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DOUBLE COMMUTANTS OF ISOMETRIES*

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Abstract. Normal operators N satisfying $\mathfrak{A}_N = \mathfrak{A}''_N$ are characterized in terms of invariant subspaces. It is shown that non-unitary isometries V always satisfy $\mathfrak{A}_V = \mathfrak{A}''_V$. Thus, since a unitary operator is normal, a complete description of isometries satisfying a double commutant theorem is achieved.

1. Introduction. In the forthcoming all Hilbert spaces will be complex, and all operators *bounded* linear transformations. The symbol $B(\mathscr{H})$ will denote the algebra of all operators on the Hilbert space $\mathscr{H}$, $\mathfrak{A}_A$ the weakly closed algebra with identity generated by $A \in B(\mathscr{H})$, $\mathfrak{A}'_A$ the commutant of A and $\mathfrak{A}''_A$ the double commutant of A. The reader is referred to [2, p. 1] for the definition of commutant and double commutant.

DEFINITION 1.1. The *class* (dc) is the class of all operators on Hilbert space satisfying $\mathfrak{A}_A = \mathfrak{A}''_A$.

This class has been studied previously in [9], [10], and [11]. In this paper we shall characterize the normal operators in the class (dc) and show that any non-unitary isometry belongs to (dc).

2. Normal operators.

THEOREM 2.1. *Let $N \in B(\mathscr{H})$ be a normal operator. Then $N \in (dc)$ if and only if every subspace of $\mathscr{H}$ invariant under N reduces N.*

PROOF. (a) Suppose $N \in (dc)$. By the Fuglede-Putnam Theorem [7, p. 9] $N^* \in \mathfrak{A}''_N$, so $N^* \in \mathfrak{A}_N$. Therefore each subspace invariant under N is invariant under N^*, i.e. reduces N.

(b) Suppose every subspace invariant under N reduces N. This says $\mathrm{Lat}\, N \subseteq \mathrm{Lat}\, N^*$. Since N is normal, by Sarason's theorem [8, p. 511] N is reflexive. (See [8] for definitions.) Therefore $\mathrm{Lat}\, N \subseteq \mathrm{Lat}\, N^* \Rightarrow N^* \in \mathfrak{A}_N$. Thus $\mathfrak{A}_N$ is a von Neuman algebra, so by the von Neuman Double Commutant Theorem, $\mathfrak{A}_N = \mathfrak{A}''_N$.

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3. Isometries. It will convenient to make use of the following rather specialized lemma.

LEMMA 3.1. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, $A \in B(\mathscr{H})$, $B \in B(\mathscr{K})$, and let $A \in (dc)$. Let $\mathscr{X} = \{X: \mathscr{H} \to \mathscr{K} \mid X$ is an operator and $BX = XA\}$. Suppose that either*

(a) $$\bigcup_{X \in \mathscr{X}} \text{Range } X = \mathscr{K}$$

or

(b) *$\bigcup_{X \in \mathscr{X}}$ Range X is dense in $\mathscr{K}$, each element of $\mathfrak{A}_A$ is the limit of a sequence of polynomials in A, and there exists a constant M such that $\|p(B)\| \leqq M \cdot \|p(A)\|$ for any polynomial p.*

Then $A \oplus B \in (dc)$.

PROOF. Let D be in the double commutant of $A \oplus B$. Then $D = E \oplus F$ where $E \in \mathfrak{A}_A'' = \mathfrak{A}_A$ and $F \in \mathfrak{A}_B''$. For any $X \in \mathscr{X}$, the operator on $\mathscr{H} \oplus \mathscr{K}$ defined by the matrix

$$\begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}$$

is in the commutant of $A \oplus B$, whence it commutes with D. This says that $FX = XE$.

Now suppose that (a) holds and that $\{p_\alpha\}$ is a net of polynomials such that $p_\alpha(A) \to E$ in the weak operator topology. Let $k \in \mathscr{K}$; then there is an $X \in \mathscr{X}$ and $h \in \mathscr{H}$ such that $k = Xh$. Therefore $p_\alpha(B)k = p_\alpha(B)Xh = Xp_\alpha(A)h$ which converges weakly to $XEh = FXh = Fk$. Thus $p_\alpha(B) \to F$ weakly, so $p_\alpha(A \oplus B) \to E \oplus F = D$ weakly.

Next suppose that (b) holds and that $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials such that $p_n(A) \to E$ weakly. By similar arguments to the above we can show that $p_n(B)k \to Fk$ weakly for k in $\bigcup_{X \in \mathscr{X}}$ Range X, a dense set. The weak convergence of $p_n(A)$ implies that $\{\|p_n(A)\|\}_{n=0}^{\infty}$ is bounded. Since $\|p_n(B)\| \leqq M \cdot \|p_n(A)\|$ for all n, we therefore know that $\{\|p_n(B)\|\}_{n=0}^{\infty}$ is bounded too. Consequently $p_n(B) \to F$ weakly.

REMARK 3.2. In what follows we actually only make use of part (b) of Lemma 3.1.

THEOREM 3.3. *Any non-unitary isometry on a Hilbert space is in (dc).*

PROOF. Let V be a non-unitary isometry and write V in its Wold decomposition (see [6, p. 3]) as $U \oplus W$, where U is the pure part and W is the unitary part of V. Since V is non-unitary the pure part U does not vanish. We may further decompose W as $W_a \oplus W_s$ where W_a is the absolutely continuous part, and W_s is the singular part of W. (See [4, p. 55].)

Now U is a (possibly infinite) direct sum of copies of U_1, the unilateral shift of multiplicity 1. Since $\mathfrak{A}_{U_1} = \mathfrak{A}''_{U_1}$ and each element of $\mathfrak{A}_{U_1}$ is the limit of a *sequence* of polynomials in U_1 ([5, prob. 117]) the same holds for $\mathfrak{A}_U$. In [3] it is shown on pp. 299–300 that $\bigcup_{W_a X = XU} \text{Range}\, X$ is dense in the domain of W_a. Furthermore, for any polynomial p, $\|p(W_a)\| = r(p(W_a)) \leqq \sup_{|z|=1} |p(z)| = \|p(U_1)\| = \|p(U)\|$, where r denotes spectral radius. Thus part (b) of Lemma 3.1 applies to show that $U \oplus W_a$ is in (dc).

In [12, p. 275] Wermer shows that a unitary operator has a non-reducing invariant subspace if and only if it contains, as a direct summand, a copy of the bilateral shift of multiplicity 1. Since W_s is singular, it can contain no such direct summand, whence all of its invariant subspaces are reducing. Therefore by Theorem 2.1 $W_s \in (dc)$.

By the Corollary in [1], $\mathfrak{A}_V = \mathfrak{A}_{U \oplus U_a} \oplus \mathfrak{A}_{W_s}$. From this it clearly follows that $V \in (dc)$.

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