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MINIMAL SUBMANIFOLDS WITH M-INDEX 2 IN RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE

TOMINOSUKE OTSUKI

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For a submanifold M in a Riemannian manifold $\bar{M}$, the minimal index (M-index) at a point of M is defined by the dimension of the linear space of all 2nd fundamental forms with vanishing trace. The geodesic codimension of M in $\bar{M}$ is defined by the minimum of codimensions of M in totally geodesic submanifolds of $\bar{M}$ containing M.

It is clear in general that for M in $\bar{M}$

$$M\text{-index} \leqq \text{geodesic codimension}.$$

In [7], the author investigated minimal submanifolds with M-index 2 in Riemannian manifolds of constant curvature and gave some typical examples of such submanifolds with geodesic codimension 3 in the space forms which is quite analogous to the case of helicoids in E^3 when $\bar{M}$ is Euclidean. In the present paper, he will give some examples of such submanifolds with geodesic codimension 4 in the space forms. In the previous case, the base surface (analogous to the helix for a helicoid) must be locally flat, but in the present case it must be of positive constant curvature.

We will use the notations in [7].

1. Preliminaries. Let $\bar{M}=\bar{M}^{n+\nu}$ be a Riemannian manifold of dimension $n+\nu$ and of constant curvature $\bar{c}$ and $M=M^n$ be an n-dimensional submanifold in $\bar{M}$. Let $\bar{\omega}_A$, $\bar{\omega}_{AB}=-\bar{\omega}_{BA}$, $A,B=1,2,\cdots,n+\nu$, be the basic and connection forms of $\bar{M}$ on the orthonormal frame bundle $F(\bar{M})$ which satisfy the structure equations

$$(1.1) \qquad d\bar{\omega}_A=\sum_B \bar{\omega}_{AB}\wedge {}_B\bar{\omega},\quad d\bar{\omega}_{AB}=\sum_C \bar{\omega}_{AC}\wedge\bar{\omega}_{CB}-\bar{c}\,\bar{\omega}_A\wedge\bar{\omega}_B\,.$$

Let B be the subbundle of $F(\bar{M})$ over M such that $b=(x,e_1,\cdots,e_n,\cdots,e_{n+\nu})\in F(\bar{M})$ and $(x,e_1,\cdots,e_n)\in F(M)$, where $F(M)$ is the orthonormal frame bundle of M with the induced Riemannian metric from $\bar{M}$, then deleting the bars of $\bar{\omega}_A$, $\bar{\omega}_{AB}$ on B, we have

$$(1.2) \qquad \omega_\alpha=0,\ \omega_{i\alpha}=\sum_j A_{\alpha ij}\omega_j,\ A_{\alpha ij}=A_{\alpha ji}\quad \alpha=n+1,\cdots,n+\nu;\ i,j=1,2,\cdots,n.$$

For any point $x \in M$, let N_x be the normal space to $M_x = T_x M$ in $\bar{M}_x = T_x \bar{M}$. For any $b \in B$, let φ_b be a linear mapping from N_x into the set of all symmetric matrices of order n defined by

$$\varphi_b\left(\sum_\alpha v_\alpha e_\alpha\right) = \sum_\alpha v_\alpha A_\alpha, \; A_\alpha = (A_{\alpha ij}) .$$

Now, we suppose that M is minimal in $\bar{M}$ and of M-index 2 at each point. Then, N_x is decomposed as

$$N_x = N'_x + O_x , \quad N'_x \perp O_x ,$$

where $O_x = \varphi_b^{-1}(0)$ and $\dim N'_x = 2$, which does not depend on the choice of b over x and is smooth with respect to x. Let B_1 be the set of b such that e_{n+1}, $e_{n+2} \in N'_x$. By means of Lemma 1 in [7], on B_1 we have

$$\omega_{n+1,\beta} \equiv \omega_{n+2,\beta} \equiv 0 \pmod{\omega_1, \cdots, \omega_n} \quad (\beta > n+2) .$$

Then, for any $v \in N'_x$, we can define a linear mapping $\psi_v : M_x \to O_x$ by

$$\psi_v(X) = \sum_{\beta > n+2} < v, \; e_{n+1}\omega_{n+1,\beta}(X) + e_{n+2}\omega_{n+2,\beta}(X) > e_\beta . \tag{1.3}$$

The mapping $\psi : M_x \times N'_x \to O_x$, $\psi(X, v) = \psi_v(X)$, may be called the 1*st torsion operator* of M in $\bar{M}$. According to Lemmas 1, 2 and Theorem 1 in [7], we have

THEOREM A. *Let M^n be minimal and of M-index 2 everywhere in $\bar{M}^{n+\nu}$ of constant curvature. Then we have the following*:

(i) *M^n is of geodesic codimension 2 if and only if $\psi \equiv 0$.*

(ii) *If $\psi \neq 0$ everywhere, then $\dim \mathfrak{l}_x = n-2$, where $\mathfrak{l}_x$ is the space of relative nullity of M^n in $\bar{M}^{n+\nu}$ at x, $\psi_v(\mathfrak{l}_x) = 0$ for any $v \in N'_x$ and ψ_v, $v \neq 0$, has a common image $\psi_v(M_x)$ whose dimension $\leqq 2$.*

When $\psi \neq 0$ at $x \in M$, we decompose M_x as

$$M_x = \mathfrak{W}_x + \mathfrak{l}_x, \qquad \mathfrak{W}_x \perp \mathfrak{l}_x .$$

We can choose frames $b \in B_1$ such that $e_1, e_2 \in \mathfrak{W}_x$, $e_3, \cdots, e_n \in \mathfrak{l}_x$ and

$$\begin{cases} \omega_{1,n+1} = \lambda\omega_1, \; \omega_{2,n+1} = -\lambda\omega_2, \; \omega_{3,n+1} = \cdots = \omega_{n,n+1} = 0 , \\ \omega_{1,n+2} = \mu\omega_2, \; \omega_{2,n+2} = \mu\omega_1, \; \omega_{3,n+2} = \cdots = \omega_{n,n+2} = 0 , \\ \omega_{i\beta} = 0, \; i = 1, \cdots, n; \; \beta > n+2, \; \lambda \neq 0, \; \mu \neq 0 \end{cases} \tag{1.4}$$

and then (1.3) can be written as

$$\psi_v(X) = \left\{\frac{1}{\lambda} < v, e_{n+1} > \omega_1(X) - \frac{1}{\mu} < v, e_{n+2} > \omega_2(X)\right\}F + \left\{\frac{1}{\lambda} < v, e_{n+1} > \omega_2(X) + \frac{1}{\mu} < v, e_{n+2} > \omega_1(X)\right\}G, \tag{1.5}$$

where $F = \sum_{\gamma>n+2} f_\gamma e_\gamma$ and $G = \sum_{\gamma>n+2} g_\gamma e_\gamma$ and

$$\lambda\omega_{n+1,\gamma} + i\mu\omega_{n+2,\gamma} = (f_\gamma + ig_\gamma)(\omega_1 - i\omega_2), \quad \gamma > n+2. \tag{1.6}$$

$\psi \neq 0$ implies $F \neq 0$ or $G \neq 0$.

Now, supposing $\psi \neq 0$ everywhere, we denote the set of $b \in B_1$ satisfying (1.4) by B_2. On B_2, we have

$$\omega_{1r} + i\omega_{2r} = (p_r + iq_r)(\omega_1 + i\omega_2), \ 2 < r \leqq n. \tag{1.7}$$

The vector fields $P = \sum_{r=3}^{n} p_r e_r$ and $Q = \sum_{r=3}^{n} q_r e_r$ of M are called the *principal* and *subprincipal asymptotic vector fields*, respectively. According to Lemmas 3, 4 and Theorem 2 in [7], we have

THEOREM B. *Let M^n be minimal and of M-index 2 everywhere in $\bar{M}^{n+v}$ of constant curvature $\bar{c}$. Supposing the 1st torsion operator $\psi \neq 0$ everywhere, we have*:

(1) *The distribution $\mathfrak{l} = \{\mathfrak{l}_x, x \in M^n\}$ is completely integrable and its integral submanifolds are totally geodesic in $\bar{M}^{n+v}$.*
(2) *The distribution $\mathfrak{W} = \{\mathfrak{W}_x, x \in M^n\}$ is completely integrable if and only if $Q \equiv 0$.*
(3) *When $Q \equiv 0$, the integral surfaces of $\mathfrak{W}$ are totally umbilic in M^n.*
(4) *When $P \neq 0$ and $Q \equiv 0$, the integral curves of the vector field P are geodesics in $\bar{M}^{n+v}$.*

Under the conditions of Theorem B and $Q \equiv 0$, on B_2 we have

$$\{d\log\lambda - < P, dx > - i(2\omega_{12} - \sigma\hat{\omega}_1)\} \wedge (\omega_1 + i\omega_2) = 0, \tag{1.8}$$

$$\{d\sigma + i(1 - \sigma^2)\hat{\omega}_1\} \wedge (\omega_1 + i\omega_2) = 0, \tag{1.9}$$

$$d\omega_{12} = - \{\|P\|^2 + \bar{c} - \lambda^2 - \mu^2\}\omega_1 \wedge \omega_2, \tag{1.10}$$

(1.11) $$d\hat{\omega}_1 = -\frac{1}{\lambda\mu}\{2\lambda^2\mu^2 - \|F\|^2 - \|G\|^2\}\omega_1\wedge\omega_2\,,$$

where $\sigma=\mu/\lambda$, $\hat{\omega}_1=\omega_{n+1,n+2}$. $\hat{\omega}_1$ is the connection form of the vector bundle $N'=\cup N'_x$, $x\in M^n$, and $<P, dx> = \sum_{r=3}^{n} <P, e_r>\omega_r$. In this case, we denote the set of frames $b\in B_2$ such that $P=p\, e_3$, $p>0$, by B_3. On B_3 we have

(1.12) $$\omega_{a3}=p\omega_a,\ \omega_{at}=0,\ p\omega_{3t}=\bar{c}\omega_t,\quad a=1, 2;\ \ 3<t\leqq n.$$

According to Lemmas 7, 8, 9, 10 and Theorem 3 in [7], we have

THEOREM C. *Let $M^n(n\geqq 3)$ be a maximal minimal submanifold in an $(n+\nu)$-dimensional space form $\bar{M}^{n+\nu}$(of constant curvature $\bar{c}$) which is of M-index 2 and whose torsion operator $\psi\neq 0$, principal asymptotic vector field $P\neq 0$ everywhere and subprincipal asymptotic vector field $Q\equiv 0$, then it is a locus of $(n-2)$-dimensional totally geodesic subspaces $L^{n-2}(y)$ in $\bar{M}^{n+\nu}$ through points y of a base surface W^2 lying in a Riemannian hypersphere in $\bar{M}^{n+\nu}$ with center z_0 such that*

(i) *$L^{n-2}(y)$ intersects orthogonally with W^2 at y and contains the geodesic radius from z_0 to y.*

(ii) *The $(n-3)$-dimensional tangent spaces to the intersection of $L^{n-2}(y)$ and the hypersphere at y are parallel along W^2 in $\bar{M}^{n+\nu}$.*

W^2 in this theorem is an integral surface of the distribution $\mathfrak{W}$ and the geodesic radius from z_0 to y is the integral curve of P.

Denoting the length along geodesic rays starting at z_0 measured from z_0 by v, we have

(1.13) $$\omega_3=-dv$$

and

(1.14) $$p=\begin{cases}\sqrt{\bar{c}}\cot\sqrt{\bar{c}}\,v & (\bar{c}>0),\\ 1/v & (\bar{c}=0),\\ \sqrt{-\bar{c}}\coth\sqrt{-\bar{c}}\,v & (\bar{c}<0).\end{cases}$$

2. The 2nd torsion oberator ψ'. In the following, we shall investigate M^n in $\bar{M}^{n+\nu}$ as in Theorem C and use the notations in §1.

If the rank of ψ is 1 everywhere, M^n is of geodesic codimension 3 by Theorem 4 in [7].

Now, we assume that the rank of ψ is 2 everywhere, that is $F \wedge G \neq 0$. At any point $x \in M^n$, we denote the 2-dimensional normal space spanned by F and G by N''_x and put $N'' = \cup N''_x$, $x \in M^n$, N'' is a 2-dimensional normal vector bundle over M^n as N'. We can orthogonally decompose N_x as

$$(2.1) \qquad N_x = N'_x + N''_x + O'_x,\ O_x = N''_x + O'_x,\ N''_x \perp O'_x .$$

By the above assumption for ψ, we denote the set of frames $b \in B_3$ such that e_{n+3}, $e_{n+4} \in N''_x$ by B_4. On B_4, we have

$$(2.2) \qquad f_\gamma = g_\gamma = 0\,, \quad \gamma > n+4, \text{ and } f_{n+3}\, g_{n+4} - f_{n+4} g_{n+3} \neq 0\,.$$

Hence, from (1.6), we have

$$(2.3) \qquad \omega_{n+1,\gamma} = \omega_{n+2,\gamma} = 0\,, \qquad \gamma > n+4\,,$$

from which we get

$$d\omega_{n+1,\gamma} = \omega_{n+1,n+3} \wedge \omega_{n+3,\gamma} + \omega_{n+1,n+4} \wedge \omega_{n+4,\gamma} = 0\,,$$

$$d\omega_{n+2,\gamma} = \omega_{n+2,n+3} \wedge \omega_{n+3,\gamma} + \omega_{n+2,n+4} \wedge \omega_{n+4,\gamma} = 0\,.$$

Using (1.6) and (2.2), we have

$$\{(f_{n+3} + i g_{n+3})\omega_{n+3,\gamma} + (f_{n+4} + i g_{n+4})\omega_{n+4,\gamma}\} \wedge (\omega_1 - i\omega_2) = 0\,,$$

and hence

$$(2.4) \qquad \omega_{n+3,\gamma} \equiv \omega_{n+4,\gamma} \equiv 0 \pmod{\omega_1, \omega_2}, \quad \gamma > n+4\,.$$

By virtue of (2.4), for any $v \in N''_x$, we can define a linear mapping ψ'_v: $M_x \to O'_x$ by

$$(2.5) \qquad \psi'_v(X) = \sum_{\gamma > n+4} < v,\ e_{n+3}\omega_{n+3,\gamma}(X) + e_{n+4}\omega_{n+4,\gamma}(X) > e_\gamma\,.$$

The mapping ψ': $M_x \times N''_x \to O'_x$, $\psi'(X, v) = \psi'_v(X)$, may be called *the 2nd torsion operator* of M in $\bar{M}$. Clearly ψ' does not depend on the choice of b over x.

LEMMA 1. *ψ'_v, $v \neq 0$, has the common image.*

PROOF. By means of the above argument, we can put

$$(f_{n+3}+ig_{n+3})\omega_{n+3,\gamma}+(f_{n+4}+ig_{n+4})\omega_{n+4,\gamma}=(f'_\gamma+ig'_\gamma)(\omega_1-i\omega_2),\quad \gamma>n+4.$$

Hence we have

$$(2.6)\qquad \begin{cases} \omega_{n+3,\gamma}=\dfrac{1}{\triangle}\{(g_{n+4}\omega_1+f_{n+4}\omega_2)f'_\gamma-(f_{n+4}\omega_1-g_{n+4}\omega_2)g'_\gamma\}\,, \\ \omega_{n+4,\gamma}=\dfrac{1}{\triangle}\{-(g_{n+3}\omega_1+f_{n+3}\omega_2)f'_\gamma+(f_{n+3}\omega_1-g_{n+3}\omega_2)g'_\gamma\} \end{cases}$$

where $\triangle=f_{n+3}\ g_{n+4}-f_{n+4}\ g_{n+3}$. Putting $F'=\sum_{\gamma>n+4}f'_\gamma e_\gamma$ and $G'=\sum_{\gamma>n+4}g'_\gamma e_\gamma$, we have

$$(2.7)\qquad \psi'_v(X)=\frac{1}{\triangle}\{v_1(g_{n+4}X_1+f_{n+4}X_2)-v_2(g_{n+3}X_1+f_{n+3}X_2)\}F'$$
$$+\frac{1}{\triangle}\{-v_1(f_{n+4}X_1-g_{n+4}X_2)+v_2(f_{n+3}X_1-g_{n+3}X_2)\}G'$$

where $v=v_1e_{n+3}+v_2e_{n+4}$ and $X=\sum_{i=1}^{n}X_ie_i$. Since

$$(g_{n+4}X_1+f_{n+4}X_2)(f_{n+3}X_1-g_{n+3}X_2)-(g_{n+3}X_1+f_{n+3}X_2)(f_{n+4}X_1-g_{n+4}X_2)$$
$$=\triangle(X_1^2+X_2^2)$$

and $\triangle\neq 0$, the image of ψ'_v, $v\neq 0$, is the space spanned by F' and G'. q.e.d.

By the lemma, we may say the rank of the 2nd torsion operator ψ' as the common rank of ψ'_v, $v\neq 0$.

THEOREM 1. *Let M^n ($n\geqq 3$) be a minimal submanifold in $\bar{M}^{n+\nu}$ of constant curvature which is of M-index 2 everywhere and $Q\equiv 0$ and the rank of $\psi\equiv 2$. Then M^n is of geodesic codimension 4 if and only if the rank of $\psi'\equiv 0$.*

PROOF. The necessity is trivial.

Let us suppose that the rank of $\psi'\equiv 0$. This is equivalent to $F'\equiv G'\equiv 0$. Hence, by (2.6), we have

$$\omega_{n+3,\gamma}=\omega_{n+4,\gamma}=0\,,\qquad \gamma>n+4\,.$$

Combining these with (2.3) and (1.4), we see that there exists an $(n+4)$-dimensional totally geodesic submanifold in $\bar{M}^{n+\nu}$ containing M^n by means of the structure equations (1.1). q. e. d.

By this theorem, if we consider the case $\psi' \equiv 0$, we may put $\nu = 4$ from the local point of view.

3. M^n in $\bar{M}^{n+4}$. In the following, we suppose $\nu=4$. On B_4, putting

$$\Phi_\gamma = \frac{1}{\lambda}(f_\gamma + ig_\gamma), \qquad \gamma > n+2, \tag{3.1}$$

(2.2) implies that

$$\Phi_{n+3} \neq 0,\ \Phi_{n+4} \neq 0,\ \Phi = \Phi_{n+4}/\Phi_{n+3} \neq \text{real}. \tag{3.2}$$

From (1.6), we have

$$\omega_{n+1,\gamma} + i\sigma\omega_{n+2,\gamma} = \Phi_\gamma(\omega_1 - i\omega_2) \tag{3.3}$$

and

$$\begin{aligned} d\omega_{n+1,\gamma} + id\sigma \wedge \omega_{n+2,\gamma} + i\sigma d\omega_{n+2,\gamma} &= d\Phi_\gamma \wedge (\omega_1 - i\omega_2) \\ &\quad + \Phi_\gamma(\omega_{12} \wedge \omega_2 + \omega_{13} \wedge \omega_3 + i\omega_{12} \wedge \omega_1 - i\omega_{23} \wedge \omega_3) \end{aligned}$$

by (1.12). Putting

$$\omega_{n+3,n+4} = \hat{\omega}_2, \tag{3.4}$$

the above equation can be written as

$$\begin{aligned} &\hat{\omega}_1 \wedge \omega_{n+2,\gamma} + \sum_{\delta>n+2} \omega_{n+1,\delta} \wedge \omega_{\delta\gamma} + id\sigma \wedge \omega_{n+2,\gamma} \\ &\quad + i\sigma\left\{-\hat{\omega}_1 \wedge \omega_{n+1,\gamma} + \sum_{\delta>n+2} \omega_{n+2,\delta} \wedge \omega_{\delta\gamma}\right\} \\ &= d\Phi_\gamma \wedge (\omega_1 - i\omega_2) + \Phi_\gamma\{i\omega_{12} \wedge (\omega_1 - i\omega_2) - p\omega_3 \wedge (\omega_1 - i\omega_2)\} \end{aligned}$$

and using (3.3) this equation becomes

$$\begin{aligned} & i\{d\sigma - i(1-\sigma^2)\hat{\omega}_1\} \wedge \omega_{n+2,\gamma} \\ &= \{d\Phi_\gamma + \Phi_\gamma(i(\omega_{12} + \sigma\hat{\omega}_1) + pdv) + \sum_{\delta>n+2} \Phi_\delta\omega_{\delta\gamma}\} \wedge (\omega_1 - i\omega_2). \end{aligned} \tag{3.5}$$

For simplicity, we put $\Phi_{n+3} = \Phi_1$, $\Phi_{n+4} = \Phi_2$. Then (3.5) are two equations as follows:

$$\frac{1}{\Phi_1} i\{d\sigma - i(1-\sigma^2)\}\,\hat{\omega}_1 \wedge \omega_{n+2,n+3}$$

$$= \{d\log\Phi_1 + i(\omega_{12} + \sigma\hat{\omega}_1) + pdv - \Phi\hat{\omega}_2\} \wedge (\omega_1 - i\omega_2),$$

$$\frac{1}{\Phi_2} i\{d\sigma - i(1-\sigma^2)\hat{\omega}_1\} \wedge \omega_{n+2,n+4}$$

$$= \left\{d\log\Phi_2 + i(\omega_{12} + \sigma\hat{\omega}_1) + pdv + \frac{1}{\Phi}\hat{\omega}_2\right\} \wedge (\omega_1 - i\omega_2).$$

LEMMA 2. *The curvature $d\hat{\omega}_2$ of N'' is not zero everywhere.*

PROOF. From (3.3) we have easily

$$\omega_{n+1,n+3} = \frac{1}{\lambda}(f_{n+3}\omega_1 + g_{n+3}\omega_2),$$

$$\omega_{n+2,n+3} = \frac{1}{\lambda\sigma}(g_{n+3}\omega_1 - f_{n+3}\omega_2),$$

$$\omega_{n+1,n+4} = \frac{1}{\lambda}(f_{n+4}\omega_1 + g_{n+4}\omega_2),$$

$$\omega_{n+2,n+4} = \frac{1}{\lambda\sigma}(g_{n+4}\omega_1 - f_{n+4}\omega_2).$$

Hence we have the curvature form of the bundle N'' given by

$$(3.6) \quad d\hat{\omega}_2 = \omega_{n+3,n+1} \wedge \omega_{n+1,n+4} + \omega_{n+3,n+2} \wedge \omega_{n+2,n+4} = -\frac{\triangle}{\lambda^2}\left(1 + \frac{1}{\sigma^2}\right)\omega_1 \wedge \omega_2.$$

Since $\triangle \neq 0$ by (2.2), $d\hat{\omega}_2 \neq 0$ everywhere. q. e. d.

COROLLARY. *The set of points where $\hat{\omega}_2 = 0$ is non dense in M^n.*

THEOREM 2. *Let M^n be a submanifold in $\bar{M}^{n+4}$ as in Theorem 1. Assuming the following conditions:*

(α) $\hat{\omega}_1 \neq 0$, $\hat{\omega}_2 \neq 0$ *and σ and Φ are constant on W^2,*
(β) *W^2 is of constant curvature c,*
where W^2 is an integral surface of the distribution $\mathfrak{W}$, we have the following

for W^2:

(i) $\sigma = 1$ *or* -1 *and* $\Phi = i$ *or* $-i$,
(ii) $<F, G> = 0$,
(iii) $c > 0$.

PROOF. Since σ is constant on W^2, we get from (1.9)

$$(1-\sigma^2)\hat{\omega}_1 \wedge (\omega_1 + i\omega_2) = 0\,,$$

hence

$$(1-\sigma^2)\hat{\omega}_1 = 0 \quad \text{on } W^2.$$

Since $\hat{\omega}_1 \neq 0$ by (α), it must be $\sigma = 1$ or -1.

Then, from (3.5) and $\sigma^2 = 1$, we have the relations

$$\text{(3.7)} \qquad \{d\log \Phi_1 + i(\omega_{12} + \sigma\hat{\omega}_1) + p\,dv - \Phi\omega_2\} \wedge (\omega_1 - i\omega_2) = 0\,,$$

$$\left\{d\log \Phi_2 + i(\omega_{12} + \sigma\hat{\omega}_1) + p\,dv + \frac{1}{\Phi}\hat{\omega}_2\right\} \wedge (\omega_1 - i\omega_2) = 0\,,$$

from which

$$\left\{d\log \Phi + \left(\Phi + \frac{1}{\Phi}\right)\hat{\omega}_2\right\} \wedge (\omega_1 - i\omega_2) = 0\,.$$

Since Φ is constant on W^2 by (α), we have

$$\left(\Phi + \frac{1}{\Phi}\right)\hat{\omega}_2 \wedge (\omega_1 - i\omega_2) = 0\,,$$

hence

$$\left(\Phi + \frac{1}{\Phi}\right)\hat{\omega}_2 = 0\,.$$

Since $\hat{\omega}_2 \neq 0$ on W^2, it must be $\Phi = i$ or $-i$, from which we obtain easily $<F, G> = 0$.

Next, from (β), we may put

$$d\omega_{12} = -c\ \omega_1 \wedge \omega_2 \qquad \text{on } W^2\,,$$

hence from (1.10) we have

$$p^2 + \bar{c} - \lambda^2 - \mu^2 = c\,.$$

Using $\sigma^2 = 1$, we have

$$2\lambda^2 = p^2 + \bar{c} - c \text{ on } W^2\,, \tag{3.8}$$

which implies that λ and μ are constant on W^2, since by means of Theorem C and (1.14), p is constant on W^2. Hence (1.8) implies

$$\hat{\omega}_1 = 2\sigma\omega_{12} \qquad \text{on } W^2\,. \tag{3.9}$$

Making use of this and (1.11), we have

$$\begin{aligned} 2c &= \frac{1}{\lambda^2}(2\lambda^4 - \|F\|^2 - \|G\|^2) \\ &= 2\lambda^2 - |\Phi_1|^2 - |\Phi_2|^2 = 2(\lambda^2 - |\Phi_1|^2)\,, \end{aligned}$$

that is

$$|\Phi_1|^2 = \lambda^2 - c\,. \tag{3.10}$$

This relation shows that Φ_1 is constant on W^2. On the other hand, from (3.7), (3.9) we have

$$i(3\omega_{12} + d\theta_1 + i\Phi\hat{\omega}_2) \wedge (\omega_1 - i\omega_2) = 0\,,$$

where θ_1 is the argument of the function Φ_1. Hence we have

$$\hat{\omega}_2 = -i\Phi(3\omega_{12} + d\theta_1) \qquad \text{on } W^2. \tag{3.11}$$

From (3.6) and (3.11), we have

$$d\hat{\omega}_2 = -3i\Phi d\omega_{12} = 3ic\ \Phi\ \omega_1 \wedge \omega_2 = -\frac{2}{\lambda^2}\triangle\omega_1 \wedge \omega_2\,,$$

hence

$$3ic\Phi = -\frac{2}{\lambda^2}(f_{n+3}g_{n+4} - f_{n+4}g_{n+3})\,,$$

that is

$$3c = 2|\Phi_1|^2 \quad \text{on } W^2. \tag{3.12}$$

This relation shows that $c>0$. q. e. d.

By (3.10) and (3.12) we have

$$2\lambda^2 = 5c, \qquad |\Phi_1|^2 = \frac{3}{2}c \quad \text{on } W^2. \tag{3.13}$$

4. Frenet formula of W^2 under (α) and (β). In this section, we shall determine the Frenet formula of W^2 in terms of an isothermal coordinate, when the conditions (α) and (β) in Theorem 2 are satisfied.

By means of (ii) in Theorem 2, we denote the set of frames b over W^2 such that

$$F = fe_{n+3}, \qquad G = ge_{n+4}, \quad f>0, \ g>0 \tag{4.1}$$

by B_5.

Without loss of generality, we may put

$$c = 1 \quad \text{and} \quad \sigma = 1.$$

Since $\Phi_1 = f/\lambda$ and $\Phi_2 = ig/\lambda$ on B_5, we have

$$\lambda = \mu = \frac{\sqrt{10}}{2}, \qquad f = g = \frac{\sqrt{15}}{2} \quad \text{on } W^2 \tag{4.2}$$

by (3.13). Furthermore, from (3.9) and (3.11) we have

$$\hat{\omega}_1 = 2\omega_{12}, \qquad \hat{\omega}_2 = 3\omega_{12} \tag{4.3}$$

and from (3.3)

$$\begin{aligned} \omega_{n+1,n+3} &= \frac{\sqrt{6}}{2}\omega_1, & \omega_{n+1,n+4} &= \frac{\sqrt{6}}{2}\omega_2, \\ \omega_{n+2,n+3} &= -\frac{\sqrt{6}}{2}\omega_2, & \omega_{n+2,n+4} &= \frac{\sqrt{6}}{2}\omega_1. \end{aligned} \tag{4.4}$$

(3.8) becomes

$$p^2 + \bar{c} = 6. \tag{4.5}$$

Now, we figure the Frenet formula of W^2. First of all we have

$$dx = e_1\omega_1 + e_2\omega_2. \tag{4.6}$$

By means of (1.4), (1.12) and (4.2), we have easily

$$\bar{D}(e_1+ie_2) = -i(e_1+ie_2)\omega_{12} + pe_3(\omega_1+i\omega_2) + \frac{\sqrt{10}}{2}(e_{n+1}+i\, e_{n+2})(\omega_1 - i\omega_2) \tag{4.7}$$

$$\bar{D}e_3 = -p(e_1\omega_1 + e_2\omega_2), \tag{4.8}$$

where $\bar{D}$ denotes the covariant differential operator in $\bar{M}^{n+4}$. Analogously, we have

$$\bar{D}(e_{n+1}+ie_{n+2}) = -\frac{\sqrt{10}}{2}(e_1+ie_2)(\omega_1+i\omega_2) - 2i(e_{n+1}+ie_{n+2})\omega_{12} + \frac{\sqrt{6}}{2}(e_{n+3}+ie_{n+4})(\omega_1 - i\omega_2) \tag{4.9}$$

by means of (1.4), (4.2), (4.3) and (4.4). Lastly we have

$$\bar{D}(e_{n+3}+ie_{n+4}) = -\frac{\sqrt{6}}{2}(e_{n+1}+ie_{n+2})(\omega_1+i\omega_2) - 3i(e_{n+3}+ie_{n+4})\omega_{12}. \tag{4.10}$$

These equations (4.6)~(4.10) constitute the Frenet formula of W^2. In order to solve these equations, we shall write these equations in terms of an isothermal coordinate of W^2.

On the other hand, for the unit sphere S^2 we have the following formula, considering it as the Gaussian complex number sphere, as is well known,

$$ds^2 = \frac{4dzd\bar{z}}{(1+z\bar{z})^2} = (\omega_1{}^*)^2 + (\omega_2{}^*)^2, \tag{4.11}$$

and

$$\omega_1^* + i\omega_2^* = \frac{2dz}{1+z\bar{z}}, \quad \omega_{12}^* = i\frac{\bar{z}dz - zd\bar{z}}{1+z\bar{z}}, \tag{4.12}$$

where ω_{12}^* is the connection form of S^2.

Since W^2 is of constant curvature 1, we may consider it locally as the unit sphere S^2. Then, we may put

$$\omega_1 + i\omega_2 = e^{-i\theta}(\omega_1^* + i\omega_2^*). \tag{4.13}$$

Substituting this into

$$d(\omega_1 + i\omega_2) = -i\omega_{12} \wedge (\omega_1 + i\omega_2),$$

we have

$$(\omega_{12} - \omega_{12}^* - d\theta) \wedge (\omega_1^* + i\omega_2^*) = 0,$$

hence

$$\omega_{12} = \omega_{12}^* + d\theta. \tag{4.14}$$

Substituting (1.13) and (4.14) into (4.6)~(4.10) and putting

$$\begin{cases} e_1^* + ie_2^* = e^{i\theta}(e_1 + ie_2),\ e_{n+1}^* + ie_{n+2}^* = e^{2i\theta}(e_{n+1} + ie_{n+2}), \\ e_{n+3}^* + ie_{n+4}^* = e^{3i\theta}(e_{n+3} + ie_{n+4}), \end{cases} \tag{4.15}$$

we have

$$dx = e_1^*\omega_1^* + e_2^*\omega_2^*, \tag{4.6*}$$

$$\bar{D}(e_1^* + ie_2^*) = -i(e_1^* + ie_2^*)\omega_{12}^* + pe_3(\omega_1^* + i\omega_2^*) + \frac{\sqrt{10}}{2}(e_{n+1}^* + ie_{n+2}^*)(\omega_1^* - i\omega_2^*), \tag{4.7*}$$

$$\bar{D}e_3 = -p(e_1^*\omega_1^* + e_2^*\omega_2^*), \tag{4.8*}$$

$$\bar{D}(e_{n+1}^* + ie_{n+2}^*) = -\frac{\sqrt{10}}{2}(e_1^* + ie_2^*)(\omega_1^* + i\omega_2^*) - 2i(e_{n+1}^* + ie_{n+2}^*)\omega_{12}^* + \frac{\sqrt{6}}{2}(e_{n+3}^* + ie_{n+4}^*)(\omega_1^* - i\omega_2^*), \tag{4.9*}$$

$$(4.10^*)\qquad \bar{D}(e_{n+3}^* + ie_{n+4}^*) = -\frac{\sqrt{6}}{2}(e_{n+1}^* + ie_{n+2}^*)(\omega_1^* + i\omega_2^*) - 3i(e_{n+3}^* + ie_{n+4}^*)\omega_{12}^*.$$

Therefore using (4.12) and putting

$$(4.16)\qquad \xi = e_1^* + ie_2^*,\quad \eta = e_{n+1}^* + ie_{n+2}^*,\quad \zeta = e_{n+3}^* + ie_{n+4}^*,$$

we have the Frenet formula of W^2 in the isothermal coordinate z as follows:

$$(4.17)\qquad \begin{cases} dx = \dfrac{1}{h}(\bar{\xi}dz + \xi d\bar{z}), \\ \bar{D}e_3 = -\dfrac{p}{h}(\bar{\xi}dz + \xi d\bar{z}), \\ \bar{D}\xi = \dfrac{1}{h}\xi(\bar{z}dz - zd\bar{z}) + \dfrac{2p}{h}e_3dz + \dfrac{\sqrt{10}}{h}\eta d\bar{z}, \\ \bar{D}\eta = -\dfrac{\sqrt{10}}{h}\xi\, dz + \dfrac{2}{h}\eta(\bar{z}dz - zd\bar{z}) + \dfrac{\sqrt{6}}{h}\zeta d\bar{z}, \\ \bar{D}\zeta = -\dfrac{\sqrt{6}}{h}\eta dz + \dfrac{3}{h}\zeta(\bar{z}dz - zd\bar{z}), \end{cases}$$

where $h = 1 + z\bar{z}$.

5. Solutions in Case $\bar{M}^{n+4} = E^{n+4}$. In this section, we shall find M^n in Euclidean space E^{n+4} as in Theorem 2, by solving the Frenet formula (4.17) of W^2.

In this case, by (4.5) we have

$$(5.1)\qquad p = \sqrt{6}.$$

From the last equation of (4.17), we have

$$\frac{\partial\zeta}{\partial\bar{z}} = -\frac{3z}{h}\zeta.$$

Hence we can put

$$(5.2)\qquad \zeta = \frac{1}{h^3}F(z),$$

where $F(z)$ is a complex holomorphic vector field. Substituting (5.2) into the 5th of (4.17), we have

$$\frac{\partial \zeta}{\partial z} = -\frac{3\bar{z}}{h^4}F(z) + \frac{1}{h^3}F'(z) = -\frac{\sqrt{6}}{h}\eta + \frac{3\bar{z}}{h}\zeta$$

$$= -\frac{\sqrt{6}}{h}\eta + \frac{3\bar{z}}{h^4}F(z)\,,$$

hence

$$\eta = \sqrt{6}\frac{\bar{z}}{h^3}F(z) - \frac{1}{\sqrt{6}\,h^2}F'(z)\,. \tag{5.3}$$

From (5.3) and (5.2), we have

$$\frac{\partial \eta}{\partial \bar{z}} = \sqrt{6}\left(\frac{1}{h^3} - \frac{3z\bar{z}}{h^4}\right)F(z) + \frac{2z}{\sqrt{6}\,h^3}F'(z)$$

and

$$\frac{\sqrt{6}}{h}\zeta - \frac{2z}{h}\eta = \frac{\sqrt{6}}{h^4}F(z) - \frac{2\sqrt{6}\,z\bar{z}}{h^4}F(z) + \frac{2z}{\sqrt{6}\,h^3}F'(z)$$

$$= \sqrt{6}\left(\frac{1}{h^3} - \frac{3z\bar{z}}{h^4}\right)F(z) + \frac{2z}{\sqrt{6}\,h^3}F'(z)\,,$$

hence

$$\frac{\partial \eta}{\partial \bar{z}} = \frac{\sqrt{6}}{h}\zeta - \frac{2z}{h}\eta\,.$$

From the 4th of (4.17), we have

$$\frac{\partial \eta}{\partial z} = -\frac{3\sqrt{6}\,\bar{z}^2}{h^4}F(z) + \frac{\sqrt{6}\,\bar{z}}{h^3}F'(z) + \frac{2\bar{z}}{\sqrt{6}\,h^3}F'(z) - \frac{1}{\sqrt{6}\,h^2}F''(z)$$

$$= -\frac{\sqrt{10}}{h}\xi + \frac{2\bar{z}}{h}\eta = -\frac{\sqrt{10}}{h}\xi + \frac{2\sqrt{6}\,\bar{z}^2}{h^4}F(z) - \frac{2\bar{z}}{\sqrt{6}\,h^3}F'(z)\,,$$

hence

$$\xi = \frac{\sqrt{15}\,\bar{z}^2}{h^3}F(z) - \frac{\sqrt{15}\,\bar{z}}{3h^2}F'(z) + \frac{1}{2\sqrt{15}\,h}F''(z)\,. \tag{5.4}$$

From (5. 4) and (5.3), we have

$$\frac{\partial\xi}{\partial\bar{z}}=\sqrt{15}\left(\frac{2\bar{z}}{h^3}-\frac{3z\bar{z}^2}{h^4}\right)F(z)-\frac{\sqrt{15}}{3}\left(\frac{1}{h^2}-\frac{2z\bar{z}}{h^3}\right)F'(z)-\frac{z}{2\sqrt{15}\,h^2}F''(z)$$

$$=\sqrt{15}\,\bar{z}\left(\frac{3}{h^4}-\frac{1}{h^3}\right)F(z)-\frac{\sqrt{15}}{3}\left(\frac{2}{h^3}-\frac{1}{h^2}\right)F'(z)-\frac{z}{2\sqrt{15}\,h^2}F''(z),$$

and

$$\frac{\sqrt{10}}{h}\eta-\frac{z}{h}\xi=\frac{2\sqrt{15}\,\bar{z}}{h^4}F(z)-\frac{\sqrt{15}}{3h^3}F'(z)-\frac{\sqrt{15}\,z\bar{z}^2}{h^4}F(z)$$

$$+\frac{\sqrt{15}\,z\bar{z}}{3h^3}F'(z)-\frac{z}{2\sqrt{15}\,h^2}F''(z)$$

$$=\sqrt{15}\,\bar{z}\left(\frac{3}{h^4}-\frac{1}{h^3}\right)F(z)-\frac{\sqrt{15}}{3}\left(\frac{2}{h^3}-\frac{1}{h^2}\right)F'(z)-\frac{z}{2\sqrt{15}\,h^2}F''(z),$$

hence

$$\frac{\partial\xi}{\partial\bar{z}}=\frac{\sqrt{10}}{h}\eta-\frac{z}{h}\xi.$$

From the 3rd of (4.17), (5.3) and (5.4), we have

$$\frac{\partial\xi}{\partial z}=-\frac{3\sqrt{15}\,\bar{z}^3}{h^4}F(z)+\frac{5\sqrt{15}\,\bar{z}^2}{3h^3}F'(z)-\frac{11}{2\sqrt{15}\,h^2}\bar{z}F''(z)+\frac{1}{2\sqrt{15}\,h}F'''(z)$$

$$=\frac{\bar{z}}{h}\xi+\frac{2p}{h}e_3=\frac{\sqrt{15}\,\bar{z}^3}{h^4}F(z)-\frac{\sqrt{15}\,\bar{z}^2}{3h^3}F'(z)+\frac{\bar{z}}{2\sqrt{15}\,h^2}F''(z)+\frac{2\sqrt{6}}{h}e_3.$$

Hence we have

$$(5.5)\quad e_3=-\frac{\sqrt{10}\,\bar{z}^3}{h^3}F(z)+\frac{\sqrt{10}\,\bar{z}^2}{2h^2}F'(z)-\frac{\bar{z}}{\sqrt{10}\,h}F''(z)+\frac{1}{12\sqrt{10}}F'''(z),$$

from which we have

$$\frac{\partial e_3}{\partial \bar{z}} = -\sqrt{10}\left(\frac{3\bar{z}^2}{h^3} - \frac{3z\bar{z}^3}{h^4}\right)F(z) + \frac{\sqrt{10}}{2}\left(\frac{2\bar{z}}{h^2} - \frac{2z\bar{z}^2}{h^3}\right)F'(z)$$

$$-\frac{1}{\sqrt{10}}\left(\frac{1}{h} - \frac{z\bar{z}}{h^2}\right)F''(z) = -\frac{3\sqrt{10}\,\bar{z}^2}{h^4}F(z) + \frac{\sqrt{10}\,\bar{z}}{h^3}F'(z)$$

$$-\frac{1}{\sqrt{10}\,h^2}F''(z) = -\frac{\sqrt{6}}{h}\xi = -\frac{p}{h}\xi .$$

If e_3 is real, then we have also

$$\frac{\partial e_3}{\partial z} = -\frac{\sqrt{6}}{h}\bar{\xi} = -\frac{p}{h}\bar{\xi} .$$

Hence, if we choose $F(z)$ so that e_3 is real, then e_3, ξ, η, ζ given by (5.5), (5.4), (5.3), (5.2), satisfy the equations (4.17) respectively except the first one.

From now we search for $F(z)$ such that e_3 is real. Since $h = 1 + z\bar{z}$ is real, it is equivalent to determine so that

$$-12\sqrt{10}\,h^3 e_3 = 120\,\bar{z}^3 F(z) - 60h\bar{z}^2 F'(z) + 12h^2\bar{z}F''(z) - h^3 F'''(z) \tag{5.6}$$
$$\equiv 6G(z, \bar{z})$$

is real. $G(z, \bar{z})$ is a polynomial in $\bar{z}$ of order at most 3, hence it is also so in z by means of $\overline{G(z, \bar{z})} = G(z, \bar{z})$.

Now, we have easily from (5.6)

$$\begin{aligned} 6G(z, \bar{z}) = {} & \{120F(z) - 60zF'(z) + 12z^2F''(z) - z^3F'''(z)\}\bar{z}^3 \\ & - 3\{20F'(z) - 8zF''(z) + z^2F'''(z)\}\bar{z}^2 \\ & + 3\{4F''(z) - zF'''(z)\}\bar{z} - F'''(z) . \end{aligned}$$

Since $G(z, \bar{z})$ is a vector valued polynomial in z and $\bar{z}$, we see from the above relation that $F'''(z)$ is a polynomial in z. Therefore, we may put

$$F(z) = A_0 + A_1 z + \cdots + A_m z^m , \tag{5.7}$$

where $A_0, A_1, \cdots, A_m$ are constant vectors in C^4. Then, by simple calculation, we have

$$120F(z) - 60zF'(z) + 12z^2F''(z) - z^3F'''(z) = 120A_0 + 60A_1z + 24A_2z^2$$
$$+ 6A_3z^3 + \cdots + (4-m)(5-m)(6-m)A_mz^m ,$$

$$20F'(z) - 8zF''(z) + z^2F'''(z) = 20A_1 + 24A_2z + 18A_3z^2$$
$$+ \cdots + m(5-m)(6-m)A_mz^{m-1} ,$$

$$4F''(z) - zF'''(z) = 8A_2 + 18A_3z + \cdots + m(m-1)(6-m)A_mz^{m-2} ,$$

hence we have

$$\begin{aligned} 6G(z,\bar{z}) = & \{120A_0+60A_1z+24A_2z^2+6A_3z^3+\cdots+(4-m)(5-m)(6-m)A_mz^m\}\bar{z}^3 \\ & -3\{20A_1+24A_2z+18A_3z^2+\cdots+m(5-m)(6-m)A_mz^{m-1}\}\bar{z}^2 \\ & +3\{8A_2+18A_3z+\cdots+m(m-1)(6-m)A_mz^{m-2}\}\bar{z} \\ & -\{6A_3+24A_4z+\cdots+m(m-1)(m-2)A_mz^{m-3}\}. \end{aligned}$$

Noticing that the polynomial inside of the first brace lacks the terms of order 4, 5 and 6 in z, we may suppose that $m=6$. Then, we have

$$\begin{aligned} G(z,\bar{z}) = & (20A_0+10A_1z+4A_2z^2+A_3z^3)\bar{z}^3 \\ & -(10A_1+12A_2z+9A_3z^2+4A_4z^3)\bar{z}^2 \\ & +(4A_2+9A_3z+12A_4z^2+10A_5z^3)\bar{z} \\ & -(A_3+4A_4z+10A_5z^2+20A_6z^3)\,. \end{aligned} \tag{5.8}$$

Hence, it must be

$$\begin{aligned} \overline{G(z,\bar{z})} = & (-20\bar{A}_6+10\bar{A}_5z-4\bar{A}_4z^2+\bar{A}_3z^3)\bar{z}^3 \\ & -(10\bar{A}_5-12\bar{A}_4z+9\bar{A}_3z^2-4\bar{A}_2z^3)\bar{z}^2 \\ & +(-4\bar{A}_4+9\bar{A}_3z-12\bar{A}_2z^2+10\bar{A}_1z^3)\bar{z} \\ & -(\bar{A}_3-4\bar{A}_2z+10\bar{A}_1z^2-20\bar{A}_0z^3)\,. \end{aligned}$$

Comparing this with (5.8), $G(z,\bar{z})=\overline{G(z,\bar{z})}$ is satisfied if and only if

$$A_3=\bar{A}_3,\quad A_4=-\bar{A}_2,\quad A_5=\bar{A}_1,\quad A_6=-\bar{A}_0\,. \tag{5.9}$$

Making use of (5.9), $G(z,\bar{z})$ can be written as

$$
\begin{aligned}
G(z,\bar{z}) &= (20A_0 + 10A_1 z + 4A_2 z^2 + A_3 z^3)\bar{z}^3 \\
&\quad - (10A_1 + 12A_2 z + 9A_3 z^2 - 4\bar{A}_2 z^3)\bar{z}^2 \\
&\quad + (4A_2 + 9A_3 z - 12\bar{A}_2 z^2 + 10\bar{A}_1 z^3)\bar{z} \\
&\quad - (A_3 - 4\bar{A}_2 z + 10\bar{A}_1 z^2 - 20\bar{A}_0 z^3) \\
&= - A_3 + 4(\bar{A}_2 z + A_2 z) + 9A_3 z\bar{z} - 10(\bar{A}_1 z^2 + A_1\bar{z}^2) \\
&\quad - 12(\bar{A}_2 z + A_2\bar{z})z\bar{z} + 20(\bar{A}_0 z^3 + A_0\bar{z}^3) \\
&\quad + 10(\bar{A}_1 z^2 + A_1\bar{z}^2)z\bar{z} - 9A_3(z\bar{z})^2 \\
&\quad + 4(\bar{A}_2 z + A_2\bar{z})(z\bar{z})^2 + A_3(z\bar{z})^3 \\
&= - A_3\{1 - 9z\bar{z} + 9(z\bar{z})^2 - (z\bar{z})^3\} \\
&\quad + 4(\bar{A}_2 z + A_2\bar{z})\{1 - 3z\bar{z} + (z\bar{z})^2\} \\
&\quad - 10(\bar{A}_1 z^2 + A_1\bar{z}^2)\{1 - z\bar{z}\} \\
&\quad + 20(\bar{A}_0 z^3 + A_0\bar{z}^3)\,.
\end{aligned}
$$

Substituting this into (5.6), we have

$$
\begin{aligned}
(5.10)\qquad e_3 = \frac{1}{2\sqrt{10}\,h^3}\{&A_3(1-9z\bar{z}+9z^2\bar{z}^2-z^3\bar{z}^3) \\
&- 4(\bar{A}_2 z + A_2\bar{z})(1 - 3z\bar{z} + z^2\bar{z}^2) \\
&+ 10(\bar{A}_1 z^2 + A_1\bar{z}^2)(1 - z\bar{z}) - 20(\bar{A}_0 z^3 + A_0\bar{z}^3)\}\,.
\end{aligned}
$$

Analogously from (5.2), we have

$$
(5.11)\qquad \zeta = \frac{1}{h^3}\{z^3A_3 + (z^2A_2 - z^4\bar{A}_2) + (zA_1 + z^5\bar{A}_1) + A_0 - z^6\bar{A}_0\}\,.
$$

On the other hand, (5.3) and (5.4) can be written as

$$
\eta = \frac{1}{\sqrt{6}\,h^3}\{6\bar{z}F(z) - (1 + z\bar{z})F'(z)\}
$$

and

$$\xi = \frac{1}{\sqrt{15}\,h^3}\{15\bar{z}^2F(z) - 5(1+z\bar{z})\bar{z}F'(z) + \frac{1}{2}(1+z\bar{z})^2F''(z)\}\ .$$

Since we have

$$\begin{aligned}6\bar{z}F(z)-(1+z\bar{z})F'(z) &= 6\bar{z}(z^3A_3+z^2A_2-z^4\bar{A}_2+zA_1+z^5\bar{A}_1+A_0-z^6\bar{A}_0)\\ &\quad -(1+z\bar{z})(3z^2A_3+2zA_2-4z^3\bar{A}_2+A_1+5z^4\bar{A}_1-6z^5\bar{A}_0)\\ &= -3(1-z\bar{z})z^2A_3+2(-1+2z\bar{z})zA_2+2(2-z\bar{z})z^3\bar{A}_2\\ &\quad +(-1+5z\bar{z})A_1+(-5+z\bar{z})z^4\bar{A}_1+6\bar{z}A_0+6z^5\bar{A}_0\,,\end{aligned}$$

$$\begin{aligned}&15\bar{z}^2F(z) - 5(1+z\bar{z})\bar{z}F'(z) + \frac{1}{2}(1+z\bar{z})^2F''(z)\\ &\qquad = 15\bar{z}^2(z^3A_3+z^2A_2-z^4\bar{A}_2+zA_1+z^5\bar{A}_1+A_0-z^6\bar{A}_0)\\ &\qquad\quad -5(1+z\bar{z})\bar{z}(3z^2A_3+2zA_2-4z^3\bar{A}_2+A_1+5z^4\bar{A}_1-6z^5\bar{A}_0)\\ &\qquad\quad +(1+2z\bar{z}+z^2\bar{z}^2)(3zA_3+A_2-6z^2\bar{A}_2+10z^3\bar{A}_1-15z^4\bar{A}_0)\\ &\qquad = 3(1-3z\bar{z}+z^2\bar{z}^2)zA_3+(1-8z\bar{z}+6z^2\bar{z}^2)A_2\\ &\qquad\quad -(6-8z\bar{z}+z^2\bar{z}^2)z^2\bar{A}_2+5(-1+2z\bar{z})\bar{z}A_1+5(2-z\bar{z})z^3\bar{A}_1\\ &\qquad\quad +15\bar{z}^2A_0-15z^4\bar{A}_0,\end{aligned}$$

η and ξ can be written as:

(5.12) $$\begin{aligned}\eta = \frac{1}{\sqrt{6}\,h^3}\{&-3(1-z\bar{z})z^2A_3+2(-1+2z\bar{z})zA_2+2(2-z\bar{z})z^3\bar{A}_2\\ &+(-1+5z\bar{z})A_1+(-5+z\bar{z})z^4\bar{A}_1+6\bar{z}A_0+6z^5\bar{A}_0\}\end{aligned}$$

and

(5.13) $$\begin{aligned}\xi = \frac{1}{\sqrt{15}\,h^3}\{&3(1-3z\bar{z}+z^2\bar{z}^2)zA_3+(1-8z\bar{z}+6z^2\bar{z}^2)A_2\\ &-(6-8z\bar{z}+z^2\bar{z}^2)z^2\bar{A}_2+5(-1+2z\bar{z})\bar{z}A_1+5(2-z\bar{z})z^3\bar{A}_1\\ &+15\bar{z}^2A_0-15z^4\bar{A}_0\}\ .\end{aligned}$$

Now, we must find the conditions such that ξ, η, ζ, e_3 make an orthonormal frame. In the case of this section, (4.17) are

$$de_3 = -\frac{\sqrt{6}}{h}(\bar{\xi}dz + \xi d\bar{z}),$$

$$\begin{cases} d\xi = \frac{1}{h}\xi(\bar{z}dz - zd\bar{z}) + \frac{2\sqrt{6}}{h}e_3dz + \frac{\sqrt{10}}{h}\eta d\bar{z}, \\ d\bar{\xi} = -\frac{1}{h}\bar{\xi}(\bar{z}dz - zd\bar{z}) + \frac{2\sqrt{6}}{h}e_3d\bar{z} + \frac{\sqrt{10}}{h}\bar{\eta}dz, \end{cases}$$

$$\begin{cases} d\eta = -\frac{\sqrt{10}}{h}\xi dz + \frac{2}{h}\eta(\bar{z}dz - zd\bar{z}) + \frac{\sqrt{6}}{h}\zeta d\bar{z}, \\ d\bar{\eta} = -\frac{\sqrt{10}}{h}\bar{\xi}d\bar{z} - \frac{2}{h}\bar{\eta}(\bar{z}dz - zd\bar{z}) + \frac{\sqrt{6}}{h}\bar{\zeta}dz, \end{cases}$$

$$\begin{cases} d\zeta = -\frac{\sqrt{6}}{h}\eta dz + \frac{3}{h}\zeta(\bar{z}dz - zd\bar{z}), \\ d\bar{\zeta} = -\frac{\sqrt{6}}{h}\bar{\eta}d\bar{z} - \frac{3}{h}\bar{\zeta}(\bar{z}dz - zd\bar{z}). \end{cases}$$

In the following calculation, "$\equiv$" denotes the equality modulus the quantities:

$$e_3\cdot\xi,\ e_3\cdot\eta,\ e_3\cdot\zeta,\ e_3\bar{\xi},\ e_3\cdot\bar{\eta},\ e_3\cdot\bar{\zeta},$$
$$\xi\cdot\xi,\ \xi\cdot\eta,\ \xi\cdot\zeta,\ \xi\cdot\bar{\eta},\ \xi\cdot\bar{\zeta},$$
$$\bar{\xi}\cdot\bar{\xi},\ \bar{\xi}\cdot\eta,\ \bar{\xi}\cdot\zeta,\ \bar{\xi}\cdot\bar{\eta},\ \bar{\xi}\cdot\bar{\zeta},$$
$$\eta\cdot\eta,\ \eta\cdot\zeta,\ \eta\cdot\bar{\zeta},\ \bar{\eta}\cdot\bar{\eta},\ \bar{\eta}\cdot\zeta,\ \bar{\eta}\cdot\bar{\zeta},\ \zeta\cdot\zeta,\ \bar{\zeta}\cdot\bar{\zeta}.$$

Then, making use of the above ralations, we have easily the relations:

$$d(e_3\cdot e_3) \equiv 0,$$
$$d(e_3\cdot\xi) \equiv \frac{\sqrt{6}}{h}(2e_3\cdot e_3 - \xi\cdot\bar{\xi})dz,$$
$$d(e_3\cdot\eta) \equiv d(e_3\cdot\zeta) \equiv 0,$$
$$d(\xi\cdot\bar{\xi}) \equiv d(\xi\cdot\xi) \equiv d(\xi\cdot\eta) \equiv d(\xi\cdot\zeta) \equiv d(\xi\cdot\bar{\zeta}) \equiv 0,$$
$$d(\xi\cdot\bar{\eta}) \equiv \frac{\sqrt{10}}{h}(\eta\cdot\bar{\eta} - \xi\cdot\bar{\xi})d\bar{z},$$
$$d(\eta\cdot\bar{\eta}) \equiv d(\eta\cdot\eta) \equiv d(\eta\cdot\zeta) \equiv 0,$$
$$d(\eta\cdot\bar{\zeta}) \equiv \frac{\sqrt{6}}{h}(\zeta\cdot\bar{\zeta} - \eta\cdot\bar{\eta})d\bar{z},$$
$$d(\zeta\cdot\bar{\zeta}) \equiv d(\zeta\cdot\zeta) \equiv 0,$$

from which we see that if we can choose A_0, A_1, A_2, A_3 so that all the above quantities 10 lines before and

$$e_3.\cdot e_3 - 1,\ \xi\cdot\bar{\xi} - 2,\ \eta\cdot\bar{\eta} - 2,\ \zeta\cdot\bar{\zeta} - 2$$

are zero at $z = 0$, then these are identically zero.

By means of (5.10), (5.11), (5.12), (5.13), when $z = 0$, we have

$$e_3 = \frac{1}{2\sqrt{10}} A_3, \qquad \xi = \frac{1}{\sqrt{15}} A_2, \qquad \eta = -\frac{1}{\sqrt{6}} A_1, \qquad \zeta = A_0 .$$

Thus, the conditions for A_0, A_1, A_2, A_3 are

$$(5.14)\qquad \begin{cases} A_3 = \bar{A}_3, \\ A_2\cdot A_2 = A_1\cdot A_1 = A_0\cdot A_0 = 0, \\ A_3\cdot A_3 = 40,\ A_2\cdot\bar{A}_2 = 30,\ A_1\cdot\bar{A}_1 = 12,\ A_0\cdot\bar{A}_0 = 2, \\ A_3\cdot A_2 = A_3\cdot A_1 = A_3\cdot A_0 = 0, \\ A_2\cdot A_1 = A_2\cdot\bar{A}_1 = A_2\cdot A_0 = A_2\cdot\bar{A}_0 = 0, \\ A_1\cdot A_0 = A_1\cdot\bar{A}_0 = 0. \end{cases}$$

Now, we give the equation of W^2 by means of the above result. First of all, we choose four constant vectors A_0, A_1, A_2, A_3 in $\boldsymbol{C}^4$ which satisfy the condition (5 14) and determine e_3 given by (5.10) which is real and a unit vector field in $E^8 \cong \boldsymbol{C}^4$. On the other hand, we may consider as

$$x + \frac{1}{p} e_3 = 0$$

by (4. 17). Hence we have a general solution of W^2 as follows:

$$(5.15)\qquad x = -\frac{1}{\sqrt{6}} e_3 = \frac{1}{4\sqrt{15}(1+z\bar{z})^3} \{-(1-9z\bar{z}+9z^2\bar{z}^2-z^3\bar{z}^3)A_3$$
$$+ 4(1-3z\bar{z}+z^2\bar{z}^2)(\bar{z}A_2 + z\bar{A}_2) - 10(1-z\bar{z})(\bar{z}^2A_1 + z^2\bar{A}_1)$$
$$+ 20(\bar{z}^3A_0 + z^3\bar{A}_0)\}.$$

If we put

$$A_3 = 2\sqrt{10}\ \partial/\partial x_7,$$
$$A_2 = \sqrt{15}\,(\partial/\partial x_1 + i\partial/\partial x_2),$$
$$A_1 = -\sqrt{6}\,(\partial/\partial x_3 + i\partial/\partial x_4),$$
$$A_0 = \partial/\partial x_5 + i\partial/\partial x_6,$$

then we can write (5.15) in the canonical coordinates $x_1, x_2, \cdots, x_7$ as follows:

$$(5.16)\qquad \begin{cases} x_1 = \dfrac{1-3z\bar z+z^2\bar z^2}{(1+z\bar z)^3}(z+\bar z), \\ x_2 = -i\dfrac{1-3z\bar z+z^2\bar z^2}{(1+z\bar z)^3}(z-\bar z), \\ x_3 = \dfrac{\sqrt{5}\,(1-z\bar z)}{\sqrt{2}\,(1+z\bar z)^3}(z^2+\bar z^2), \\ x_4 = -i\dfrac{\sqrt{5}\,(1-z\bar z)}{\sqrt{2}\,(1+z\bar z)^3}(z^2-\bar z^2), \\ x_5 = \dfrac{\sqrt{5}}{\sqrt{3}\,(1+z\bar z)^3}(z^3+\bar z^3), \\ x_6 = -i\dfrac{\sqrt{5}}{\sqrt{3}\,(1+z\bar z)^3}(z^3-\bar z^3), \\ z_7 = -\dfrac{1-9z\bar z+9z^2\bar z^2-z^3\bar z^3}{\sqrt{6}\,(1+z\bar z)^3}. \end{cases}$$

Finally, we show how ts construct M^n in E^{n+4} as in Theorem 2. First of all, we consider as

$$E^{n+4} = R^{n-4}\times R^8,\ R^8 \cong C^4$$

and construct a surface W^2 given by (5.15) in C^4. This surface is clearly of geodesic codimension 5 in R^8. Hence, we may consider as

$$W^2 \subset R^7 \qquad \text{and} \qquad C^4 = R\times R^7.$$

For any point $y \in W^2$, we denote a linear subspace $L^{n-2}(y)$ through y such that

$$L^{n-2}(y) \,\|\, R^{n-4}\times R \quad \text{and} \quad L^{n-2}(y) \,\|\, e_3(z),\ y = y(z).$$

Then, the locus of points on the moving $L^{n-2}(y)$ makes an n-dimensional submanifold M^n in E^{n+4} which is minimal and of M-index 2 everywhere and satisfies the conditions in Theorem 2.

Remark. As is well known, the Veronese surface is given by

$$x_1 = \sqrt{3}\, u_2u_3, \quad x_2 = \sqrt{3}\, u_3u_1, \quad x_3 = \sqrt{3}\, u_1u_2,$$

$$x_4 = \frac{\sqrt{3}}{2}(u_1u_1 - u_2u_2), \quad x_5 = \frac{1}{2}(3u_1u_1 + 3u_2u_2 - 2),$$

where $u_1u_1 + u_2u_2 + u_3u_3 = 1$. Through the stereographic projection, we put

$$u_1 = \frac{z+\bar{z}}{1+z\bar{z}}, \quad u_2 = -i\frac{z-\bar{z}}{1+z\bar{z}}, \quad u_3 = \frac{z\bar{z}-1}{1+z\bar{z}}$$

and substituting these into the above equations we have

$$(5.17)\qquad \begin{cases} x_1 = i\sqrt{3}\,\dfrac{1-z\bar{z}}{(1+z\bar{z})^2}(z-\bar{z}), \\[2mm] x_2 = -\sqrt{3}\,\dfrac{1-z\bar{z}}{(1+z\bar{z})^2}(z+\bar{z}), \\[2mm] x_3 = -i\sqrt{3}\,\dfrac{1}{(1+z\bar{z})^2}(z^2-\bar{z}^2), \\[2mm] x_4 = \sqrt{3}\,\dfrac{1}{(1+z\bar{z})^2}(z^2+\bar{z}^2), \\[2mm] x_5 = -\dfrac{1-4z\bar{z}+z^2\bar{z}^2}{(1+z\bar{z})^2}. \end{cases}$$

Comparing (5.16) multiplied by $\sqrt{6}$ with (5.17), we see that W^2 may be considered as a generalization of the Veronese surface. It is minimal in a 6-dimensional sphere as the Veronese surface is minimal in the 4-dimensional unit sphere. Both of them are isometric imbeddings of the projective plane with a canonical metric of constant curvature.

6. Solutions in Case $\bar{M}^{n+4} = S^{n+4}(R)$. In this section, we shall find M^n in $(n+4)$-dimensional sphere as in Theorem 2.

In this case, we regard as $\bar{M}^{n+4} = S^{n+4}(R) \subset E^{n+5}$, where $\frac{1}{R^2} = \bar{c}$. Putting

$$\frac{x}{R} = e_{n+5}\,, \tag{6.1}$$

we have

$$dx = Rde_{n+5} = e_1{}^*\omega_1{}^* + e_2{}^*\omega_2{}^*\,.$$

Hence, denoting the ordinary differential operator in E^{n+5} by d, we have easily

$$de_3 = \overline{D}e_3 = -\frac{p}{h}(\bar{\xi}dz + \xi d\bar{z})\,, \tag{6.2}$$

and

$$d\xi = \overline{D}\xi - \frac{1}{R}(\omega_1{}^* + i\omega_2{}^*)e_{n+5}\,,$$

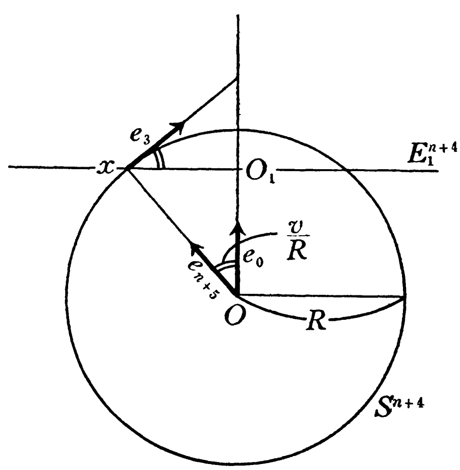
i. e.

$$d\xi = \frac{1}{h}\xi({}^{-}dz - zd\bar{z}) + \frac{2p}{h}e_3dz + \frac{\sqrt{10}}{h}\eta d\bar{z} - \frac{2}{Rh}e_{n+5}dz \tag{6.3}$$

by (4.17) and (4.12).

On the other hand, we have

$$p = \sqrt{c}\ \cot\ \sqrt{c}\,v = \frac{1}{R}\cot\frac{v}{R} \tag{6.4}$$



Since the point

$$x+\frac{1}{p}e_3 = R\left(e_{n+5}+e_3 \tan\frac{v}{R}\right)$$

is a fixed point, the unit vector

$$e_0 = e_{n+5} \cos\frac{v}{R}+e_3 \sin\frac{v}{R}$$

is fixed on W^2. Hence W^2 lies on the linear space E_1^{n+4} which is orthogonal to e_0 and passes through the point $O_1 = e_0 R\cos\frac{v}{R}$.

Now, we have

$$\overrightarrow{O_1 x} = -e_3{}^* R \sin\frac{v}{R},$$

where

$$e_3{}^* = e_3 \cos\frac{v}{R} - e_{n+5} \sin\frac{v}{R}. \tag{6.5}$$

Since we have

$$pe_3-\frac{1}{R}e_{n+5} = \frac{1}{R}e_3 \cot\frac{v}{R}-\frac{1}{R}e_{n+5} = \frac{1}{R \sin\frac{v}{R}}e_3{}^*$$

and

$$p^2+\bar{c} = \left(\frac{1}{R}\cot\frac{v}{R}\right)^2+\frac{1}{R^2} = \frac{1}{\left(R \sin\frac{v}{R}\right)^2} = 6$$

by (4.5), (6.3) can be written as

$$d\xi = \frac{1}{h}\xi(\bar{z}dz - zd\bar{z}) + \frac{2\sqrt{6}}{h}e_3{}^* dz + \frac{\sqrt{10}}{h}\eta\, d\bar{z}. \tag{6.6}$$

Next, we compute de_3^* on W^2. By means of (6.1), (6.2) and (6.5), we have

$$de_3^* = \cos\frac{v}{R}de_3 - \sin\frac{v}{R}de_{n+5}$$

$$= -\left(\frac{p}{h}\cos\frac{v}{R} + \frac{1}{Rh}\sin\frac{v}{R}\right)(\bar{\xi}dz + \xi d\bar{z})$$

and

$$\frac{p}{h}\cos\frac{v}{R} + \frac{1}{Rh}\sin\frac{v}{R} = \frac{1}{Rh\,\sin\frac{v}{R}} = \frac{\sqrt{6}}{h},$$

hence

$$de_3^* = -\frac{\sqrt{6}}{h}(\bar{\xi}dz + \xi d\bar{z}). \tag{6.7}$$

Therefore, the Frenet formula (4.17) of W^2 in $S^{n+4}(R)$ becomes the following one in E_1^{n+4}:

$$\left\{\begin{aligned} dx &= \frac{1}{h}(\bar{\xi}dz + \xi d\bar{z}), \\ de_3^* &= -\frac{\sqrt{6}}{h}(\bar{\xi}dz + \xi d\bar{z}), \\ d\xi &= \frac{1}{h}\xi(\bar{z}dz - zd\bar{z}) + \frac{2\sqrt{6}}{h}e_3^*dz + \frac{\sqrt{10}}{h}\eta d\bar{z}, \\ d\eta &= -\frac{\sqrt{10}}{h}\xi dz + \frac{2}{h}\eta(\bar{z}dz - zd\bar{z}) + \frac{\sqrt{6}}{h}\zeta d\bar{z}, \\ d\zeta &= -\frac{\sqrt{6}}{h}\eta dz + \frac{3}{h}\zeta(\bar{z}dz - zd\bar{z}), \end{aligned}\right. \tag{6.8}$$

which is completely identical with the system of equations in Case $\bar{M}^{n+4} = E^{n+4}$.

We can construct a minimal submanifold M^n with M-index 2 of geodesic codimension 4 in the sphere $S^{n+4}(R)$ by means of the results of the previous sections.

7. Solutions in Case $\bar{M}^{n+4} = H^{n+4}(\bar{c})$. In this section, we shall find M^n in $(n+4)$-dimensional hyperbolic space $H^{n+4}(\bar{c})$ of curvature $\bar{c}$ as in Theorem 2.

In this case, (4.5) and (1.14) imply

$$\bar{c} = 6 - p^2 = 6 + \bar{c} \coth^2 \sqrt{-\bar{c}}\, v,$$

i. e.

$$(7.1) \qquad -\bar{c} = 6 \sinh^2 \sqrt{-\bar{c}}\, v.$$

We use the Poincare representation of $H^{n+4}(\bar{c})$ in the unit disk in E^{n+4} with the canonical coordinates $x_1, x_2, \cdots, x_{n+4}$. Its line element, as is well known, is given by

$$(7.2) \qquad ds^2 = \frac{4\, R^2\, dx\cdot dx}{(1 - x\cdot x)^2}, \qquad R = \sqrt{\frac{1}{-\bar{c}}}$$

Since the components of the Riemannian metric are

$$g_{ij} = \frac{4R^2}{L^2}\delta_{ij}, \qquad g^{ij} = \frac{L^2}{4R^2}\delta_{ij},$$

where

$$L = 1 - x\cdot x,$$

we have its components of the connection:

$$(7.3) \qquad \Gamma^k_{ij} = 2(\delta^k_i x_j + \delta^k_j x_i - \delta_{ij} x_k)/L.$$

For any two tangent vectors X and Y, we have

$$< X, Y > = \frac{4R^2}{L^2} X\cdot Y,$$

where $< X, Y >$ and $X\cdot Y$ denote the inner products of X and Y in $H^{n+4}(\bar{c})$ and E^{n+4}, respectively. Hence, if $(x, e_1, \cdots, e_{n+4})$ is an orthonormal frame in $H^{n+4}(\bar{c})$, then $\left(x, \frac{2R}{L}e_1, \cdots, \frac{2R}{L}e_{n+4}\right)$ is the one in E^{n+4}.

Now, for any tangent vector field $X = \sum_{j=1}^{n+4} X^j \partial/\partial x^j$, by means of (7.3) we have easily

$$(7.4) \qquad \bar{D}x = \frac{L}{2R}\left[d\left(\frac{2R}{L}X\right) + \frac{2}{L}\left\{\left(x\cdot\frac{2R}{L}X\right)dx - x\left(\frac{2R}{L}X\cdot dx\right)\right\}\right].$$

Putting

$$e_3^* = \frac{2R}{L} e_3, \quad \xi^* = \frac{2R}{L} \xi, \quad \eta^* = \frac{2R}{L} \eta, \quad \zeta^* = \frac{2R}{L} \zeta, \tag{7.5}$$

we rewrite the formula (4. 17) in these terms. First of all, we have

$$dx = \frac{L}{2Rh} (\bar{\xi}^* dz + \xi^* d\bar{z}) . \tag{7.6}$$

From the 2nd of (4. 17) and (7. 4),

$$de_3^* + \frac{2}{L} \{(x \cdot e_3^*) dx - x(e_3^* \cdot dx)\} = -\frac{p}{h} (\bar{\xi}^* dz + \xi^* d\bar{z}) .$$

By (7. 6) and $(e_3^* \cdot dx) = 0$, the above equation becomes

$$de_3^* = -\left\{p + \frac{1}{R} (x \cdot e_3^*)\right\} \frac{1}{h} (\bar{\xi}^* dz + \xi^* d\bar{z}) .$$

Now, from the third of (4. 17), we have analogously

$$d\xi^* + \frac{2}{L} \{(x \cdot \xi^*) dx - x(\xi^* \cdot dx)\} = \frac{1}{h} \xi^* (\bar{z} dz - z d\bar{z}) + \frac{2p}{h} e_3^* dz + \frac{\sqrt{10}}{h} \eta^* d\bar{z} .$$

Since we have

$$\xi^* \cdot dx = \frac{L}{2Rh} \xi^* \cdot (\bar{\xi}^* dz + \xi^* d\bar{z}) = \frac{L}{Rh} dz ,$$

the above equation becomes

$$d\xi^* = \frac{1}{h} \xi^* (\bar{z} dz - z d\bar{z}) + \frac{2}{h} \left(p e_3^* + \frac{1}{R} x\right) dz + \frac{\sqrt{10}}{h} \eta^* d\bar{z} - \frac{1}{Rh} (x \cdot \xi^*)(\bar{\xi}^* dz + \xi^* d\bar{z}) . \tag{7.7}$$

Next, from the fourth of (4.17), we have

$$d\eta^* + \frac{2}{L}\{(x\cdot\eta^*)dx - x(\eta^*\cdot dx)\}$$

$$= -\frac{\sqrt{10}}{h}\xi^* dz + \frac{2}{h}\eta^*(\bar{z}dz - zd\bar{z}) + \frac{\sqrt{6}}{h}\zeta^* d\bar{z}.$$

Since $\eta^*\cdot dx = 0$, the above relation becomes

$$(7.8) \qquad d\eta^* = -\frac{\sqrt{10}}{h}\xi^* dz + \frac{2}{h}\eta^*(\bar{z}dz - zd\bar{z}) + \frac{6}{h}\zeta^* d\bar{z}$$

$$-\frac{1}{Rh}(x\cdot\eta^*)(\bar{\xi}^* dz + \xi^* d\bar{z}).$$

Last of all, we have from the fifth of (4.17) and (7.4)

$$d\zeta^* + \frac{2}{L}\{(x\cdot\zeta^*)dx - x(\zeta^*\cdot dx)\} = -\frac{\sqrt{6}}{h}\eta^* dz + \frac{3}{h}\zeta^*(\bar{z}dz - zd\bar{z}),$$

that is

$$(7.9) \quad d\zeta^* = -\frac{\sqrt{6}}{h}\eta^* dz + \frac{3}{h}\zeta^*(\bar{z}dz - zd\bar{z}) - \frac{1}{Rh}(x\cdot\zeta^*)(\bar{\xi}^* dz + \xi^* d\bar{z}).$$

On the other hand, any geodesic starting from the origin $O = (0, \cdots, 0)$ in $H^{n+4}(\bar{c})$ is a Euclidean straight line segment in the unit disk. The arc lengths v and r in $H^{n+4}(\bar{c})$ and E^{n+4} have the relations as follows:

$$v = R\log\frac{1+r}{1-r}, \qquad r = \tanh\frac{v}{2R}.$$

Since any W^2 is congruent to others under the hyperbolic motions, we may suppose the forcal point z_0 in Theorem C is the origin O. Then, we have

$$x = -e_3^*\, r = -e_3 \tanh\frac{v}{2R},$$

and hence

$$x\cdot\xi^* = x\cdot\eta^* = x\cdot\zeta^* = 0,$$

$$L = 1 - x\cdot x = 1 - r^2 = 1 - \tanh^2\frac{v}{2R} = \frac{1}{\cosh^2\frac{v}{2R}},$$

and

$$p + \frac{1}{R}(x\cdot e_3{}^*) = p - \frac{r}{R} = \frac{1}{R}\coth\frac{v}{R} - \frac{1}{R}\tanh\frac{v}{2R}$$

$$= \frac{1}{R\sinh\frac{v}{R}} = \sqrt{6}$$

by (1.14) and (7.1).

Making use of these relations, (7.6)~(7.9) can be written as

$$(7.10)\quad \begin{cases} dx = \dfrac{1}{\left(\cosh\frac{v}{R}+1\right)R}\dfrac{1}{h}(\bar{\xi}^*dz + \xi^*d\bar{z}), \\ de_3{}^* = -\dfrac{\sqrt{6}}{h}(\bar{\xi}^*dz + \xi^*d\bar{z}), \\ d\xi^* = \dfrac{1}{h}\xi^*(\bar{z}dz - zd\bar{z}) + \dfrac{2\sqrt{6}}{h}e_3{}^*dz + \dfrac{\sqrt{10}}{h}\eta^*d\bar{z}, \\ d\eta^* = -\dfrac{\sqrt{10}}{h}\xi^*dz + \dfrac{2}{h}\eta^*(\bar{z}dz - zd\bar{z}) + \dfrac{\sqrt{6}}{h}\zeta^*d\bar{z}, \\ d\zeta^* = -\dfrac{\sqrt{6}}{h}\eta^*dz + \dfrac{3}{h}\zeta^*(\bar{z}dz - zd\bar{z}), \end{cases}$$

which is completely identical with the system of equations for W^2 in Case $\bar{M}^{n+4} = E^{n+4}$ except the first one.

Therefore, we can construct W^2 in $H^{n+4}(\bar{c})$ by the formula (5.10) and

$$(7.11)\quad x = -\frac{1}{\sqrt{6}R\left(\cosh\frac{v}{R}+1\right)}e_3{}^*.$$

Then, we can construct a minimal submanifold M^n with M-index 2 of geodesic condimension 4, taking W^2 as the base surface, according to Theorem C.

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
TOKYO, JAPAN