Tôhoku Math. Journ.
23(1971), 353-362.

ON A SATURATION THEOREM OF TURECKII

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(Received October 7, 1970)

1. Introduction. Let $C^*[-\pi,\pi]$ denote the space of $2\pi-$periodic continuous functions and $\|\cdot\|$ the supremum norm on $[-\pi,\pi]$. Many of the classical linear methods of approximating functions in $C^*[-\pi,\pi]$ are given by a sequence (L_n) of positive convolution operators. That is, L_n has the form

$$L_n(f,x) = (f*d\mu_n)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)d\mu_n(t) \tag{1.1}$$

where $d\mu_n$ is a non-negative, even Borel measure on $[-\pi,\pi]$, with $\frac{1}{\pi}\int_{-\pi}^{\pi} d\mu_n(t) = 1$.

An important concept in the study of the approximation properties of such operators is that of saturation. We say that the sequence (L_n) is saturated if there is a positive sequence of real numbers $(\phi(n))$ which tend to 0, $(n\to\infty)$, such that

i. $\|f-L_n(f)\| = o(\phi(n)), (n\to\infty)$, if and only if f is constant.

and

ii. there is a non-constant function f_0 in $C^*[-\pi,\pi]$ such that $\|f_0-L_n(f_0)\| = O(\phi(n)), (n\to\infty)$.

The sequence $(\phi(n))$ is then called the saturation order of (L_n) and the set $S(L_n)$ of those functions in $C^*[-\pi,\pi]$ which satisfy ii, is called the saturation class of (L_n). For a general discussion of saturation in Fourier Analysis, we refer the reader to the book o fP.L. Butzer and R.J. Nessel [2] or the expository article of P.L. Butzer and E. Görlich [1].

In this paper, we are interested in examining when the second moments

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{t}{2}d\mu_n(t)$$

* This research was supported by NSF Grant GP 19620

determine the saturation of (L_n). If f is in $C^*[-\pi, \pi]$ we let

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\, dt \qquad k=0, \pm 1, \cdots$$

and for a Borel measure $d\mu$

$$\rho_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\, d\mu(t) \qquad k = 0, \pm 1, \cdots$$

Of course when $d\mu$ is even $\rho_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kt\, d\mu(t)_o, k = 0, \pm 1, \pm 2, \cdots$

A.H.Tureckii [7], [8] has established the following sufficient condition for the first Fourier coefficients to determine the saturation of (L_n).

THEOREM (Tureckii). *If (L_n) is a sequence of linear operators of the form* (1.1) *and if*

$$\lim_{n\to\infty} \frac{1-\rho_{k,n}}{1-\rho_{1,n}} = k^2, \; k = \pm 1, \pm 2, \cdots \tag{1.2}$$

then (L_n) is saturated with order $(1-\rho_{1,n})$ and saturation class $S(L_n) = \{f: f' \in \mathrm{Lip}\ 1\}$.

The condition (1.2) has many equivalent formulations. A general accounting of these can be found in the papers of E. Görlich and E.L. Stark [4,5]. In particular the condition (1.2) is equivalent to

$$\int_{-\pi}^{\pi} t^4 d\mu_n(t) = o\left(\int_{-\pi}^{\pi} t^2 d\mu_n(t)\right) (n\to\infty). \tag{1.3}$$

The condition (1.3) indicates more clearly the behavior of the measures $d\mu_n$ which is used in the proof of Tureckii's Theorem. Indeed, what is needed is that for each $\varepsilon>0$

$$\int_{[-\pi,\pi]\setminus[-\varepsilon,\varepsilon]} d\mu_n(t) = o\left(\int_{-\pi}^{\pi} \sin^2 \frac{t}{2} d\mu_n(t)\right) (n\to\infty). \tag{1.4}$$

In other words, the integrals of the measures $d\mu_n$ outside each neighborhood of 0 must be negligible in comparison with the saturation order.

The object of this paper is to extend the Theorem of Tureckii by replacing the conditions (1.2) and (1.4) by the following weaker conditions.

A. *There exists a constant* $C_A>0$, *such that for each integer* k, *there is an* $N(k)$, *for which*

$$\frac{1-\rho_{k,n}}{1-\rho_{1,n}} \geqq C_A k^2 \qquad \textit{when } n \geqq N(k)$$

B. *There exists a constant* $C_B>0$, *such that for each* $\varepsilon>0$, *there is an* $N(\varepsilon)$, *for which*

$$\int_{-\varepsilon}^{\varepsilon} \sin^2\frac{t}{2}\, d\mu_n(t) \geqq C_B \int_{-\pi}^{\pi} \sin^2\frac{t}{2}\, d\mu_n(t) \ \textit{ when } n \geqq N(\varepsilon)\,.$$

These two conditions are equivalent and this is shown in Section 2. In Section 3 we shall prove the following extension of Tureckii's Theorem

THEOREM 1. *If* (L_n) *is a sequence of operators of the form* (1.1) *and if either condition A or condition B is satisfied then* (L_n) *is saturated with order* $(1-\rho_{1,n})$ *and saturation class* $S(L_n)=\{f\colon f'\in \mathrm{Lip}\ 1\}$.

Although Tureckii's Theorem determines the saturation properties of many classical methods of approximation (e.g. the Jackson and Korovkin operator (see [1, p.375]), it is easy to construct sequences of operators for which B holds but (1.4) is not satisfied. Indeed, if $(d\mu_n)$ is a sequence of measures for which (1.4) holds then each measure $d\mu_n$ can be altered slightly so that (1.4) is no longer true while B still is satisfied. We will now illustrate this point with the following example. Let K_n denote the Jackson kernel of degree $2n-2$ [6]

$$K_n(t) = C_n\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^4$$

with C_n the normalizing constant. The trigonometric polynomials

$$\Lambda_n(t) = \frac{n^2}{n^2+1}K_n(t) + \frac{1}{2n^2}(K_n(t-\pi)+K_n(t+\pi))$$

generate by convolution a sequence of operators $(L_n(f)=f*\Lambda_n)$, which satisfy condition B but do not satisfy (1.4). It is well known [1] that for each $\varepsilon > 0$

$$\lim_{n\to\infty} n^2 \int_{-\varepsilon}^{\varepsilon} \sin^2 \frac{t}{2} K_n(t)dt = \frac{3}{4}$$

so that B is satisfied for (Λ_n). However

$$\frac{1}{\pi} \int_{\pi/2}^{3\pi/2} K_n(t-\pi)dt \to 1 \quad (n\to\infty)$$

and thus (1.4) does not hold.

This example also answers a question of Görlich and Stark [4], who asked whether every sequence (T_n) of non-negative even trigonometric polynomials, with T_n of degree $\leqq n$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} T_n(t)dt = 1$, which satisfy

$$\int_{-\pi}^{\pi} \sin^2 \frac{t}{2} T_n(t)dt = O\left(\frac{1}{n^2}\right) \tag{1.5}$$

must also satisfy (1.3). The above example shows that this is not true. However, it can be shown that each such sequence must satisfy A and B (see [3]). Thus the saturation properties of operators generated by convolution with the polynomials T_n are determined by Theorem 1. A more general treatment of saturation of trigonometric convolution operators is given in [3]. This paper also contains most of the techniques which will be used here.

2. LEMMA 1. *The conditions* A *and* B *are equivalent.*

PROOF. We first show that B implies A. Let k be a non-zero integer and choose $0 < \varepsilon < \pi/|k|$. Then

$\sin^2 \frac{kt}{2} \geqq \left(\frac{2}{\pi}\right)^2 k^2 \sin^2 \frac{t}{2}$ on $(-\varepsilon, \varepsilon)$, and so, if we let $N(\varepsilon)$ be as given in B we have

$$\int_{-\pi}^{\pi} \sin^2 \frac{kt}{2} d\mu_n(t) \geqq \int_{-\varepsilon}^{\varepsilon} \sin^2 \frac{kt}{2} d\mu_n(t) \geqq \left(\frac{2}{\pi}\right)^2 k^2 \int_{-\varepsilon}^{\varepsilon} \sin^2 \frac{t}{2} d\mu_n(t)$$

$$\geqq \left(\frac{2}{\pi}\right)^2 k^2 C_B \int_{-\pi}^{\pi} \sin^2 \frac{t}{2} d\mu_n(t), \text{ for } n \geqq N(\varepsilon).$$

Therefore, A holds with $N(k)=N(\varepsilon)$, and $C_A=\left(\frac{2}{\pi}\right)^2 C_B$.

We will now show that A implies B with $C_B=C_A/4$. Suppose B does not hold with $C_B=C_A/4$. Then there is an $\varepsilon_0>0$ and a sequence (n_j) such that

$$(2.1)\qquad \int_{-\varepsilon_0}^{\varepsilon_0} \sin^2\frac{t}{2}\,d\mu_{n_j}(t)\leqq\frac{C_A}{4}\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,d\mu_{n_j}(t),\ j=1,2,\cdots.$$

Let $\phi(n)=\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,d\mu_n(t)$ and consider the measures $d\nu_{n_j}$ which are $\frac{1}{\phi(n_j)}d\mu_{n_j}$ on $[-\pi,\pi]\backslash(-\varepsilon_0,\varepsilon_0)$, and 0 on$(-\varepsilon_0,\varepsilon_0)$, $j=1,2,\cdots$. Then $\int_{-\pi}^{\pi}d\nu_{n_j}(t)\leqq\frac{1}{\sin^2\varepsilon_{0/2}}\frac{1}{\phi(n_j)}$ $\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,d\mu_{n_j}(t)=\frac{1}{\sin^2\varepsilon_{0/2}}$. Thus the sequence of measures $(d\nu_{n_j})$ lies in a compact subset of the dual space of $C^*[-\pi,\pi]$ with the weak* topology. Hence, there is a subsequence $(n_j')\subseteq(n_j)$ and a measure $d\nu$ such that $d\nu_{n_j'}$ converges weak* to $d\nu$. In particular for eack k

$$(2.2)\qquad \lim_{n'_i\to\infty}\int_{-\pi}^{\pi}\sin^2\frac{kt}{2}\,d\nu_{n_j}'(t)=\int_{-\pi}^{\pi}\sin^2\frac{kt}{2}\,d\nu\leqq\int_{-\pi}^{\pi}d\nu\,.$$

Now, choose k_0 so large that

$$(2.3)\qquad \frac{C_Ak_0^2}{2}\geqq\int_{-\pi}^{\pi}d\nu\,.$$

Then by virtue of (2.3), we have that for n_j' sufficiently large, say$\geqq N$

$$\frac{1}{\phi(n_j')}\int_{-\varepsilon_0}^{\varepsilon_0}\sin^2\frac{k_0t}{2}\,d\mu_{n'_j}(t)=\frac{1}{\phi(n_j')}\int_{-\pi}^{\pi}\sin^2\frac{k_0t}{2}\,d\mu_{n'_j}(t)-\int_{-\pi}^{\pi}\sin^2\frac{k_0t}{2}\,d\mu_{n'_j}(t)$$
$$\geqq\frac{1}{\phi(n_j')}\left(\int_{-\pi}^{\pi}\sin^2\frac{k_0t}{2}\,d\mu_{n'_j}(t)\right)-C_A\frac{k_0^2}{2}.$$

Thus, using condition A we have for $n_j'\geqq\max\ (N,N(k_0))$

$$\int_{-\varepsilon_0}^{\varepsilon_0} \sin^2 \frac{k_0 t}{2} d\mu_{n'_j}(t) \geqq \frac{C_A k_0^2}{2} \int_{-\pi}^{\pi} \sin^2 \frac{t}{2} d\mu_{n'_j}(t).$$

Finally, since $\sin^2 \frac{k_0 t}{2} \leqq k_0^2 \sin^2 \frac{t}{2}$, we have that

$$\int_{-\varepsilon_0}^{\varepsilon_0} \sin^2 \frac{t}{2} d\mu_{n'_j}(t) \geqq \frac{1}{k_0^2} \int_{-\varepsilon_0}^{\varepsilon_0} \sin^2 \frac{k_0 t}{2} d\mu_{n'_j}(t) \geqq \frac{C_A}{2} \int_{-\pi}^{\pi} \sin^2 \frac{t}{2} d\mu_{n'_j}(t)$$

which is the desired contradiction to (2.1) and the Lemma is proved.

3. PROOF OF THEOREM 1. Suppose (L_n) is a sequence of positive linear operators of the form (1.1) which satisfy either A or B. By virtue of Lemma 1, both A and B are satisfied, and we will use them interchangeably. We first wish to show that (L_n) is saturated with order $(1-\rho_{1,n})$. Suppose $f \in C^*[-\pi, \pi]$ and

$$\|f - L_n(f)\| = o(1-\rho_{1,n}) \qquad (n \to \infty),$$

then $\hat{f}(k) - \hat{f}(k)\rho_{k,n} = o(1-\rho_{1,n})$ $(n \to \infty)$. Since $1-\rho_{k,n} \geqq C_A k^2 (1-\rho_{1,n})$ $n \geqq N(k)$ we have $\hat{f}(k) = 0$, $k = \pm 1, \pm 2, \cdots$. Therefore, f is a constant function. The function $f_0(t) = \sin^2 \frac{t}{2}$ is clearly a non-constant function for which

$$\|f_0 - L_n(f_0)\| = O(1-\rho_{1,n}) \qquad (n \to \infty).$$

Thus, (L_n) is saturated with order $(1-\rho_{1,n})$.

We now wish to characterize the saturation class $S(L_n)$. A function $f \in C^*[-\pi, \pi]$ is in $S(L_n)$ if and only if

$$\left\| \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x+t) + f(x-t) - 2f(x)) d\mu_n(t) \right\| = O(1-\rho_{1,n}) \quad (n \to \infty)$$

where we have used the fact that each $d\mu_n$ is even. Equivalently, $f \in S(L_n)$ if and only if

$$\left\| \int_{-\pi}^{\pi} \frac{f(x+t) + f(x-t) - 2f(x)}{\sin^2 \frac{t}{2}} d\psi_n(t) \right\| = O(1) \quad (n \to \infty)$$

where $d\psi_n(t) = \frac{1}{\pi}(1-\rho_{1,n})^{-1} \sin^2\frac{t}{2} d\mu_n(t)$.

Since $\int_{-\pi}^{\pi} d\psi_n(t)=1/2$, $n=1,2,\cdots$, it is clear that if $f' \in \mathrm{Lip}\ 1$, then f is in $S(L_n)$.

We need to show that if $f \in S(L_n)$ then $f' \in \mathrm{Lip}\ 1$. We shall first show that if f is twice continuously differentiable and

$$\|f-L_n(f)\| \leqq M(1-\rho_{1,n}) \qquad (n\to\infty) \tag{3.1}$$

then

$$\|f''\| \leqq C(M+\|f\|) \tag{3.2}$$

where C is a constant independent of f.

Since each measure $d\psi_n$ has norm $1/2$ there is a subsequence (n_j) and a measure $d\psi$ such that $(d\psi_{n_j})$ converges weak* to $d\psi$. Using Condition B and the weak* convergence we have for each $\varepsilon > 0$

$$\int_{-\varepsilon}^{\varepsilon} d\psi \geqq \lim_{j\to\infty}\int_{-\varepsilon}^{\varepsilon} d\psi_{n_j} \geqq C_B\,. \tag{3.3}$$

Choose ε_0 so small that

$$\int_{(-\varepsilon_0,\varepsilon_0)\setminus\{0\}} d\psi \leqq \frac{C_B}{\pi^2}\,. \tag{3.4}$$

Now, if f is twice continuously differentiable and satisfies (3.1), then

$$\left\|\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}\, d\psi(t)\right\|$$

$$\leqq \lim_{n_j\to\infty}\left\|\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}\, d\psi_{n_j}(t)\right\| \leqq M\,.$$

Thus, we have

$$\left\|\int_{-\varepsilon_0}^{\varepsilon_0} \frac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}\, d\psi(t)\right\| \tag{3.5}$$

$$\leqq M+\left\|\int_{[-\pi,\pi]\backslash(\varepsilon_0,\varepsilon_0)}\frac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}d\psi(t)\right\|$$

$$\leqq M+\frac{4\|f\|}{\sin^2\frac{\varepsilon_0}{2}}.$$

Since $\dfrac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}$ has the value $4f''(x)$ at $t=0$, we have from (3. 3) that

$$\left\|\int_{-\varepsilon_0}^{\varepsilon_0}\frac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}d\psi(t)\right\| \tag{3.6}$$

$$\geqq 4C_B\|f''\|-\left\|\int_{(-\varepsilon_0,\varepsilon_0)\backslash\{0\}}\frac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}d\psi(t)\right\|.$$

Since $\left|\dfrac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}\right|\leqq\|f''\|\dfrac{t^2}{\sin^2\frac{t}{2}}\leqq\pi^2\|f''\|,\ t>0,$

we have from (3.6) and (3.4) that

$$\left\|\int_{-\varepsilon_0}^{\varepsilon_0}\frac{f(x+t)+f(x-t)-2f(x)}{\sin^2\frac{t}{2}}d\psi(t)\right\|\geqq 4C_B\|f''\| \tag{3.7}$$

$$-C_B\|f''\|\geqq 3C_B\|f''\|$$

Using (3.7) with (3.5) gives

$$\|f''\|\leqq\frac{1}{3C_B}\left(M+\frac{4}{\sin^2\frac{\varepsilon_0}{2}}\|f\|\right)$$

which establishes (3.2).

Finally let f be any function in $S(L_n)$, such that

$$\|f - L_n(f)\| \leqq M(1 - \rho_{1,n}) \qquad n = 1, 2, \cdots.$$

Consider the twice continuously differentiable function $f_m = f * K_m$ where K_m is the Jackson kernel of degree $2m-2$. Then for f_m, we have

$$\|f_m - L_n(f_m)\| = \|f*K_m - f*K_m*d\mu_n\| = \|(f - f*d\mu_n)*K_m\|$$
$$\leqq \|f - f*d\mu_n\| \frac{1}{\pi}\int_{-\pi}^{\pi} K_m(t)dt \leqq M(1 - \rho_{1,n}) \quad n = 1, 2, \cdots.$$

Thus, from (3.2) and the fact that $\|f_m\| \leqq \|f\|$, $m = 1, 2, \cdots$, we have

$\|f''_m\| \leqq C(M + \|f_m\|) \leqq C(M + \|f\|)$.

If $|t| > 0$, $x \in [-\pi, \pi]$

$$\left|\frac{f_m(x+t) + f_m(x-t) - 2f_m(x)}{t^2}\right| \leqq C(M + \|f\|), \quad m = 1, 2, \cdots. \tag{3.8}$$

Taking a limit as $(m \to \infty)$ in (3.8) shows that

$$\left|\frac{f(x+t) + f(x-t) - 2f(x)}{t^2}\right| \leqq C(M + \|f\|), \ x \in [-\pi, \pi], \ |t| > 0$$

which is equivalent to $f' \in \text{Lip } 1$.

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