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SUMMATION FORMULAS AND BAND-LIMITED SIGNALS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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There is an extensive literature on band-limited signals, i.e. functions f of the form

$$(1) \qquad f(t) = \int_{-\pi}^{\pi} e^{-itu} F(u) du ,$$

and their representation by the sampling theorem (known to mathematicians as the cardinal series),

$$(2) \qquad f(t) = \sum_{-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} ,$$

which does represent f under reasonable conditions on F. I take the band-width to be π to simplify the formulas; the general case,

$$f(t) = \int_{-T}^{T} e^{-itu} F(u) du$$

can be dealt with by a change of variable.

The point of the present note is that many results from this circle of ideas can be obtained rather quickly from well-known summation formulas. I shall illustrate this by using Poisson's summation formula to derive not only the sampling theorem (2) itself, but also an estimate, due to Weiss [7] and Brown [2], for the error when (2) is applied to an f that is not in fact band-limited, and some related formulas obtained by Jagerman [4]. (One can give similar applications of the Euler-Maclaurin summation formula.) Finally I give (independently of summation formulas) a very short proof of (2) when F is integrable.

1. Poisson's formula. The form usually given [6, p. 60] is that if

$$(3) \qquad g(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-iux} G(u) du , \qquad G(u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(x) e^{iux} dx ,$$

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then

$$\sum_{k=-\infty}^{\infty} g(k) = (2\pi)^{1/2} \sum_{k=-\infty}^{\infty} G(2\pi k) . \tag{4}$$

Sufficient conditions for (4) are that g is continuous, of bounded variation, zero at $\pm\infty$, with $\int_{-\infty}^{\infty} g(x)dx$ convergent (which implies, since g is of bounded variation, that $\sum g(k)$ converges); or, of course, that G satisfies the same conditions. There is a more general form, obtainable from (4) by appropriate substitutions (see [5, p. 217]), which G. H. Hardy used to give in lectures: if A, B, a, b are real and $AB = 2\pi$ then

$$A^{1/2}e^{\pi abi} \sum_{n=-\infty}^{\infty} e^{2n\pi ai}G[A(n+b)] = B^{1/2}e^{-\pi abi} \sum_{n=-\infty}^{\infty} e^{2n\pi bi}g[B(n-a)] . \tag{5}$$

2. Poisson's formula and $\sum f(np+q)$. Let us change the notation in (1) so that it now reads

$$f(t) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} e^{-itu}F(u)du , \tag{6}$$

with $F(u) = 0$ for $|u| \geqq \pi$. Suppose that $f \in L$, i.e. that $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$. It is known from the theory of entire functions [1, pp. 101, 211] that $f' \in L$ and $f(x) \to 0$ as $|x| \to \infty$. This makes (4) applicable to $g = f$, $G = F$. All the terms on the right of (4) are zero except the one with $k = 0$, and this term is

$$(2\pi)^{1/2}F(0) = \int_{-\infty}^{\infty} f(t)dt ,$$

so that

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(t)dt , \tag{7}$$

as obtained by Jagerman [4] under more restrictive hypotheses. In fact, we see that (7) holds even if $F(u) = 0$ for $|u| \geqq 2\pi$, i.e. if the band-width is allowed to be twice as large. (Added in proof: Formula (7) was obtained earlier, in the same way, by N. Wiener; see [8], [9].)

More generally, if we drop the assumption that $F(u) = 0$ for $|u| \geqq \pi$, but suppose that $f \in L$, $f' \in L$, $f(\pm\infty) = 0$ (which would certainly be true if $f(u) = 0$ for $|u| \geqq A$ with some A), then

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(t)dt + (2\pi)^{1/2} \sum_{|k|>0} F(2\pi k) ,$$

and so the difference $\left|\sum f(n) - \int f(t)dt\right|$ does not exceed $(2\pi)^{1/2}\sum_{|k|>0} |F(2\pi k)|$.

We can obtain other formulas like (7) by specializing the parameters in (5) in various ways.

Take $a = 0$, $b = 1/2$, $A = 2\pi$, $B = 1$, $g = f$ as in (6); then

$$\sum(-1)^n f(n) = (2\pi)^{1/2} \sum F\left(2\pi\left(n + \frac{1}{2}\right)\right) ;$$

but all the terms on the right are zero and we obtain

$$\sum_{-\infty}^{\infty} f(2n) = \sum_{-\infty}^{\infty} f(2n+1) . \tag{8}$$

Next take $a = 0$, $b = 1/2$, $A = \pi$, $B = 2$. Then we have

$$2^{1/2} \sum (-1)^n f(2n) = \pi^{1/2}\{F(\pi/2) + F(-\pi/2)\} .$$

Similarly if $b = 0$, $A = 2\pi/3$, $B = 3$,

$$3^{1/2} \sum f(3(n-a)) = (2\pi/3)^{1/2}\{F(0) + e^{2\pi a i}F(2\pi/3) + e^{-2\pi a i}F(-2\pi/3)\} ,$$

and in particular when $a = 0$ or $1/2$,

$$3 \sum f(3n) = (2\pi)^{1/2}\{F(0) + F(2\pi/3) + F(-2\pi/3)\} ,$$

$$3 \sum f\left(3\left(n - \frac{1}{2}\right)\right) = (2\pi)^{1/2}\{F(0) - F(2\pi/3) - F(-2\pi/3)\} .$$

3. Poisson's formula and the sampling theorem. In (5) take $g = f$, $a = 0$, $b = t/(2\pi)$, $A = 2\pi$, $B = 1$, so that Poisson's formula reads (formally)

$$(2\pi)^{1/2} \sum_{n=-\infty}^{\infty} F(2n\pi + t) = \sum_{n=-\infty}^{\infty} f(n)e^{int} . \tag{9}$$

Suppose that $F \in L$. Since

$$\infty > \int_{-\infty}^{\infty} | F(t) | \, dt = \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{2(k+1)\pi} | F(t) | \, dt = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} | F(2n\pi + t) | \, dt ,$$

the series on the left of (9) converges for almost all t to an integrable function, whose kth Fourier coefficient is

$$(2\pi)^{-1/2} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} F(2n\pi + t)e^{-ikt}dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ikt}F(t)dt = f(k)$$

(the term-by-term integration being permitted by dominated convergence). Hence the series on the right of (9) is the Fourier series of the function on the left, and we can multiply both sides by e^{-ixt} and integrate over $(-\pi, \pi)$. The result is

$$\begin{aligned}(2\pi)^{1/2} &\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-ixt}F(2\pi n + t)dt \\ &= \sum_{n=-\infty}^{\infty} f(n)\int_{-\pi}^{\pi} e^{i(n-x)t}dt = 2\pi \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)} ,\end{aligned}$$

i.e.

$$\sum_{-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)} = (2\pi)^{-1/2}\int_{-\pi}^{\pi} e^{-ixt}F(2n\pi + t)dt$$
$$= (2\pi)^{-1/2}\sum_{n=-\infty}^{\infty}\int_{(2n-1)\pi}^{(2n+1)\pi} e^{-ixt}e^{2n\pi ix}f(t)dt\ .$$

But we have

$$f(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ixt}F(t)dt = (2\pi)^{-1/2}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\pi}^{(2k+1)\pi} e^{-ixt}F(t)dt\ ,$$

and so

$$(10)\quad f(x) - \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)} = (2\pi)^{-1/2}\sum_{k=-\infty}^{\infty}(1 - e^{2k\pi ix})\int_{(2k-1)\pi}^{(2k+1)\pi} F(t)dt\ .$$

If $F(t) = 0$ for $|t| \geqq \pi$, we have (1). In general, the absolute value of the left side of (10), i.e. of the difference of the two sides of (1), does not exceed

$$2(2\pi)^{-1/2}\sum_{k=-\infty}^{\infty} |\sin k\pi x| \int_{(2k-1)\pi}^{(2k+1)\pi} |F(t)|\, dt = 2(2\pi)^{-1/2}\int_{|t|>\pi} |F(t)|\, dt\ .$$

This is equivalent to Brown's result [2], which is, as he shows, the best result of its kind. However, it can be sharpened if x is not one of the points $(2n + 1)\pi/2$, since $|\sin k\pi x| < 1$ except at these points.

We could also obtain the same formulas by applying (5) to

$$g(u) = f(u)\frac{\sin \pi(x-u)}{\pi(x-u)}$$

(with a fixed x).

5. Short proof of the sampling theorem. If we just want to prove the sampling theorem under rather general hypotheses, the following proof is quite simple. Let

$$f(t) = \int_{-\pi}^{\pi} e^{-itu}F(u)du\ , \qquad F \in L\ .$$

This means that $f(n)/(2\pi)$ are the Fourier coefficients of F repeated with period 2π. Since a Fourier series can be multiplied by the bounded function e^{-ixt} and integrated term by term, we have

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)(2\pi)^{-1}\int_{-\pi}^{\pi} e^{int-ixt}dt = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(n-x)}{\pi(n-x)}\ .$$

The same method could be applied to functions represented by finite Fourier transforms of distributions (Campbell [3]). (Added in proof: The same proof has been found independently by Pollard and Shisha [10].)

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