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ON A SEMI-DIRECT PRODUCT DECOMPOSITION OF AFFINE GROUPS OVER A FIELD OF CHARACTERISTIC 0

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Let G be an abstract group and $f_1: G \to GL_k(V_1)$ and $f_2: G \to GL_k(V_2)$ be two finite dimensional representation of G over a field k of characteristic 0. It is well known that if f_1 and f_2 are semi-simple then $f_1 \otimes f_2$ is also semi-simple [**3**, IV, §3, 3.5; **4**, Th. 12.2]. In view of this fact we first show

PROPOSITION 0. *The coradical R of a commutative Hopf algebra A over a field k of characteristic* 0 *is a sub-Hopf algebra.*

PROOF. The Jacobson radical of a ring is the intersection of its maximal left (or right) ideals. Dually the coradical of a coalgebra C over a field is identical with the socle of C as a right (or left) C-comodule. Since k is perfect, the coradical of $\bar{k} \otimes_k A$ is $\bar{k} \otimes_k R$, where $\bar{k}$ is the algebraic closure of k. Hence we can assume that $k = \bar{k}$. Moreover A can be assumed to be finitely generated as a k-algebra. Let V_i, $i = 1, 2$ be two finite dimensional right A-comodules. Becauce $G(A^0) = \mathrm{Alg}_k(A, k)$ is dense in $A^* = \mathrm{Hom}_k(A, k)$ [**6**, Lem. 3.6], V_i is a semisimple A-comodule iff V_i is a semisimple left $G(A^0)$-module. Hence by the remark above if V_i are semisimple, then $V_1 \otimes V_2$ is also semi-simple. This means that $R \otimes R$ is a semisimple right A-comodule. Since the multiplication $\mu: A \otimes A \to A$ is a right A-comodule map, $R \cdot R$ is contained in R. Clearly R is stable under the antipode of A. Hence R is a sub-Hopf algebra of A.

The purpose of this paper is to prove

THEOREM 1. *Let A be a commutative Hopf algebra over a field k of characteristic* 0 *and R its coradical. Then there exists a Hopf algebra map $\pi: A \to R$ such that π = identity on R.*

This follows from the following three propositions, where A and R are as in Theorem 1. We assume k is of characteristic 0 throughout this paper.

PROPOSITION 2. *Let B be a sub-Hopf algebra of A which contains R. If $B \neq A$, then there exists a sub-Hopf algebra C of A which contains B properly such that $C/(C \cdot B^+)$ is cocommutative, where* $B^+ = \mathrm{Ker}(\varepsilon: B \to k)$.

PROPOSITION 3. *Let B be a sub-Hopf algebra of A which contains R. If $\pi: B \to R$ is a Hopf algebra map such that π = identity on R, then the ideal I of A generated by $\pi^{-1}(0)$ is a Hopf ideal of A and we have $I \cap R=0$.*

PROPOSITION 4. *Theorem 1 is valid when $A/(A \cdot R^+)$ is cocommutative.*

First we show that Propositions 2, 3 and 4 imply Theorem 1. Indeed consider the set of pairs (B, π), where B is a sub-Hopf algebra of A which contains R and π is a Hopf algebra map from B to R which is identity on R. Introducing the usual ordering on this set, take a maximal element (B, π) by Zorn's lemma. Assume $B \neq A$. Then there exists a sub-Hopf algebra C of A as in Proposition 2. Let I be the ideal of C generated by $\pi^{-1}(0)$. Then by Proposition 3 we have $R \subsetneq C/I$. R can be identified with the coradical of C/I [5, Exercise 4), p. 182]. Because we have $B^+ = \pi^{-1}(0) + R^+$, $(C/I)/((C/I) \cdot R^+)$ is a quotient Hopf algebra of a cocommutative Hopf algebra $C/(C \cdot B^+)$ and therefore $R \subsetneq C/I$ satisfies the condition in Proposition 4. Hence there exists a Hopf algebra map $\rho: C/I \to R$ such that $\rho = 1$ on R. Since we have $b - \pi(b) \in I$ for any $b \in B$, the composite $B \subsetneq C \to C/I \xrightarrow{\rho} R$ is identical with π. Hence (C, ρ) is properly larger than (B, π). This is a contradiction. So B is equal to A and the proof of Theorem 1 is complete.

Let B be a sub-Hopf algebra of A which contains R. Then $H = A/(A \cdot B^+)$ is an irreducible Hopf algebra since its coradical is contained in the image of $R \to H$ [5, Exercise 4), p. 182]. We have shown in [6, Lemma 4.2] that H is a quotient A-comodule of A under the left A-comodule structure

$$\rho: A \to A \otimes A,\ a \mapsto \sum a_{(1)}S(a_{(3)}) \otimes a_{(2)} .$$

LEMMA 5. *$P(H) = \{$the primitive elements of $H\}$ is a sub-A-comodule of H. If H is cocommutative, then H is a left B-comodule, i.e. $\rho(H) \subset B \otimes H$, where ρ is the left A-comodule structure map of H.*

PROOF. It is easy to see that we can assume k is algebraically closed and A is finitely generated. Then $G(A^0)$ is dense in A^* since k is of characteristic 0. Hence it suffices to show that $P(H)$ is $G(A^0)$-stable. But this is clear because the $G(A^0)$-action on H is compatible with the Hopf algebra structure of H. Next we suppose that H is cocommutative. Since A is faithfully flat over B [6, Th. 3.1], it suffices to show we have

$$\sum a_{(1)}S(a_{(3)}) \otimes 1 \otimes a_{(2)} = \sum 1 \otimes a_{(1)}S(a_{(3)}) \otimes a_{(2)}$$

in $A \otimes_B A \otimes H$ for any $a \in A$ in order to prove $\rho(H) \subset B \otimes H$. But we have an isomorphism [6, Lem. 3.9]

$$A \otimes_B A \to A \otimes H,\ x \otimes y \mapsto \sum xy_{(1)} \otimes y_{(2)}\ .$$

Through this isomorphism the equation above reduces to

$$\sum a_{(1)}S(a_{(3)}) \otimes 1 \otimes a_{(2)} = \sum a_{(1)}S(a_{(5)}) \otimes a_{(2)}S(a_{(4)}) \otimes a_{(3)}$$

in $A \otimes H \otimes H$, which is valid by the cocommutativity of H.

PROOF OF PROPOSITION 2. In the notation of Lemma 5, $H = A/(A \cdot B^+) \neq k$ implies $P(H) \neq 0$ [**5**, Cor. 11.0.2]. On the other hand Lemma 5 means that the kernel J of $A \to H \to H/H \cdot P(H)$ is a normal Hopf ideal of A [**6**, Def. 4.1]. Hence there exists a sub-Hopf algebra C of A such that $J = A \cdot C^+$ [**6**, Th. 4.3]. Because $B \subset C \subset A$, we have $C/(C \cdot B^+) \subset A/(A \cdot B^+)$ [**6**, Proof of Th. 3.1]. In view of

$$H/(H \cdot (C/(C \cdot B^+))^+) = A/(A \cdot C^+) = H/(H \cdot P(H))$$

we have $C/(C \cdot B^+) = k[P(H)]$ [**6**, Th. 3.10]. Because $P(H) \neq 0$, C contains B properly. Hence the proof of Proposition 2 is complete.

PROOF OF PROPOSITION 3. I is clearly a Hopf ideal of A. Because A is faithfully flat over B [**6**, Th. 3.1], we have $I \cap B = \pi^{-1}(0)$ [**2**, I, § 3, n° 5, Prop. 9, d)]. Hence $I \cap R = 0$.

PROOF OF PROPOSITION 4. First we show that $H = A/(A \cdot R^+)$ can be assumed to be finitely generated. Indeed in the argument below Proposition 4, we can assume $C/(C \cdot B^+)$ is finitely generated. Then $(C/I)/((C/I) \cdot R^+)$ is also finitely generated, and therefore ρ: $C/I \to R$ exists. Now $V = P(H)$ is a finite dimensional vector space over k and H is isomorphic as a Hopf algebra to $U(V)$, the universal enveloping algebra of V, since H is irreducible cocommutative [**5**, Th. 13.0.1]. Hence $\mathfrak{Sp}\,(H)$, the affine k-group represented by H, is isomorphic to $\mathfrak{D}_a(V) = (V^*)_a$ [**3**, II, § 1,2.1]. Since [**3**, III, § 4,6.6] can be extended to "torseurs durs" [**3**, III, § 5,1.4], the following exact sequence

$$(*) \qquad 0 \to \mathfrak{Sp}\,(H) \to \mathfrak{Sp}\,(A) \to \mathfrak{Sp}\,(R) \to 1$$

in $\widetilde{\widetilde{Gr}}_k$ [**3**, III, § 3,7.2] is an "H-extension" [**3**, II, § 3,2.1]. On the other hand by Lemma 5, $P(H)$ is a sub-R-comodule of H. This means that the action of $\mathfrak{Sp}\,(R)$ on $\mathfrak{Sp}\,(H) = \mathfrak{D}_a(V)$, which is determined naturally by (*), is linear [**3**, II, § 2,1.1]. Since R is co-semi-simple, the Hochschild cohomology $H_0^2(\mathfrak{Sp}\,(R), \mathfrak{Sp}\,(H))$ is zero [**3**, II, § 3,3.7]. By [**3**, II, § 3,2.3] the extension (*) splits and the proof of Theorem 1 is complete.

COMMENT 6. Theorem 1 is essentially proved in [**4**, Th. 14.2]. But the proof there is due to Levi's theorem on the structure of Lie algebra [**4**, Th. 13.5]. Our proof is free from Lie algebra theory. Abe and Doi announce the same results in [1, (3.5) and (3.6)]. But I think their proof has gaps. Now we have used the extension theory of affine algebraic k-groups in [3, III, §6] to prove Proposition 4. But it is easy to translate it into the Hopf algebra language and prove it again.

COROLLARY 7. *Let A be a commutative Hopf algebra over a field of characteristic* 0 *and R its coradical. Let $\pi: A \to R$ be a Hopf algebra map such that $\pi = 1$ on R. Then we have an isomorphism of k-algebras*

$$f: A \to (A/(A \cdot R^+)) \otimes R,\ a \mapsto \sum a_{(1)} \otimes \pi(a_{(2)}) .$$

(If we introduce the concept of "semidirect product of Hopf algebras" of [1, §4], *this becomes a Hopf algebra isomorphism. Compare our proof with that of* [1, §5].*)*

PPOOF. Because A is faithfully flat over R [6, Th. 3.1] and f is clearly R-linear, it suffices to construct the inverse of

$$f \otimes_R A: A \otimes_R A \to (A/(A \cdot R^+)) \otimes A,\ x \otimes y \mapsto \sum x_{(1)} \otimes \pi(x_{(2)})y .$$

But the algebra map

$$h: A \otimes_R A \leftarrow A \otimes A,\ \sum x_{(1)} \otimes S(\pi(x_{(2)}))y \leftarrow\!\shortmid x \otimes y$$

is 0 on $(A \cdot R^+) \otimes A$, and induces the inverse of $f \otimes_R A$.

REFERENCES

[1] E. ABE and Y. DOI, Decomposition theorem for Hopf algebras and proaffine algebraic groups, to appear.

[2] N. BOURBAKI, Algèbre commutative, Hermann, Paris, 1961.

[3] M. DEMAZURE et P. GABRIEL, Groupes algébriques, Masson & Cie., Paris; North-Holland, Amsterdam, 1970.

[4] G. HOCHSCHILD, Introduction to affine algebraic groups, Holden-Day, San Francisco, 1971.

[5] M. E. SWEEDLER, Hopf algebras, Benjamin, New York, 1969.

[6] M. TAKEUCHI, A correspondence between Hopf ideals and sub-Hopf algebras, to appear.

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