Tôhoku Math. Journ.
24 (1972), 105-108.

AN EXAMPLE OF RIEMANNIAN MANIFOLDS SATISFYING $R(X, Y)\cdot R = 0$ BUT NOT $\nabla R = 0$

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(Received Nov. 20, 1971)

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

$$(*) \qquad R(X, Y)\cdot R = 0 \text{ for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Conversely, does this algebraic condition (*) on the curvature tensor field R imply that M is locally symmetric (i.e. $\nabla R = 0$)? For this problem, K. Nomizu conjectured that the answer is affirmative in the case where M is irreducible and complete and dim $M \geqq 3$.

In the present paper, we shall show that, in a 4-dimensional Euclidean space E^4, there exists an irreducible and complete hypersurface M which satisfies the condition (*) but is not locally symmetric.

The author wishes to express his sincere thanks to Prof. S. Sasaki and Prof. S. Tanno who gave him many valuable suggestions and guidances.

1. Reduction of condition (*). Let M be a 3-dimensional Riemannian manifold which is isometrically immersed in a Euclidean space E^4. Let U be a neighborhood of a point $p_0 \in M$ on which we can choose a unit vector field N normal to M. For any vector fields X and Y tangent to M, we have the formulas of Gauss and Weingarten:

$$(1.1) \qquad \begin{aligned} D_X Y &= \nabla_X Y + H(X, Y)N\,, \\ D_X N &= -AX\,, \end{aligned}$$

where D_X and ∇_X denote covariant differentiations for the Euclidean connection of E^4 and the Riemannian connection on M, respectively. A is a field of symmetric endomorphisms which corresponds to the second fundamental form H, that is, $H(X, Y) = g(AX, Y)$ for tangent vectors X and Y, g being the Riemannian metric induced from E^4. The equation of Gauss expresses the curvature tensor R of M by means of A:

$$R(X, Y)Z = g(Z, AY)AX - g(Z, AX)AY\,.$$

The type number $t(p)$ at $p \in M$ is, by definition, the rank of A at p. At a point $p \in M$, let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the tangent space $T_p(M)$ such that $Ae_i = \lambda_i e_i$ $(i = 1, 2, 3)$.

LEMMA 1.1. (cf. [2]) *At a point p, the condition (*) is equivalent to $\lambda_i\lambda_j\lambda_k(\lambda_i - \lambda_j) = 0$ for $k \neq i, j$ where $i \neq j$. Thus, the condition (*) is satisfied at p if either*

(a) $$t(p) = 3 \text{ and } \lambda_1 = \lambda_2 = \lambda_3$$

or

(b) $$t(p) \leqq 2 .$$

From now on, we shall assume that the type number $t(p)$ at p is not greater than 2 for any point $p \in M$ and that there exists at least one point $p_0 \in M$ such that $t(p_0)$ is equal to 2. By continuity of the eigenvalues of A, there exists a neighborhood W on M at p_0 such that $t(p) = 2$ for all $p \in W$.

We shall now define a 1-dimensional distribution on W as follows:

$$T_0(p) = \{X \in T_p(M) \colon AX = 0\} .$$

LEMMA 1.2. (cf. [2]) *T_0 is differentiable and totally geodesic.*

Let $V \in T_0$ be a unit vector field on W, then we have $\nabla_V V = 0$ by the above lemma and hence $D_V V = 0$ by (1.1). This means that an integral curve of V is a piece of a straight line in E^4.

LEMMA 1.3. *T_0 is parallel in M if and only if the family of integral curves of V is parallel in E^4.*

PROOF. From (1.1), we have $D_X V = \nabla_X V$ for all vector fields X tangent to M.

LEMMA 1.4. *T_0 is parallel in M, if M is either a locally symmetric space or a locally product space as a Riemannian manifold.*

PROOF. Assume that M is a locally product space, then it is locally of the form $M^2 \times M^1$, where M^2 and M^1 are a 2-dimensional space and a 1-dimensional space, respectively. Then, the Ricci tensor field S of M is recurrent (i.e. $\nabla_X S = \alpha(X)S$ for a certain 1-form α and for all vector fields X on M). On the other hand, since S is given by

$$S(Z, Y) = g(AZ, Y) \operatorname{trace} A - g(A^2Z, Y)$$

for vector fields Y and Z, we have $S(V, X) = 0$ for any X. It is easy to show that the rank of S is 2 on W, that is, $S(Z, X) = 0$ for any X implies that $Z = \beta V$ for a certain scalar field β. Since

$$\alpha(Y)S(V, X) = (\nabla_Y S)(V, X) = \nabla_Y(S(V, X)) - S(\nabla_Y V, X) - S(V, \nabla_Y X) ,$$

we have $S(\nabla_Y V, X) = 0$ and hence $D_Y V = \nabla_Y V = 0$ because V is unit. The proof for a locally symmetric case is similar.

2. An example of Riemannian manifolds satisfying $R(X, Y)\cdot R = 0$ but not $\nabla R = 0$. In this section, the hypersurface M in consideration will be one defined by the form $w = f(x, y, z)$ where (x, y, z, w) is a Cartesian coordinate system in E^4 and f is a C^∞ real valued function defined on E^3. Of course, M is deffeomorphic to E^3 and (x, y, z) is a coordinate system globally defined on M.

M is represented by a position vector P as follows:

$$P = (x, y, z, f(x, y, z)) .$$

Since $P_x = (1, 0, 0, f_x)$, $P_y = (0, 1, 0, f_y)$ and $P_z = (0, 0, 1, f_z)$ are tangent to M, the unit normal vector field on M is represented by

$$N = 1/h(-f_x, -f_y, -f_z, 1) ,$$

where $h = (1 + f_x^2 + f_y^2 + f_z^2)^{1/2}$. Using the formula of Gauss, the second fundamental form H is represented by the coordinates x, y, z as follows:

$$H = 1/h\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} . \tag{2.1}$$

Then,

$$\det\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = 0 \text{ for each } (x, y, z) \in E^3$$

is a condition of the type number $t(p) \leqq 2$ for each point $p \in M$.

Using a theory of implicit functions, we have the following

LEMMA 2.1. *Let $F(\xi, \eta, \zeta)$ be a real valued C^∞ function defined on E^3 which has no singular point* (i.e. $F_\xi^2 + F_\eta^2 + F_\zeta^2 \neq 0$ *anywhere in* E^3). *If f satisfies a partial differential equation*

$$F(f_x, f_y, f_z) = 0 ,$$

then $t(p) \leqq 2$ for each point $p \in M$.

PROOF. It is obvious.

LEMMA 2.2. *Let W be a neighborhood on M such that for each $p \in W$, $t(p) = 2$. Then, T_0 is parallel if and only if there exist real constants a, b, c and d $(a^2 + b^2 + c^2 + d^2 = 1)$ such that $af_x + bf_x + cf_z = d$ on W.*

PROOF. The condition $af_x + bf_y + cf_z = d$ is equivalent to $N \cdot V = 0$ for the parallel vector field $V = (a, b, c, d)$ in E^4, which means that V is tangent to M. And moreover, $V \in T_0$ (i.e. $AV = 0$) is easily seen from (2.1). Then, by lemma 1.3, T_0 is parallel. The converse is clear.

Now, let us consider the hypersurface M defined by

$$w = (x^2z - y^2z - 2xy)/2(z^2 + 1)$$

or

$$2z^2w - x^2z + y^2z + 2w + 2xy = 0 ,$$

which satisfies the non-linear partial differential equation

$$w_x^2 - w_y^2 + 2w_z = 0 .$$

By lemma 2.1, the type number $t(p) \leqq 2$ at each point of M. In fact, $t(p) = 2$ almost everywhere on M. Then the condition (*) is satisfied by lemma 1.1. And there exists a neighborhood W such that $t(p) = 2$ for each $p \in W$. But, by lemma 2.2, T_0 is not parallel on W and hence M is irreducible and not locally symmetric by lemma 1.4. Since M is isometrically immersed and closed in E^4, M is complete.

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