Tôhoku Math. Journ.
24 (1972), 571-579.

MINIMAL SUBMANIFOLDS AND CONVEX FUNCTIONS

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(Received February 8, 1972)

In this note we describe some results concerning minimal submanifolds in complete Riemannian manifolds of non-negative curvature. Our main theorem is stated as follows:

MAIN THEOREM. *Any compact minimal hypersurface in a complete non-compact Riemannian manifold of non-negative curvature is totally geodesic.*

The author was motivated to study minimal submanifolds by Nakagawa-Shiohama [7] in which they suggested to investigate the relation between compact minimal submanifolds and "souls" of complete non-compact Riemannian manifolds of non-negative curvature. For souls, see Cheeger-Gromoll [2] as well as Shiohama [9]. In the course of our investigation we also deal with a property of the distance function α_N from points on a minimal submanifold N to a totally geodesic hypersurface H in a complete Riemannian manifold of non-negative curvature. Roughly speaking, this property is that α_N is superharmonic on N, which may be seen as the dual of the case where the ambient manifold has non-positive curvature, compare Hermann [6]. Making use of this property we are able to determine the relation between N and H under some conditions, for related results see Frankel [3], [4].

In section 1, we describe some lemmas concerning convex sets and convex functions under fairly general situations. Lemmas 1 and 2 are originally mentioned in Cheeger-Gromoll [2], which we will use without proof. Lemma 4 is an important link in our arguments. In section 2, assuming that the ambient manifold has non-negative curvature, we construct several convex functions to apply Lemma 4. Finally in section 3, we consider the non-compact case to obtain our main theorem, see Nakagawa-Shiohama [7] and Shiohama [8]. Lemmas 5 and 10 are originally proved in Cheeger-Gromoll [2]. For all basic concepts and tools in Riemannian geometry that will be used without comment, we refer to Gromoll-Klingenberg-Meyer [5].

I am very grateful to Professor S. Sasaki and Professor S. Tanno for their advice and encouragement. I would also like to thank Mr. T. Sakai

and Mr. K. Kenmotsu for many helpful criticisms and suggestions.

1. Preliminaries. The most important notion we use is that of convexity. A subset B of a complete Riemannian manifold M is called *strongly convex* if for any points $p, q \in B$ there is a unique minimal geodesic $c: [0, 1] \to M$ from p to q and $c([0, 1]) \subset B$. Recall that there is a positive continuous function $r: M \to (0, \infty]$, the convexity radius, such that any open metric ball $B_\varepsilon(q) \subset B_{r(p)}(p)$ is strongly convex. We say that a subset C of M is *convex* if for any $p \in \bar{C}$ there is a number $\varepsilon(p)$ $(0 < \varepsilon(p) < r(p))$ such that $C \cap B_{\varepsilon(p)}(p)$ is strongly convex, where $\bar{C}$ is the closure of C. In their paper [2], Cheeger and Gromoll studied the structure of convex sets. One of their results is:

LEMMA 1. (Structure Theorem for Convex Sets) *Let C be a connected closed convex subset of a Riemannian manifold M. Then C carries the structure of an imbedded k-dimensional submanifold of M with smooth totally geodesic interior* int C *and (possibly non-smooth) boundary ∂C.*

For our present purpose it will suffice to consider the case where C is n-dimensional ($n := \dim M$). Hyperplanes in a tangent space M_p mean always hyperplanes through the origin o. For a hyperplane H_p in M_p, let H_p^+ denote a closed half-space defined by

$$H_p^+ := \{v \in M_p \mid \langle v, u \rangle \geqq 0\} ,$$

where $u \in M_p$ is a unit normal vector of H_p.

Let $D_r(p)$ denote the open disc in M_p of radius $r > 0$ centered at o, that is,

$$D_r(p) := \{V \in M_p \mid \|v\| < r\} .$$

The following lemma was proved in Cheeger-Gromoll [2] under more general conditions.

LEMMA 2. *Let C be an n-dimensional closed convex subset of an n-dimensional Riemannian manifold M. Then for any $p \in \partial C$, there is a closed half-space H_p^+ in M_p such that*

$$C \cap B_{\varepsilon(p)}(p) \subset \exp_p (H_p^+ \cap D_{\varepsilon(p)}(p)) .$$

We call such H_p^+ a supporting half-space of C at p. The non-empty boundary ∂C of an n-dimensional closed convex subset C in an n-dimensional Riemannian manifold M is a (possibly non-smooth) hypersurface of M.

LEMMA 3. *Let M and C be as above. If $N := \partial C$ is smooth in a neighbourhood of $p \in N$, then the 2-nd fundamental form l_N^u of N with respect to the unit normal vector u of N at p pointing toward the interior* *of C is negative semi-definite.*

PROOF. Put

$$H_p := \{v \in M_p \mid \langle v, u \rangle = 0\} = N_p ,$$
$$H_p^+ := \{v \in M_p \mid \langle v, u \rangle \geqq 0\} .$$

Then H_p^+ is the unique supporting half-space of C at p.

$$H := \exp_p (H_p \cap D_{\varepsilon(p)}(p))$$

is a (non-complete) smooth hypersurface of M, which is totally geodesic at p. For sufficiently small ε $(0 < \varepsilon < \varepsilon(p))$ and $v \in H_p$, $\|v\| = 1$, define a geodesic

$$\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M;\ \gamma(s) := \exp_p(sv) \in H$$

and for a fixed $q := \exp_p(au) \in \operatorname{int} C$ $(0 < a < \varepsilon(p))$, let $c_s\colon [0, a] \to M$ be the minimal geodesic from q to $\gamma(s)$. Then we have a smooth one-parameter variation $\mathscr{V}_H$ of c_0 defined as follows:

$$\mathscr{V}_H\colon [0, a] \times (-\varepsilon, \varepsilon) \longrightarrow M;\ \mathscr{V}_H(t, s) := c_s(t) .$$

Since H_p^+ is a supporting half-space of C and $q \in \operatorname{int} C$, $d(p, q) < \varepsilon(p)$, each geodesic $c_s\colon [0, a] \to M$ intersects N at a unique parameter value $t_s \in (0, a]$ and the function $s \to t_s$ is smooth.

Making use of $\mathscr{V}_H$, we define another variation

$$\mathscr{V}_N\colon [0, a] \times (-\varepsilon, \varepsilon) \longrightarrow M;\ \mathscr{V}_N(t, s) := \exp_q \left(\frac{t_s}{a} t\dot{c}_s(0)\right) .$$

Let $L_H(s)$ and $L_N(s)$ denote the lengths of geodesics $t \longrightarrow \mathscr{V}_H(t, s)$ and $t \longrightarrow \mathscr{V}_N(t, s)$ respectively. Then L_H and L_N are smooth and moreover

$$L_H \geqq L_N,\ L_H(0) = L_N(0),\ L'_H(0) = L'_N(0) = 0 ,$$

therefore we have

$$L''_H(0) \geqq L''_N(0) .$$

Now let J_H and J_N denote the Jacobi fields along c_0 induced from $\mathscr{V}_H$ and $\mathscr{V}_N$ respectively. Then

$$J_H(0) = J_N(0) , \qquad J_H(a) = J_N(a) = v ,$$

since q and p are not conjugate along c_0 they must coincide;

$$J := J_H = J_N .$$

Since H is totally geodesic at p, by the 2-nd variation formula we have

$$L''_H(0) = \int_0^a [\langle J', J' \rangle - \langle R(J, \dot{c}_0)\dot{c}_0, J \rangle] dt$$

and

$$L''_N(0) = \int_0^a [\langle J', J' \rangle - \langle R(J, \dot{c}_0)\dot{c}_0, J \rangle] dt + l_N^u(v, v) ,$$

where $J' := \nabla_{\dot{c}_0} J$. Hence we have

$$l_N^u(v, v) = L''_N(0) - L''_H(0) \leqq 0 .$$

DEFINITION. A continuous function α on a Riemannian manifold M is said to be *convex* or *superharmonic* provided that for any $\varepsilon > 0$, there exists a neighbourhood U_p of p in M and a smooth function $\alpha_{p,\varepsilon}$ on U_p which satisfies the following (i) and (ii) or (i) and (ii)′ respectively.

(i) $\alpha_{p,\varepsilon}(x) \geqq \alpha(x)$ for any $x \in U_p$, $\alpha_{p,\varepsilon}(p) = \alpha(p)$,

(ii) $\langle \nabla_X \nabla \alpha_{p,\varepsilon}, X \rangle \leqq \varepsilon$ for any $X \in M_p$, $\|X\| = 1$,

(ii)′ $\Delta \alpha_{p,\varepsilon} \leqq \varepsilon$ at p,

where Δ is the Laplacian.

Superharmonic functions satisfy the minimum principle, for the proof see Calabi [1].

LEMMA 4. *Let α be a convex function on a Riemannian manifold M and N a minimal submanifold of M. Then the restriction $\alpha|N$ of α on N is superharmonic.*

PROOF. Fix an arbitrary $p \in N$. For any smooth function β in a neighbourhood of p in M and any tangent vector $X \in N_p$, we have

$$\begin{aligned} l_N^{(\nabla\beta)^\perp}(X, X) &:= \langle \nabla_X \nabla \beta, X \rangle \\ &= \langle \nabla_X \nabla \beta, X \rangle - \langle \overset{N}{\nabla}_X \overset{N}{\nabla}(\beta|N), X \rangle , \end{aligned}$$

where $l_N^{(\nabla\beta)^\perp}$ is the 2-nd fundamental form of N at p with respect to $(\nabla\beta)^\perp$. Fix an orthonormal basis $X_1, \cdots, X_k$ of N_p ($k := \dim N$). Then by the definition of the Laplacian $\overset{N}{\Delta}$ of N, we have

$$\begin{aligned} \overset{N}{\Delta}(\beta|N)|_p &= \sum_{i=1}^k \langle \overset{N}{\nabla}_{X_i} \overset{N}{\nabla}(\beta|N), X_i \rangle \\ &= \sum_{i=1}^k \langle \nabla_{X_i} \nabla\beta, X_i \rangle - \sum_{i=1}^k l_N^{(\nabla\beta)^\perp}(X_i, X_i) \\ &= \sum_{i=1}^k \langle \nabla_{X_i} \nabla\beta, X_i \rangle , \end{aligned}$$

since we have assumed that N is minimal. Now choose $\alpha_{p,\varepsilon}$ satisfing (i) and (ii) in Definition. Then we get

$$\overset{N}{\Delta}(\alpha_{p,\varepsilon}|N)|_p = \sum_{i=1}^{k} \langle \nabla_{X_i} \nabla \alpha_{p,\varepsilon}, X_i \rangle \leqq k\varepsilon$$

and the lemma follows.

Under the same assumption as Lemma 4, if $\alpha|N$ attains the minimum at some $p \in N$, then $\alpha|N$ is constant. In particular if N is compact, then $\alpha|N$ is constant. Lemma 4 makes it possible to relate minimal submanifolds with convex functions. For example, let M be a complete simply connected Riemannian manifold of non-positive curvature. Then for fixed $p \in M$, the distance function $x \to -d(x, p)$ is convex. By Lemma 4, the well-known result follows, i.e., M contains no compact minimal submanifold of positive dimension.

2. Applications for manifolds of non-negative curvature. Now let us construct convex functions in complete Riemannian manifolds of non-negative curvature. The following lemma is a rather special case of a result mentioned in Cheeger-Gromoll [2].

LEMMA 5. *Let C be an n-dimensional connected closed convex subset with $\partial C \neq \emptyset$ of an n-dimensional complete Riemannian manifold M of nonnegative curvature. Then the function*

$$\alpha: C \to \boldsymbol{R}\ ; \qquad \alpha(x) := d(x, \partial C)$$

is convex.

PROOF. Since α is continuous, it will suffice to show that α is convex in $\operatorname{int} C = C - \partial C$.

Fix $q \in \operatorname{int} C$, $b := d(q, \partial C) > 0$, and let c: $[0, b] \to M$ be a shortest connection between q and ∂C. c runs in int C except $p := c(b) \in \partial C$. Put

$$H_p := \{v \in M_p | \langle v, \dot{c}(b) \rangle = 0\}\ ,$$
$$H_p^+ := \{v \in M_p | \langle v, -\dot{c}(b) \rangle \geqq 0\}\ .$$

First we show that H_p^+ is the unique supporting half-space of C at p. If there is a unit vector $w \in M_p$ such that

$$\langle w, \dot{c}(b) \rangle < 0,\ W_\varepsilon := \{\exp_p(tw) | t \in [0, \varepsilon]\} \not\subset C$$

for any $\varepsilon > 0$, then we can choose an arbitrary small $\delta > 0$ such that the minimal geodesic from $c(b - \delta)$ to W_ε $(\varepsilon < \varepsilon(p))$ is orthogonal to W_ε at $\exp_p(t_\delta w)$ $(0 < t_\delta < \varepsilon)$ and $\exp_p(t_\delta w) \notin C$. But this contradicts that c is a shortest connection between q and ∂C. Fix $a \in (0, b)$ and extend c on the interval $[-a, b]$. Let $X_1, \cdots, X_{n-1}, \dot{c}$ be an orthonormal parallel fields along c. Since $c([-a, b])$ is compact, there is a small $r > 0$ such that

$$F: D_r(c) \to M;\ F(v) := \exp(v)$$

is a diffeomorphism into M, where

$$D_r(c) := \bigcup_{t\in[-a,b]} \{v \in M_{c(t)} \mid v \perp \dot{c}(t),\ \|v\| < r\} .$$

For any $x \in F(D_r(c))$ let

$$F^{-1}(x) = \sum_{i=1}^{n-1} \pi_i(x) X_i(t_x) \qquad (\pi_i(x) \in \boldsymbol{R})$$

and define a smooth curve

$$c_x\colon [t_x, b] \to M;\ c_x(t) := F\left\{\sum_{i=1}^{n-1} \pi_i(x) X_i(t)\right\} .$$

Then the length function

$$L\colon F(D_r(c)) \to \boldsymbol{R} ; \qquad L(x) := [\text{length of } c_x]$$

is smooth in $B_r(q)$. Since c_x connects x and a point on $H := \exp_p(H_p \cap D_r(p))$ and H_p^+ is a supporting half-space, we have

$$L(x) \geqq \alpha(x) \text{ for any } x \in B_r(q),\ L(q) = \alpha(q) .$$

Now for any parallel vector field X along c with $X \perp \dot{c}$ we have by the 2-nd variation formula

$$\langle \nabla_{X(0)} \nabla L, X(0)\rangle = \int_0^b [\langle X', X'\rangle - \langle R(X, \dot{c})\dot{c}, X\rangle] dt \leqq 0 ,$$

because H is totally geodesic at p and the curve $t \to F(tX(0))$ is a geodesic which is orthogonal to c at q. Clearly

$$\langle \nabla_{\dot{c}(0)} \nabla L, \dot{c}(0)\rangle = 0 ,$$

hence α is convex.

Under the same situation as Lemma 5, it follows that for any $a \geqq 0$ the subset

$$C^a := \{x \in C \mid d(x, \partial C) \geqq a\}$$

is closed and convex.

Combining Lemmas 4 and 5 we have:

THEOREM 6. *Let M be an n-dimensional complete Riemannian manifold of non-negative curvature, C an n-dimensional connected closed convex subset of M with $\partial C \neq \emptyset$, and N a minimal submanifold of M which is contained in C. Suppose that on N there exists a closest point to ∂C, then each point of N lies at the same distance from ∂C.*

In particular, a compact minimal submanifold contained in C lies at the same distance from ∂C. If N intersects with ∂C, then N is contained in ∂C. Theorem 6 may be seen as a generalization of Theorem 5.3 in Yano-Ishihara [10], in which they consider the case where M is the n-dimensional sphere S^n and C a closed hemi-sphere.

Combining Theorem 6, Lemmas 3 and 5 we have:

THEOREM 7. *Let M be an n-dimensional complete Riemannian manifold of non-negative curvature, C an n-dimensional connected closed convex subset of M with $\partial C \neq \emptyset$, and N a minimal hypersurface of M which is contained in C. Suppose that on N there exists a closest point to ∂C, then N is totally geodesic.*

PROOF. Let $a := d(N, \partial C)$. Then N is contained in the closed convex subset $C^a := \{x \in C \mid d(x, \partial C) \geqq a\}$. Let D be a connected component of C^a which contains N. Then D is an $(n-1)$- or n-dimensional (possibly non-smooth) submanifold with totally geodesic interior. In the case $\dim D = n-1$, N is contained in D as an open submanifold, hence the theorem follows. On the other hand, if $\dim D = n$, then $\partial D \neq \emptyset$ and by Theorem 6, N is contained in ∂D as an open submanifold. Since N is minimal, by Lemma 3 the proof is completed.

A Riemannian manifold M with non-empty boundary ∂M is said to be convex, if the 2-nd fundamental form of ∂M with respect to the inward normal vector on ∂M is negative semidefinite. Making use of the same argument as above, we have:

PROPOSITION 8. *Let M be a convex Riemannian manifold of non-negative curvature, and N a minimal submanifold of M. Suppose that on N there exists a closest point to ∂M, then N lies at the same distance from ∂M. Moreover if N is a hypersurface, then it is totally geodesic.*

Now let us consider the relation between minimal submanifolds and totally geodesic hypersurfaces. Let H be a hypersurface in a complete Riemannian manifold M. Then for any $p \in H$ there is a small $\delta(p) > 0$ such that $B_\delta(p) - H$ has exactly two connected components for any $0 < \delta < \delta(p)$.

We shall say that a submanifold N of M satisfies the condition (H) relative to H provided that for any $p \in N \cap H$, there is a small $0 < \delta'(p) < \delta(p)$ such that $N \cap B_{\delta'(p)}(p)$ lies in the closure of a connected component of $B_{\delta'(p)}(p) - H$.

THEOREM 9. *Let H be a totally geodesic hypersurface in a complete Riemannian manifold M of non-negative curvature, and N a minimal submanifold of M which satisfies the condition (H) relative to H. Suppose that on N there is a closest point to H, then N lies at the same distance from H.*

PROOF. By Lemma 5, the distance function

$$\alpha: M \to \boldsymbol{R}\ ; \qquad \alpha(x) := d(x, H)$$

is seen to be convex in $M - H$. Fix an arbitrary $p \in H$ and choose $\delta(p) < r(p)$ as above, where $r(p)$ is the convexity radius of M at p. Let $H^+_{p,\delta}$ denote one of the connected components of $B_\delta(p) - H$ $(0 < \delta < \delta(p))$. Since H is totally geodesic, the closure $\overline{H^+_{p,\delta}}$ of $H^+_{p,\delta}$ is an n-dimensional closed convex subset of M $(n := \dim M)$ and for any $x \in H^+_{p,\delta/2}$ the shortest connection from x to $\partial\overline{H^+_{p,\delta}}$ coincides with the shortest connection from x to H. Hence the distance function

$$\alpha_p: H^+_{p,\delta/2} \to \boldsymbol{R}\ ; \qquad \alpha_p(x) := d(x, \partial\overline{H^+_{p,\delta}})$$

coincides with $\alpha | H^+_{p,\delta/2}$ and is convex. It follows that by Lemma 4, if N satisfies the condition (H) relative to H, then $\alpha | H$ is superharmonic, and the theorem follows.

3. An application for non-compact manifolds of non-negative curvature. Recall that a ray in a complete non-compact Riemannian manifold M is a normal geodesic $\gamma: [0, \infty) \to M$, each segment of which is minimal.

With each ray γ in M we associate a function ρ_γ as follows: For $t \geqq 0$, let

$$\rho_t(x) := d(x, \gamma(t)) - t \qquad (x \in M)\ .$$

It follows from the triangle inequality that the family $\{\rho_t\}$ is uniformly equi-continuous. For fixed x, the function $t \to \rho_t(x)$ is decreasing on $[0, \infty)$ and bounded below by $-d(x, \gamma(0))$. Hence, for $t \to \infty$, $\{\rho_t\}$ converges uniformly on compact subsets to a continuous function ρ_γ on M.

The following lemma was proved in Cheeger-Gromoll [2] and used effectively to construct the structure theorem of complete non-compact Riemannian manifolds of non-negative curvature.

LEMMA 10. *Let M be a complete non-compact Riemannian manifold of non-negative curvature. Then for any ray γ in M the associated function ρ_γ is convex.*

Now we are able to generalize Theorem 2.1 and Theorem 2.2 in Nakagawa-Shiohama [7] as follows:

THEOREM 11. *Any compact minimal submanifold of a complete non-compact Riemannian manifold M of non-negative curvature is contained in a level surface of ρ_γ, where γ is any ray in M.*

Finally our main theorem follows by Lemma 10 and Theorem 7, for the proof it will suffice to notice that the subset

$$C_t := \{x \in M \mid \rho_\gamma(x) \geqq t\}$$

is closed and convex.

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