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GENERATORS OF W^*-ALGEBRAS III

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Two reasons motivate the study of generators of W^*-algebras. Firstly it is obvious that if a W^*-algebra $\mathfrak{A}$ is generated by the operators A_1, $A_2, \dots$ with certain algebraic properties, then any W^*-algebra isomorphic to $\mathfrak{A}$ is likewise generated by such operators. Thus the generating properties of W^*-algebras might yield useful invariants for this algebra. Secondly the study of generators of W^*-algebras leads to classes of operators which are rather pathological algebraically.

Unfortunately the results in [1, 2, 4, 5] indicate, that at least in the case of properly infinite W^*-algebras, generators will hardly lead to useful invariants, because these algebras are all singly generated by operators satisfying rather restrictive conditions like subnormality or polynomial identities. The present note continues the study of special generators of W^*-algebras begun in [1] and [5]. In particular we show that generators of properly infinite W^*-algebras may be chosen from subclasses of partial isometries, roots of shifts or contractions similar to unitary operators. Some of our results were motivated by the recent work of Wogen [5].

For obvious reasons we consider only W^*-algebras on separable Hilbert spaces. For $k = 1, 2, \dots, M_k$ denotes the algebra of all $k \times k$ matrices. We shall also use $M_\infty = B(H)$, H a separable Hilbert space. For $T_1, T_2, \dots \in B(H)$, $\mathfrak{R}(T_1, T_2, \dots)$ denotes the W^*-algebra generated by $T_1, T_2, \dots$.

Let $\mathfrak{A}$ be a properly infinite W^*-algebra and U a unitary operator in the center of $\mathfrak{A}$. Then the results in [1] show that there exist generators T of $\mathfrak{A}$ with $T^n = U, n \geqslant 3$. Conversely $T^n = U$ unitary implies that U is in the center of $\mathfrak{R}(T)$. Extending these results one might ask whether generators may be chosen from the class of roots of isometries. If $T^n = S$ is a completely non unitary isometry of finite multiplicity, $\mathfrak{R}(T)$ is obviously of type I. Thus there remains the case when S is a shift of infinite multiplicity.

THEOREM 1. *Any properly infinite W^*-algebra $\mathfrak{A}$ on a separable Hilbert space H is generated by an n-th root, $n \geqslant 2$, of a unilateral shift S of infinite multiplicity.*

PROOF. a) The case $n = 2$ is more difficult. We write $\mathfrak{A} = \mathfrak{B} \otimes M_\infty$, with $\mathfrak{B}$ properly infinite. Let

$$S = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & \ddots \end{bmatrix} .$$

Then any $T \in \mathfrak{A}$ commuting with S must be of the form

$$T = \begin{bmatrix} a_1 & & & \\ a_2 & a_1 & & \\ a_3 & a_2 & \ddots & \\ \vdots & a_3 & \ddots & \ddots \end{bmatrix} \qquad a_i \in \mathfrak{B}$$

and we have $T^2 = S$ only if

$$a_1^2 = 0 \, , \quad 1 = a_2 a_1 + a_1 a_2 \, , \quad 0 = a_n a_1 + a_{n-1} a_2 + \ldots + a_1 a_n \quad \text{for } n \geqslant 3 \, .$$

Assume we can choose $a_i \in \mathfrak{B}$ with these properties and $\mathfrak{R}(a_1, a_2, \ldots) = \mathfrak{B}$. Then since $S \in \mathfrak{R}(T)$ any $C = C^* \in \mathfrak{R}(T)'$ is diagonal $C = \operatorname{diag}(c, c, \ldots)$. $TC = CT$ implies $ca_i = a_i c$ or $c \in \mathfrak{B}'$. Thus $\mathfrak{R}(T) = \mathfrak{A}$. Therefore it remains find generators of $\mathfrak{B}$ satisfying the above relations. Write $\mathfrak{B} = \mathfrak{C} \otimes M_2$ and let a and b be positive invertible contractive generators of $\mathfrak{C}$. Then consider

$$a_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} , \quad a_2 = \begin{pmatrix} ba & a^{-1} \\ 0 & -ab \end{pmatrix} , \quad a_3 = \begin{pmatrix} 0 & -bab \\ 0 & 0 \end{pmatrix} , \quad a_4 = \ldots = 0 \, .$$

Simple computations show that the above relations are indeed satisfied. $\mathfrak{R}(a_1, a_2, a_3) = \mathfrak{B}$ is likewise trivial.

b) Higher roots of S are much easier to construct. Again write $\mathfrak{A} = \mathfrak{B} \otimes M_\infty$ with $\mathfrak{B}$ properly infinite. Let

$$T = \begin{bmatrix} 0 & & & \\ a_1 & 0 & & \\ & a_2 & 0 & \\ & & a_3 & \ddots \\ & & & \ddots \end{bmatrix} \qquad a_i \in \mathfrak{B}$$

with $a_{i+nk} = a_i$ and $a_1 = a$, $a_2 = b$, $a_3 = \dots = a_{k-1} = 1$, $a_k = a^{-1}b^{-1}$. Where a and b are positive invertible generators of $\mathfrak{B}$. Then T^k is an isometry. $\mathfrak{R}(T) = \mathfrak{A}$ is shown as in [1].

Generators may also be chosen from the class of partial isometries.

THEOREM 2. *Any properly infinite W^*-algebra $\mathfrak{A}$ on a separable Hilbert space H is generated by a partial isometry T such that TT^* and T^*T commute.*

PROOF. Write $\mathfrak{A} = \mathfrak{B} \otimes M_3$. Then $\mathfrak{B}$ is singly generated by an operator b which we may choose such that b and $1 - b^*b$ are invertible. Now let $a = (1 - b^*b)^{1/2}$, $c = (b^*b)^{1/2}$ and $d = -bac^{-1}$ and consider

$$T = \begin{bmatrix} 0 & a & c \\ & b & d \\ & & 0 \end{bmatrix}.$$

Then $TT^* = \operatorname{diag}(1, 1, 0)$ and $T^*T = \operatorname{diag}(0, 1, 1)$. Thus any

$$C = C^* \in \mathfrak{R}(T)$$

must be diagonal. Now continue as in [1].

This result is also valid for certain singly generated W^*-algebras of type II_1. We should add that $\operatorname{Sp} T = \operatorname{Sp} b \cup \{0\}$. Thus the spectrum of T can be prescribed rather arbitrarily. This fact was observed for partial isometries by Halmos [3]. A partial isometry T will be called of class n, if $T, T^2, \dots, T^n$ are partial isometries.

THEOREM 3. *Any properly infinite W^*-algebra $\mathfrak{A}$ on a separable Hilbert space is generated by a partial isometry T of class n. Here n is an arbitrary finite number.*

PROOF. We use induction on n. For $n = 1$ let $\mathfrak{A} = \mathfrak{B} \otimes M_2$ and

$$T = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \qquad a, b \in \mathfrak{B},$$

where a is a proper contractive generator of $\mathfrak{B}$ and $b = (1 - a^*a)^{1/2}$. Then T is a partial isometry which generates $\mathfrak{A}$. Moreover 1-T^*T is equivalent to 1. For the step from n to $n + 1$ write again $\mathfrak{A} = \mathfrak{B} \otimes M_2$ and let a be a partial isometry of class n which generates $\mathfrak{B}$ such that $1 - a^*a$ is equivalent to 1. Now consider T as above with $bb^* = 1$ and $b^*b = 1 - a^*a$. Then $T^*T = \operatorname{diag}(1, 0)$ and any $C = C^* \in \mathfrak{R}(T)'$ must be diagonal, $C = \operatorname{diag}(c_1, c_2)$. $CT = TC$ gives further $ac_1 = c_1a$ or $c_1 \in \mathfrak{B}'$. We also have $c_1b = bc_1 = c_2b$ or $c_1 = c_1bb^* = c_2bb^* = c_2$. Thus $\mathfrak{R}(T) = \mathfrak{A}$.

A simple computation shows further for $k \leqslant n+1$ $T^{k*}T^k = \operatorname{diag}(a^{(k-1)*}a^{k-1}, 0)$. Thus T has the required properties.

THEOREM 4. *Let $\mathfrak{A}$ be a properly infinite W^*-algebra on the separable Hilbert space H, let K be a nonempty compact set in the unit disc and let $\varepsilon > 0$. Then there exists a generator T of $\mathfrak{A}$ with* $\operatorname{Sp} T = K$ *and* $|T| \leqslant 1 + \varepsilon$.

PROOF. If $K = \{\lambda\}$ let $T = \lambda + N$ with N a 3-nilpotent generator of $\mathfrak{A}$ of suitably small norm [1]. Thus we may assume that K contains at least two points. By scaling and rotation in the complex plane we can even achieve $1 \in K$. We begin with the case $0 \notin K$ and write $\mathfrak{A} = \mathfrak{B} \otimes M_2$. Since $\mathfrak{B}$ is properly infinite, $\mathfrak{B}$ is generated by a projection p with $1 \sim 1 - p$ and a positive invertible operator b with $|b| < \varepsilon$. Now let

$$T = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

where a is a diagonal operator in $\mathfrak{B}$ with $\operatorname{Sp} a = K$ [3]. Since K contains at least two points we may also assume $p \in \Re(a)$. Obviously $\operatorname{Sp} T = \operatorname{Sp} a \cup \{1\} = K$ and $|T| \leqq 1 + \varepsilon$. Considering $(T^2 - T)^*(T^2 - T)$ we see $\operatorname{diag}(1, 0) \in \Re(T)$. Using this it is easy to see that $\Re(T) = \mathfrak{A}$. If $0 \in K$ let

$$T = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

with $\operatorname{Sp} a = K$, $p \in \Re(a)$ and b as above. Then $\operatorname{Sp} T = Sp\, a \cup \{0\} = K$ and $\Re(T) = \mathfrak{A}$ is shown as in [1].

This theorem is a strengthening of a result of Wogen [5]. However the proof is different an simpler.

COROLLARY. *Let $\varepsilon > 0$ and let K be a compact set containing* 0 *inside the disc of radius $1 - \varepsilon$, then there exists a partial isometry T of class n, $n < \infty$, with* $\operatorname{Sp} T = K$ *and* $\Re(T) = \mathfrak{A}$.

PROOF. We use induction on n. For $n = 1$ let $\mathfrak{A} = \mathfrak{B} \otimes M_2$ and let a be a generator of $\mathfrak{B}$ with $\operatorname{Sp} a = K$ and $|a| \leqslant 1 - \varepsilon/2$. Let $b = (1 - a^*a)^{1/2}$ and consider

$$T = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}.$$

Then $T^*T = \operatorname{diag}(1, 0)$, $\Re(T) = \mathfrak{A}$ and $\operatorname{Sp} T = \operatorname{Sp} a \cup \{0\} = K$. The induction is completed as in the proof of Theorem 3.

There are cases where the ε in Theorem 4 is not necessary, even if K is on the circumference of the disc. This is shown by the next theorem.

THEOREM 5. *Any properly infinite W^*-algebra $\mathfrak{A}$ on a separable Hilbert space H is generated by a contraction T similar to a unitary operator.*

PROOF. Write $\mathfrak{A} = \mathfrak{B} \otimes B(l^2(Z))$, Z the integers and $\mathfrak{B}$ properly infinite. $\mathfrak{B}$ is generated by two positive invertible contractions a and b with $1 > b > 3/4$, $1/2 > a > 1/4$. Now let $S = \operatorname{diag}(\ldots, s_{-1}, s_0, s_1, \ldots)$ with $s_k = 1$ for $k \leqslant 0$, $s_1 = a$ and $s_2 = s_3 = \ldots = ba$. Let U be the bilateral shift, then $SUS^{-1} = T$ has the matrix elements $T_{i+1,i} = s_{i+1}s_i^{-1} = t_i$ with $t_0 = a$, $t_1 = b$ and $t_i = 1$ otherwise. T is a contraction, because $(T^*T)_{i,j} = \delta_{i,j}t_i^*t_i$ is bounded by 1 for all i. It remains to show $\mathfrak{R}(T) = \mathfrak{A}$. By considering $T^*T\text{-}TT^*$ one shows that $\mathfrak{R}(T)$ contains the operator

$$V = \operatorname{diag}(\ldots, v_{-1}, v_0, v_1, \ldots)$$

with $v_0, v_1, v_2 = 1$ and $v_i = 0$ otherwise. Using polynomials in T and V one can finally show that $\mathfrak{R}(T)$ contains any diagonal operator

$$D_n = \operatorname{diag}(\ldots d_{-1}, d_0, d_1 \ldots)$$

with $d_i = \delta_{i,n}$. Thus any $C = C^* \in \mathfrak{R}(T)'$ is diagonal and the proof is completed as in [1].

It is clear from [3, problem 165] and [1, remark following Theorem 2] that contractions similar to unitary operators generate C^*-algebras which have no factor representations of type II_1.

Finally let us give another kind of special generator of a properly infinite W^*-algebra. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be properly infinite W^*-algebras on H_1 respective H_2. Then there exists a generator $T = T_1 \otimes T_2$ for $\mathfrak{A}_1 \otimes \mathfrak{A}_2$. Simply let T_1 be a generator of $\mathfrak{A}_1$ with $T_1^3 = 1$ and let T_2 be a generator of $\mathfrak{A}_2$ with $T_2^9 = 1$, $\mathfrak{R}(T_2^3) = \mathfrak{A}_2$ and $\operatorname{Sp} T_2 = \{1, e^{2\pi i/9}, e^{4\pi i/9}\}$. The existence of such generators follows from [1]. Since $T^3 = 1 \otimes T_2^3 \in \mathfrak{R}(T)$ we also have $1 \otimes T_2^{-1} \in \mathfrak{R}(T)$. Thus $1 \otimes T_2$ and $T_1 \otimes 1 \in \mathfrak{R}(T)$.

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