and $g = \det(g_{ij})$. We also denote simply by dx the volume element $\sqrt{g}\ dx^1 \cdots dx^m$. The Laplace-Beltrami operator is then

$$\Delta \cdot = \frac{1}{\sqrt{g}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{m} \sqrt{g}\ (x) g^{ij}(x) \frac{\partial \cdot}{\partial x^j} \right).$$

We always assume that the function P in the operator

$$A^P = \Delta - P$$

is of class C^1, $P \geqq 0$, and $\not\equiv 0$ in R, unless otherwise stated. We are interested in the vector space structure of $PX(R)$ ($X=B$, BD, BE, D, or E). Observe the following:

The space $PBD(R)$ (resp. $PBE(R)$) is dense in $PD(R)$ (resp. $PE(R)$) with respect to the topology τ_D(resp. τ_E) given by the simultaneous convergence in $D_R(\cdot)$(resp. $E_R(\cdot)$) and uniform convergence on every compact set in R. In particular

$$\mathcal{O}_{PD} = \mathcal{O}_{PBD} \ (\textit{resp. } \mathcal{O}_{PE} = \mathcal{O}_{PBE}). \tag{4}$$

The D-part of this statement is the author's recent result ([8], [9]). The E-part was obtained by Royden [11](see also Glasner-Katz [1]). In view of these results we will only study the class $PB(R)$ and its subspaces $PBD(R)$ and $PBE(R)$.

We also mention:

Any function in $PX(R)$ is a difference of two nonnegative functions in $PX(R)$.

2. The Green's function $G^P(x, y)$ of A^P on R is characterized as the smallest positive function on R such that

$$-A_x^P G^P(x, y) = \delta_y, \tag{5}$$

where δ_y is the Dirac measure. Since $P \geqq 0$ and $\not\equiv 0$, $G_R^P(x, y)$ always exists (cf. e. g. Sario-Nakai [12; Appendix]). This result was obtained by Myrberg [6], who also proved that there always exists a strictly positive solution of $A_x^P u = 0$ on R.

We will call a subregion Ω of R *regular* if the closure $\overline{\Omega}$ of Ω is compact and the relative boundary $\partial\Omega$ of Ω consists of a finite number of disjoint C^∞ hypersurfaces. The Green's function $G_\Omega^P(x, y)$ of A^P on Ω always exists.

Let Q be another C^1 function on R such that $Q \geqq 0$ and $\not\equiv 0$ on R. Consider the integral operator $T_\Omega = T_\Omega^{PQ}$:

$$T_\Omega \varphi = \int_\Omega G_\Omega^Q(\cdot, y)(Q(y) - P(y))\varphi(y)dy \tag{6}$$

for functions φ on Ω such that the integral on the right is defined in the sense of Lebesgue. We also consider $S_\Omega = S_\Omega^{PQ}$:

$$S_\Omega = I_\Omega - T_\Omega, \tag{7}$$

where I_Ω is the identity. If φ is bounded and continuous on Ω, then it is easy to see that $T_\Omega \varphi \in C(\overline{\Omega})$ and

$$(T_\Omega \varphi) \mid \partial\Omega = 0. \tag{8}$$

If φ is bounded and locally uniformly Hölder continuous on Ω, then $T_\Omega \varphi$ is of class C^2 and

$$\Delta T_\Omega \varphi = -(Q - P)\varphi + Q T_\Omega \varphi \tag{9}$$

on Ω(cf. e. *g*. Itô [3], Miranda [5]). Therefore by (8) and Green's formula we deduce

$$D_\Omega(T_\Omega \varphi) = -\int_\Omega T_\Omega \varphi(x) \cdot \Delta_x T_\Omega \varphi(x) dx.$$

By (9) the Fubini theorem implies that

$$D_\Omega(T_\Omega \varphi) = \langle \varphi, \varphi \rangle_\Omega^{PQ} - \int_\Omega Q(x)(T_\Omega \varphi(x))^2 dx \tag{10}$$

where

$$(11)\qquad <\varphi,\psi>_{\Omega}^{PQ}=\int_{\Omega\times\Omega} G_{\Omega}^{Q}(x,y)(Q(x)-P(x))(Q(y)-P(y))\varphi(x)\psi(y)dxdy\,.$$

3. Let $u\in PB(\Omega)$. By (9), $S_{\Omega}u=S_{\Omega}^{PQ}u\in QB(\Omega)$. Since $u-S_{\Omega}u=T_{\Omega}u$, the relation (8) and the maximum principle imply

$$(12)\qquad \|S_{\Omega}u\|_{\Omega}=\|u\|_{\Omega}\,,$$

where $\|\cdot\|_{\Omega}$ is the supremum norm considered on Ω. Let $\overline{S}_{\Omega}=S_{\Omega}^{QP}$. Then $\overline{S}_{\Omega}S_{\Omega}u\in PB(\Omega)$. Since $u-\overline{S}_{\Omega}S_{\Omega}u\in PB(\Omega)$ and $u-\overline{S}_{\Omega}S_{\Omega}u=T_{\Omega}^{PQ}u+T_{\Omega}^{QP}S_{\Omega}u$, the relation (8) implies that $u-\overline{S}_{\Omega}S_{\Omega}u\equiv 0$ on Ω. Therefore

$$(13)\qquad S_{\Omega}^{QP}\circ S_{\Omega}^{PQ}=I_{\Omega}^{Q},\ S_{\Omega}^{PQ}\circ S_{\Omega}^{QP}=I_{\Omega}^{P}\,.$$

We have thus proved that

$S_{\Omega}=S_{\Omega}^{PQ}$ is an isometric isomorphism from the class $PB(\Omega)$ onto the class $QB(\Omega)$.

4. For regular regions $\Omega\subset R$, the classes $PBD(\Omega)$ and $PBE(\Omega)$ are always identical. Observe that

$$(14)\qquad \begin{cases}(D_{\Omega}(S_{\Omega}^{PQ}u))^{1/2}\leqq (D_{\Omega}(u))^{1/2}+(<u,u>_{\Omega}^{PQ})^{1/2},\\ (D_{\Omega}(u))^{1/2}\leqq (D_{\Omega}(S_{\Omega}^{PQ}u))^{1/2}+(<u,u>_{\Omega}^{PQ})^{1/2}\end{cases}$$

for every $u\in PB(\Omega)$. By Green's formula we also deduce

$$(15)\qquad \begin{cases}E_{\Omega}^{Q}(S_{\Omega}^{PQ}u)+E_{\Omega}^{Q}(T_{\Omega}^{PQ}u)=E_{\Omega}^{P}(u)+\displaystyle\int_{\Omega}(Q(x)-P(x))(u(x))^{2}dx\,,\\ E_{\Omega}^{P}(u)+E_{\Omega}^{P}(T_{\Omega}^{PQ}u)=E_{\Omega}^{Q}(S_{\Omega}^{PQ}u)+\displaystyle\int_{\Omega}(P(x)-Q(x))(S_{\Omega}^{PQ}u(x))^{2}dx\,,\end{cases}$$

where $E_{\Omega}^{P}(u)=D_{\Omega}(u)+\int_{\Omega}P(x)(u(x))^{2}dx$. From (14) it follows that

$S_{\Omega}=S_{\Omega}^{PQ}$ is an isometric (with respect to $\|\cdot\|_{\Omega}$) isomorphism from the class $PBD(\Omega)=PBE(\Omega)$ onto the class $QBD(\Omega)=QBE(\Omega)$.

5. We proceed to the comparison of $PX(R)$ and $QX(R)$ for $X=B$, BD, and BE. Consider the integral operator $T=T^{PQ}$:

(16) $$T\varphi = \int_R G^Q(x, y)(Q(y) - P(y))\varphi(y)dy$$

for functions φ on R such that the integral on the right is defined in the sense of Lebesgue. We will say that *the ordered pair* (P, Q) *satisfies the condition* (B) *if*

(B) $$\int_R G^Q(x, y)|Q(y) - P(y)|dy < \infty .$$

By the Harnack inequality (B) is satisfied for every $x \in R$ if and only if (B) is valid for some $x \in R$. In this no. 5 we assume that (P, Q) and (Q, P) satsfy (B). If φ is bounded and continuous on R, then $T\varphi$ is defined and continuous on R. If moreover φ is locally uniformly Hölder continuous, then $T\varphi$ is of class C^2 and

(17) $$A^Q T\varphi = -(Q - P)\varphi$$

on R (cf.(9)). We also consider $S = S^{PQ}$:

(18) $$S = I - T ,$$

where I is the identity operator.

Let $\{\Omega\}$ be a directed set of regular regions Ω such that the union of $\{\Omega\}$ is R. For a continuous function φ_Ω on Ω we use the same notation φ_Ω for the function which is φ_Ω on Ω and 0 on $R - \Omega$. Assume that

$$\|\varphi_\Omega\| = \sup_R |\varphi_\Omega| < k < \infty$$

for every Ω. Moreover suppose there exists a bounded continuous function φ on R such that $\lim_{\Omega\to R}\varphi_\Omega = \varphi$ uniformly on each compact set in R. Then

(19) $$S\varphi = \lim_{\Omega\to R} S_\Omega \varphi_\Omega$$

uniformly on each compact set in R. In fact,

$$\begin{aligned}|S\varphi(x) - S_\Omega\varphi_\Omega(x)| &\leqq |S\varphi(x) - S_\Omega\varphi(x)| + |S_\Omega\varphi(x) - S_\Omega\varphi_\Omega(x)| \\ &\leqq (|T| - |T_\Omega|)|\varphi|(x) + |\varphi(x) - \varphi_\Omega(x)| + |T_\Omega||\varphi - \varphi_\Omega|(x) .\end{aligned}$$

Here $|T|\varphi = \int_R G^Q(., y)|Q(y) - P(y)|\varphi(y)dy$ and $|T_\Omega|$ is similarly defined. Since $G^Q_\Omega(x, y) \leqq G^Q(x, y)$ and $\lim_{\Omega\to R} G^Q_\Omega(x, y) = G^Q(x, y)$ on R, we infer that

$$|S\varphi(x)-S_{\Omega}\varphi_{\Omega}(x)| \leqq (|T|-|T_{\Omega}|)|\varphi|(x)+|\varphi(x)-\varphi_{\Omega}(x)|+|T\|\varphi-\varphi_{\Omega}|(x)$$

and by the Lebesgue convergence theorem the right-hand side of the above inequality converges to 0 on R. By the Harnack inequality applied to $G^{Q}-G^{Q}_{\Omega}$ and G^{Q}, we conclude that the convergence is uniform on each compact set in R. Therefore (19) is established.

6. We will first prove a comparison theorem for $PB(R)$ and $QB(R)$. This result is already suggested in the author's earlier paper [7] (see also[9] and Maeda [4]):

THEOREM 1. *If* (P, Q) *and* (Q, P) *satisfy the condition* (B), *then* S^{PQ} *is an isometric isomorphism of* $PB(R)$ *onto* $QB(R)$.

PROOF. Let $u \in PB(R)$. From (17) it follows that $Su \in Q(R)$. By the identity (12) we deduce $\|S_{\Omega}u\|_{\Omega} = \|u\|_{\Omega} \leqq \|u\|$ and a fortiori

$$\|Su\| \leqq \|u\| , \tag{20}$$

i. e. $Su \in QB(R)$. Suppose $Su=0$. By (13) and (19), $S^{QP}Su=u$ and a fortiari $u\equiv 0$. Thus S is an isomorphism of $PB(R)$ into $QB(R)$.

To prove that S is surjective let $v \in QB(R)$ and $u_{\Omega} = S^{QP}_{\Omega}v$. Observe that $u_{\Omega} \in PB(\Omega)$, $\|u_{\Omega}\|_{\Omega} \leqq \|v\|$, and by (13), $v = S_{\Omega}u_{\Omega}$. Let $\{\Omega\}$ be a directed set of regular subregions Ω such that

$$u = \lim_{\Omega\to R} u_{\Omega} \in PB(R)$$

uniformly on each compact set in R. By (19) we infer that

$$Su = \lim_{\Omega\to R} S_{\Omega}u_{\Omega} = v ,$$

i. e. S is surjective. Since $\|Su\| \geqq \|v\|_{\Omega} = \|S_{\Omega}u_{\Omega}\| = \|u_{\Omega}\|$, we deduce $|Su\| \geqq \|u\|$. This with (20) implies that S is isometric. Q.E.D.

COROLLARY 1.1. *Since* P *satisfies*

$$\int_R G^{P}(x, y)P(y)dy < \infty \tag{21}$$

(*cf.*[4]), $PB(R)$ *and* $(cP)B(R)$ *are isomorphic for* $c>0$.

PROOF. The condition (21) implies that (cP, P) and (P, cP) satisfy the condition (B). Therefore $S^{(cP)P}$ is an isometric isomorphism of $(cP)B(R)$ onto $PB(R)$. Q.E.D.

Royden [11] proved the following comparison theorem entirely different in nature from ours:

If there exists a finite constant $c>1$ such that $c^{-1}Q \leqq P \leqq cQ$ outside a compact set in R, then there exists an isometric isomorphism of $PB(R)$ onto $QB(R)$.

7. We turn to a comparison theorem for $PBD(R)$ and $QBD(R)$. We will say that *the ordered pair* (P, Q) *satisfies the condition* (D) if

$$\text{(D)} \qquad \int_{P\times R} G^Q(x, y)\,|Q(x)-P(x)|\cdot|Q(y)-P(y)|\,dxdy < \infty\,.$$

It is clear that (E) implies (B). In this no. 7 we always assume that (P, Q) and (Q, P) satisfy (D). In accordance with (11) we set

$$(22) \quad <\varphi, \psi>^{PQ} = \int_{R\times R} G^Q(x,y)(Q(x)-P(x))(Q(y)-P(y))\varphi(x)\psi(y)dxdy\,.$$

This is well defined for bounded continuous functions φ and ψ on R. By the Lebesgue convergence theorem we deduce

$$(23) \qquad <\varphi, \psi>^{PQ} = \lim_{\Omega\to R} <\varphi, \psi>_\Omega^{PQ}.$$

THEOREM 2. *If* (P, Q) and (Q, P) *satisfy the conidtion* (D), *then* S^{PQ} *is an isometric isomorphism of* $PBD(R)$ *onto* $QBD(R)$.

PROOF. Since (D) implies (B), Theorem 1 implies that $S=S^{PQ}$ is an isometric isomorphism of $PB(R)$ onto $QB(R)$. Let $u \in PBD(R)$. By (14) we have

$$(24) \qquad (D({}_\Omega S_\Omega u))^{1/2} \leqq (D_\Omega(u))^{1/2} + (<u, u>_\Omega)^{1/2}\,.$$

From (19) for $\varphi = u \in PB(R)$ it follows that

$$(25) \qquad \lim_{\Omega\to R} dS_\Omega u \wedge *dS_\Omega u = dSu \wedge *dSu$$

on R. By (23) and the Fatou lemma, we deduce from (24)

$$(D_R(Su))^{1/2} \leqq (D_R(u))^{1/2} + (<u, u>)^{1/2} < \infty\,.$$

Therefore $S(PBD(R)) \subset QBD(R)$. To obtain the reversed inclusion let $u \in PB(R)$ and $Su \in QBD(R)$. Since $u = Su + Tu$ on R,

(26) $$(D_R(u))^{1/2} \leqq (D_R(Su))^{1/2} + (D_R(Tu))^{1/2}.$$

By (25), $|\mathrm{grad}\ T_\Omega u|^2$ converges to $|\mathrm{grad}\ Tu|^2$ on R. By the Fatou lemma and the relations (10) and (23), we infer that

$$\begin{aligned} D_\Omega(Tu) &\leqq \liminf_{\Omega\to R} D_\Omega(T_\Omega u) \\ &\leqq \lim_{\Omega\to R} \langle u, u\rangle_\Omega = \langle u, u\rangle < \infty . \end{aligned}$$

From (26) it follows that $D_R(u) < \infty$, i.e. $S(PBD(R)) = QBD(R)$. Q.E.D.

COROLLARY 2.1. *If P satisfies*

(27) $$\int_R G^P(x, y)P(x)P(y)dxdy < \infty ,$$

then $PBD(R)$ and $(cP)BD(R)$ are isomorphic for $c > 0$.

PROOF. The condition (27) implies that (cP, P) and (P, cP) satisfy the condition (D). Therefore $S^{(cP)P}$ is an isometric isomorphism of $(cP)BD(R)$ onto $PBD(R)$. Q.E.D.

8. We turn to a comparison theorem for $PBE(R)$ and $QBE(R)$. We will say that *the ordered pair (P, Q) satisfies the condition* (E) if

(E) $$\int_R |Q(x) - P(x)|dx < \infty .$$

It is clear that (E) implies (B). The following comparison theorem was obtained by [11] (see also Glasner-Katz [1]):

THEOREM 3. *If (P, Q) satisfies the condition (E), then S^{PQ} is an isometric isomorphism of $PBE(R)$ onto $QBE(R)$.*

PROOF. Since (E) implies (B), Theorem 1 entails that $S = S^{PQ}$ is an isometric isomorphism of $PB(R)$ onto $QB(R)$. Let $u \in PBE(R)$. From (15) it follows that

$$E_\Omega^Q(S_\Omega u) \leqq E_\Omega^P(u) + \|u\|^2 \int_\Omega |Q(x) - P(x)|dx .$$

By (25) and the Fatou lemma, we obtain

$$E_R^Q(Su) \leqq E_R^P(u) + \|u\|^2 \int_R |Q(x) - P(x)| dx < \infty ,$$

i. e. $S(PBE(R)) \subset QBE(R)$. Conversely let $u \in PB(R)$ and $Su \in QBE(R)$. By (15) and $\|S_\Omega u\| = \|u\|$, we have

$$E_\Omega^P(u) \leqq E_\Omega^Q(S_\Omega u) + \|u\|^2 \int_\Omega |Q(x) - P(x)| dx .$$

On setting $S_\Omega u = u$ on $R-\Omega$ we infer by Green's formula that

$$E_\Omega^Q(S_\Omega u - S_{\Omega'} u) = E_\Omega^Q(S_\Omega u) - E_{\Omega'}^Q(S_{\Omega'} u)$$

for $\Omega' \supset \Omega$. Therefore $E_\Omega^Q(S_\Omega u) \to E_R^Q(Su)$ as $\Omega \to R$, and a fortiori

$$E_R^P(u) \leqq E_R^Q(Su) + \|u\|^2 \int_\Omega |Q(x) - P(x)| dx < \infty .$$

We have shown that $S(PBE(R)) = QBE(R)$. Q.E.D.

COROLLARY 3.1. *If P satisfies*

$$\int_R P(x) dx < \infty , \tag{28}$$

then $PBE(R)$ and $(cP)BE(R)$ are isomorphic for $c > 0$.

PROOF. The condition (28) implies that (cP, P) and (P, cP) satisfy the condition (E). Therefore $S^{(cP)P}$ is an isometric isomorphism of $(cP)BE(R)$ onto $PBE(R)$. Q.E.D.

9. As usual we denote by $H(R)$ the space of harmonic functions u on R, i. e. $\Delta u = 0$. Comparison theorems between $PX(R)$ and $HX(R)$ for $X = B$, BD, and BE can be obtained on replacing Q by 0 in nos. 1—8. We will denote by $G(x, y) = G_R(x, y)$ the harmonic Green's function on R. If $R \in O_G$, then $PB(R) = \{0\}$ (Ozawa [10], Royden [11]). Therefore excluding trivial cases, we assume in this no. 9 that $R \notin O_G$. We will say that P satisfies the condition (B_0), (D_0), or (E_0) if

(B$_0$) $$\int_R G(x,y)P(y)dy < \infty\,,$$

(D$_0$) $$\int_{R\times R} G(x,y)P(x)P(y)dxdy < \infty$$

or

(E$_0$) $$\int_R P(x)dx < \infty\,.$$

Since $G^P(x,y)<G(x,y)$, the conditions (B$_0$), (D$_0$), and (E$_0$) imply(21), (27), and (28), respectively.

Dicussions in no. 6 are valid if Q is replaced by 0:

COROLLARY 1.2. *If P satisfies the condition* (B$_0$), *then S^{P0} is an isometric isomorphism of $PB(R)$ onto $HB(R)$.*

The replacement of Q by 0 does not affect the validity of the reasoning in nos. 7 and 8. With this in view we maintain:

COROLLARY 2.2. *If P satisfies the condition* (D$_0$), *then S^{P0} is an isometric isomorphism of $PBD(R)$ onto $HBD(R)$.*

COROLLARY 3.2. *If P satisfies the condition* (E$_0$), *then S^{P0} is an isometric isomorphism of $PBE(R)$ onto $HBD(R)$.*

Equations on Euclidean spaces.

10. Hereafter we take the Euclidean space $E^m(m\geqq 3)$ as the base Riemannian manifold for the equation $\Delta u=Pu$. We fix an orthogonal coordinate so that the metric tensor is (δ_{ij}). For a point $x\in E^m$, its coordinate will be denoted by $(x^1,\cdots,x^m)$. The volume element is thus $dx=dx^1\cdots dx^m$. We also write $|x|=\left(\sum_{i=1}^m (x^i)^2\right)^{1/2}$.

The harmonic Green's function $G(x,y)$ on E^m is given by

$$c_m G(x,y) = |x-y|^{2-m}\,, \tag{29}$$

where $c_m=(m-2)\omega_m$ with ω_m the surface area $2\pi^{m/2}/\Gamma(m/2)$ of the unit ball in E^m. We first observe the following elementary identity (a special case of the Riesz composition theorem):

$$\int_{E^m} G(x,y)|y|^{-\alpha}dy = a|x|^{-(\alpha-2)}(m > \alpha > 2)\,, \tag{30}$$

where $a = a(m,\alpha)$ is a finite strictly positive constant depending on m and α but not on $x \neq 0$.

In fact let $z = \Lambda(y)$ be an affine transformation of E^m given by

$$z^i = \Lambda^i(y) = |x|^{-1} \sum_{j=1}^{m} p_{ij}(y^j - x^j)(i = 1,\cdots,m) \tag{31}$$

where (p_{ij}) is an orthonormal matrix such that

$$\delta^{1i} = -\sum p_{ij}|x|^{-1}x^j(i = 1,\cdots,m)\,. \tag{32}$$

From (31) and (32) it follows that

$$|y-x| = |x|\,|z|,\ |y| = |x|\,|z-e| \tag{33}$$

with $e = (1,0,\cdots,0)$. The Jacobian of Λ is

$$J = \det\left(\frac{\partial z^i}{\partial y^j}\right) = \det(|x|^{-1}p_{ij}) = |x|^{-m}$$

and therefore $dz = |x|^{-m}dy$. Hence

$$\begin{aligned}\int_{E^m} G(x,y)|y|^{-\alpha}dy &= c_m^{-1}\int_{E^m} |x-y|^{2-m}|y|^{-\alpha}dy \\ &= c_m^{-1}\int_{E^m} |x|^{2-m}|z|^{2-m}\cdot|x|^{-\alpha}|z-e|^{-\alpha}\cdot|x|^m dz \\ &= a|x|^{-(\alpha-2)},\end{aligned}$$

where

$$a = c_m^{-1}\int_{E^m} |z|^{2-m}|z-e|^{-\alpha}dz < \infty$$

if $\alpha > 2$.

11. Let $\lambda(t)$ be a real-valued C^2 function on $[0,\infty)$ such that $\frac{d}{dt}\lambda(t) \geqq 0, \frac{d^2}{dt^2}\lambda(t) \geqq 0$, $\lambda(t) \geqq t$, and

$$\begin{cases}\lambda(t) \equiv \varepsilon & (t \in [0, \varepsilon/2]) , \\ \lambda(t) \equiv t & (t \in , [\varepsilon+\delta, \infty)) ,\end{cases} \tag{34}$$

where ε and δ are arbitrarily fixed positive number. Consider the equation

$$\Delta u(x) = Q_\alpha(x)u(x),\ Q_\alpha(x) = \lambda(|x|)^{-\alpha} \tag{35}$$

where $\alpha \in (-\infty, \infty)$ and $\Delta\cdot = \sum_{i=1}^{m} \frac{\partial^2 \cdot}{\partial x^{i2}}$. We maintain:

$$\dim Q_\alpha B(E^m) \leqq 1 \tag{36}$$

for every $\alpha \in (-\infty, \infty)$.

For the proof let $\dim Q_\alpha B(E^m) > 0$. Take two positive functions u_i in $Q_\alpha B(E^m) (i = 1, 2)$. Let $\Omega(n) = \{x \in E^m \mid |x| < n\} (n = 1, 2, \cdots)$ and $S_n = S_{\Omega(n)}^{Q\alpha 0}$, $S = S_R^{Q\alpha 0}$. Then

$$S_n u_i(x) = u_i(x) + \int_{\Omega(n)} G_{\Omega(n)}(x, y) Q_\alpha(y) u_i(y) dy .$$

Observe that $S_n u_i \in HB(\Omega(u))$ and $\|S_n u_i\|_{\Omega(n)} = \|u_i\|_{\Omega(n)} \leqq \|u_i\|$. Since $u_i > 0$, we obtain by the Lebesgue-Fatou convergence theorem that

$$Su_i(x) = u_i(x) + \int_{E^m} G(x, y) Q_\alpha(y) u_i(y) dy \tag{37}$$

and $Su_i \in HB(E^m)$. Since

$$HB(E^m) = E^1 , \tag{38}$$

$Su_i \equiv c_i > 0$. Set $w = c_2 u_1 - c_1 u_2 \in Q_\alpha B(E^m)$. Then by (37)

$$w(x) = -\int_{E^m} G(x, y) Q_\alpha(y) w(y) dy = -(Tw)(x)$$

and consequently $|w| \leqq T|w|$ on E^m. Since $|w|$ is subharmonic and $T|w|$ is a potential, we obtain $|w| \equiv 0$. Thus u_1 and u_2 are linearly dependent. The space $Q_\alpha B(E^m)$ is generated by positive functions in $Q_\alpha B(E^m)$. We conclude that $\dim Q_\alpha B(E^m) = 1$.

12. We have seen that either $\dim Q_\alpha B(E^m) = 0$ or 1. We next study for what α the first or the second alternative occurs. Let $\omega = (\omega_{ij})$ be an orthonormal matrix and f_ω be the function defined by $f_\omega(x) = f(x\omega)$ for a given function f on E^m. Here x is viewed as the matrix of type $(1, m)$. Since $(Q_\alpha)_\omega = Q_\alpha$, *rotation free*, we conclude that $u_\omega \in Q_\alpha B(E^m)$ for $u \in Q_\alpha B(E^m)$. Because of (36), we must have $u = u_\omega$ for every ω. Therefore:

Every function $u \in Q_\alpha B(E^m)$ is rotation free.

A fortiori there exists a C^2 function $\varphi_u(t)$ on $[0, \infty]$ such that

$$u(x) = \varphi_u(|x|). \tag{39}$$

Suppose $\dim Q_\alpha B(E^m) = 1$. Then for $u \in Q_\alpha B(E^m)$ such that $u > 0$ we maintain:

$$\liminf_{|x| \to \infty} u(x) > 0. \tag{40}$$

If this were not the case, there would exist an increasing divergent sequence $\{r_n\} \subset E^m$ such that $\varphi_u(r_n) \to 0$ as $n \to \infty$. Let $\Omega(r_n) = \{x \in E^m \mid |x| < r_n\}$. The maximum principle implies that $\|u\|_{\Omega(r_n)} = \varphi_u(r_n)$ and a fortiori $u \equiv 0$, a contradiction.

By (37)

$$Su(x) = u(x) + \int_{E^m} G(x,y) Q_\alpha(y) u(y) dy.$$

Since (40) and the maximum principle imply that $\inf_{E^m} u = b > 0$,

$$\int_{E^m} G(x, y) Q_\alpha(y) \mathrm{d}y \leqq b^{-1}(Su(x) - u(x)) < \infty,$$

i.e. Q_α satisfies the condition (B_0). Conversely if Q_α satisfies the condition (B_0), then by Corollary 1.2, $Q_\alpha B(E^m)$ is isomorphic to $HB(E^m)$ and therefore $\dim Q_\alpha B(E^m) = 1$.

We have shown that $(E^m, Q_\alpha) \in \mathcal{O}_{PB}$ is equivalent to

$$c_x = c_m \int_{E^m} G(x, y) Q_\alpha(y) dy = \infty. \tag{41}$$

Clearly there exists a constant $d_x>1$ such that

$$d_x^{-1}c_x \leqq e = \int_{|y|>\varepsilon+\delta} \frac{1}{|y|^{m-2}}\cdot\frac{1}{|y|^{\alpha}}dy \leqq d_x c_x .$$

By using the polar coordinate we infer that $e=c_m\int_{\varepsilon}^{\infty} r^{-(\alpha-1)}dr=\infty$ if and only if $(\alpha-1)\leqq 1$, i.e. $\alpha\leqq 2$.

The conclusion of this no. 12 is:

$$(E_m, Q_\alpha)\in \mathcal{O}_{PB}\ (\alpha\leqq 2), \qquad (E^m, Q_\alpha)\notin \mathcal{O}_{PB}\ (\alpha>2). \tag{42}$$

13. Since $Q_\alpha BD(E^m)\subset Q_\alpha B(E^m)$, (36) implies that either dim $Q_\alpha BD(E^m)=0$ or 1. Suppose the latter alternative is the case. Let $u>0$ be the generator of $Q_\alpha BD(E^m)$. From (37) it follows that

$$u(x) = c-\int_{E^m} G(x,y)Q_\alpha(y)u(y)dy, \tag{43}$$

where $c\in E^1$. Let $\Omega(n)=\{x\in E^m \mid |x|<n\}$ and $G_n=G_{\Omega(n)}$. Since $u|\partial\Omega(n)=c_n$, a constant, we also have

$$u(x) = c_n-\int_{\Omega(n)} G_n(x,y)Q_\alpha(y)u(y)dy .$$

By (10), we infer

$$D_{\Omega(n)}(u) = \int_{\Omega(n)\times\Omega(n)} G_n(x,y)Q_\alpha(x)Q_\alpha(y)u(x)u(y)dxdy .$$

Since the integrand is nonnegative and converges increasingly to $G(x,y)Q_\alpha(x)Q_\alpha(y)\times u(x)u(y)$ on $E^m\times E^m$, the Lebesgue-Fatou theorem yields

$$D_{E^m}(u) = \int_{E^m\times E^m} G(x,y)Q_\alpha(x)Q_\alpha(y)u(x)u(y)dxdy . \tag{44}$$

As in no. 12, $\inf_{E^m}u=b>0$. Thus

$$\int_{E^m\times E^m} G(x,y)Q_\alpha(x)Q_\alpha(y)dxdy \leqq b^{-2}D_{E^m}(u)<\infty ,$$

i. e. Q_α satisfies the condition (D_0). Conversely if Q_α satisfies the condition (D_0), then by Corollary 2.2, $Q_\alpha BD(E^m)$ is isomorphic to $HBD(E^m)$. A fortiori dim $Q_\alpha BD(E^m)=1$.

We have seen that $(E^m, Q_\alpha)\in \mathcal{O}_{PBD}=\mathcal{O}_{PD}$ is equivalent to

$$c = c_m\int_{E^m\times E^m} G(x,y)Q_\alpha(x)Q_\alpha(y)dxdy = \infty\,. \tag{45}$$

In view of (42) and the relation $\mathcal{O}_{PB}\subset\mathcal{O}_{PD}$, we only have to consider the case $\alpha>2$. Clearly there exists a constant $d>1$ such that

$$d^{-1}c\leqq l = c_m\int_{(E^m x - V)\times E^m y} G(x,y)Q_\alpha(x)\,|y|^{-\alpha}dxdy\leqq dc\,,$$

where $V=\{|x|\leqq\varepsilon+\delta\}$. Let c_m be as in no. 10. Assume $\alpha<m$. By (30),

$$l = c_m\int_{E^m-V}\left(\int_{E^m} G(x,y)\,|y|^{-\alpha}dy\right)Q_\alpha(x)dx = \alpha c_m\int_{E^m-V}|x|^{-(\alpha-2)}\cdot\,|x|^{-\alpha}dx$$
$$= \alpha c_m^2\int_{\varepsilon+\delta}^{\infty} r^{-2\alpha+m+1}dr\,.$$

The condition $l=\infty$ is then equivalent to $-2\alpha+m+1\geqq -1$, i. e. $\alpha\leqq(m+2)/2$ for $\alpha<m$. Clearly $l<\infty$ for $\alpha\geqq m$.

The conclusion of this no. 13 is:

$$(E^m, Q_\alpha)\in\mathcal{O}_{PBD}\ (\alpha\leqq(m+2)/2),\ \ (E^m, Q_\alpha)\notin\mathcal{O}_{PBD}\ (\alpha>(m+2)/2)\,. \tag{46}$$

14. Since $Q_\alpha BE(E^m)\subset Q_\alpha B(E^m)$, (36) implies that either dim $Q_\alpha BE(E^m)=0$ or 1. Suppose that the latter is the case. Let $u>0$ be the generator of $Q_\alpha BE(E^m)$. Recall that $\inf_{E^m}u=b>0$ (no. 12). Since

$$E_{E^m}^{Q_\alpha}(u) = D^{E^m}(u) + \int_{E^m} Q_\alpha(x)(u(x))^2dx\,,$$

we infer that

$$\int_{E^m} Q_\alpha(x)dx\leqq b^{-2}E_{E^m}^{Q_\alpha}(u)<\infty\,,$$

i. e. Q_α satisfies the condition (E_0). Conversely if Q_α satisfies the condition (E_0), then by Corollary 3. 2, $Q_\alpha BE(E^m)$ is isomorphic to $HBD(E^m)$. A fortiori dim $Q_\alpha BE(E^m)=1$.

We have seen that $(E^m, Q_\alpha)\in \mathcal{O}_{PBE}=\mathcal{O}_{PE}$ is equivalent to

$$c=\int_{E^m} Q_\alpha(x)dx=\infty\,. \tag{47}$$

Let $V=\{x \mid |x|\leqq \varepsilon+\delta\}$. Clearly there exists a constant $d>1$ such that

$$d^{-1}c<p=\int_{E^m-V} Q_\alpha(x)dx<c\,.$$

Using c_m in no. 10, we deduce

$$p=\int_{E^m-V} |x|^{-\alpha}dx=c_m\int_{\varepsilon+\delta}^{\infty} r^{-\alpha+m-1}dr$$

and therefore $p=\infty$ if and only if $-\alpha+m-1\geqq -1$, i.e. $\alpha\leqq m$.

The conclusion of this no. 14 is:

$$(E^m, Q_\alpha)\in \mathcal{O}_{PBE}\ (\alpha\leqq m), \qquad (E^m, Q_\alpha)\notin \mathcal{O}_{PE}\ (\alpha>m). \tag{48}$$

15. From the results obtained in nos. 10-14, we have the following strict inclusion relations:

$$\mathcal{O}_G<\mathcal{O}_{PB}<\mathcal{O}_{PD}=\mathcal{O}_{PBD}<\mathcal{O}_{PE}=\mathcal{O}_{PBE} \tag{49}$$

where $\mathfrak{A}<\mathfrak{B}$ means that $\mathfrak{A}$ is a proper subset of $\mathfrak{B}$. It is perhaps more or less trival to merely establish the strict inclusions in (49) but we are interested in this paper in giving a unified way for finding counter examples. The strict inclusion $\mathcal{O}_G<\mathcal{O}_{PB}$ was remarked by Royden [11] for $m=2$. Glasner-Katz-Nakai [2] remarked $\mathcal{O}_{PB}<\mathcal{O}_{PD}$ for $m\geqq 2$ except for $m=3$.

16. We next study the equation

$$\Delta u(x)=P_\alpha(x)u(x),\ P_\alpha(x)\sim |x|^{-\alpha}(|x|\to\infty) \tag{50}$$

on $E^m(m\geqq 3)$. Here $P_\alpha(x)\sim|x|^{-\alpha}(|x|\to\infty)$ means that there exist positive constants $c>1$ and $\rho>1$ such that

(51) $$c^{-1}|x|^{-\alpha} \leqq P_\alpha(x) \leqq c|x|^{-\alpha}(|x| \geqq \rho).$$

Thus $P_\alpha(x)$ is "almost rotation free." We are assuming that $P_\alpha(x)$ is of class C^1 and $P_\alpha(x) \geqq 0$ on E^m.

THEOREM 4. *The following degeneracy relations are valid*

(52) $$(E^m, P_\alpha) \in \mathcal{O}_{PB} - \mathcal{O}_G \textit{ for every } \alpha \in (-\infty, 2];$$

(53) $$(E^m, P_\alpha) \in \mathcal{O}_{PB} - \mathcal{O}_{PD} \textit{ for every } \alpha \in (2, (m+2)/2];$$

(54) $$(E^m, P_\alpha) \in \mathcal{O}_{PE} - \mathcal{O}_{PD} \textit{ for every } \alpha \in ((m+2)/2, m];$$

(55) $$(E^m, P_\alpha) \notin \mathcal{O}_{PE} \textit{ for every } \alpha \in (m, \infty).$$

PROOF. Since $m \geqq 3$, E^m always carries the harmonic Green's function given by (29). Therefore $(E^m, P_\alpha) \notin \mathcal{O}_G$ for every $\alpha \in E^1$. Observe that there exist some positive constants $c > 1$ and $\rho > 1$ such that

(56) $$c^{-1}Q_\alpha(x) \leqq P_\alpha(x) \leqq cQ_\alpha(x)$$

on $\Lambda(\rho) = \{x \in E^m \mid |x| > \rho\}$. In particular

(57) $$P_\alpha(x) \leqq cQ_\alpha(x)$$

everywhere on E^m. By Royden's comparison theorem referred to in no. 6,

(58) $$\dim P_\alpha B(E^m) = \dim Q_\alpha B(E^m).$$

Therefore (42) implies (52) and a half of (53), i. e. $(E^m, P_\alpha) \notin \mathcal{O}_{PB}$ for $\alpha > 2$.

Hereafter we always assume $\alpha > 2$. Then $\dim P_\alpha B(E^m) = \dim Q_\alpha B(E^m) = \dim (cQ_\alpha)B(E^m)$. Let p_α and q_α be positive generators of $P_\alpha B(E^m)$ and $(cQ_\alpha)B(E^m)$ respectively. We set $S = S^{P_\alpha(cQ_\alpha)}$. Since

(59) $$|P_\alpha(x) - cQ_\alpha(x)| \leqq (c-1)Q_\alpha(x)$$

and $\int_{E^m} G(x,y)Q_\alpha(x)dy < \infty$ for $\alpha > 2$, (P_α, cQ_α) satisfies (B) and a fortiori S is an isometric isomorphism of $P_\alpha B(E^m)$ onto $(cQ_\alpha)B(E^m)$. We may assume

$$q_\alpha = Sp_\alpha = p_\alpha - \int_{E^m} G^{cQ_\alpha}(\cdot, y)(cQ_\alpha(y) - P_\alpha(y))p_\alpha dy < p_\alpha.$$

Observe that q_α is rotation free and thus the maximum principle implies $\inf_{E^m} q_\alpha > 0$ (see (40)). Therefore

$$\inf_{E^m} p_\alpha = d > 0. \tag{60}$$

If $\alpha \in (2, (m+2)/2]$, then by (44)

$$D_{E^m}(p_\alpha) = \int_{E^m \times E^m} G(x, y) P_\alpha(x) P_\alpha(y) p_\alpha(x) p_\alpha(y) \; dxdy$$
$$\geqq d^2 \int_{E^m \times E^m} G(x, y) P_\alpha(x) P_\alpha(y) dxdy.$$

If $D_{E^m}(p_\alpha)$ were finite, then (56) would imply that

$$\int_{E^m \times E^m} G(x, y) Q_\alpha(x) Q_\alpha(y) dxdy < \infty.$$

This contradicts (46). A fortiori $(E^m, P_\alpha) \in \mathcal{O}_{PD}$ for $\alpha \in (2, (m+2)/2]$. This establishes (53).

Let $\alpha \in ((m+2)/2, m]$. From(46), (59), and no. 7, it follows that $(E^m, P_\alpha) \notin \mathcal{O}_{PD}$. Suppose $E_{E^m}^{P_\alpha}(p_\alpha) < \infty$. Then the relation

$$E_{E^m}^{P_\alpha}(p_\alpha) > \int_{E^m} P_\alpha(x) (p_\alpha(x))^2 dx \geqq d^2 \int_{E^m} P_\alpha(x) dx$$

and (56) imply $\int_{E^m} Q_\alpha(x) dx < \infty$, in violation of (48). The relation (54) is thus proved.

Finally if $\alpha \in (m, \infty)$, then (48), (59), and no. 8 imply the assertion (55). Q. E. D.

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DEPARTMENT OF MATHEMATICS
NAGOYA UNIVERSITY
NAGOYA, JAPAN