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CONVERGENCE IN PERTURBED NONLINEAR SYSTEMS

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C. Avramescu studied in [2] the existence and properties of convergent solutions to perturbed linear systems of the form

$$\text{(I)} \qquad x' = A(t)x + f(t, x) ,$$

where $A(t)$ is a continuous $n \times n$ matrix and $f(t, x)$ a continuous n-vector-valued function.

Hallam [3] studied the problem of the maintenance of the convergence properties of solutions to the nonlinear equation

$$\text{(II)} \qquad y' = A(t, y)$$

under the effect of a perturbation term $F(t, y)$. Hallam made extensive use of Alekseev's formula [1], which can be applied only if the function $A(t, u)$ is continuously differentiable with respect to u. The author studied in [6] the asymptotic relationship between the system (II) and the system

$$\text{(III)} \qquad x' = A(t, x) + F(t, x)$$

in the case in which $A(t, u)$ is not necessarily differentiable with respect to u. Our purpose here is to study, by means of our considerations in [6], the convergence properties of the system (III) in connection with the unperturbed system (II).

In Section 1 we give some definitions and preliminary facts. In Section 2 we study the convergence properties of systems of the form (III). In Section 3 we give a theorem, which ensures the existence of convergent solutions of the system (III) with $F(t, x) = G(t, x)x$, where G is a continuous $n \times n$ matrix.

We note here that the present method can be applied equally well in admissibility problems and problems concerning the existence of periodic, or almost periodic solutions.

1. Let C_{t_0}, $t_0 \geqq 0$ be the space of all continuous n-vector-valued functions on the interval $[t_0, +\infty)$. By $C_{t_0}^b$ we denote the space of all functions in C_{t_0}, which are bounded on $[t_0, +\infty)$, under the norm

$$||f||_b = \sup_{t \in [t_0, +\infty)} \{||f(t)||\} ,$$

where $||\cdot||$ is the Euclidean norm in R^n. $C_{t_0}^l$ will be the space consisting of all functions in C_{t_0}, which have a finite limit as $t \to +\infty$. The space C_{t_0} is a Fréchet space if its topology is that of the uniform convergence on compact subintervals of $[t_0, +\infty)$. The spaces $C_{t_0}^b$, $C_{t_0}^l$ are Banach spaces. For $x \in C_{t_0}^l$, let $l_x = \lim_{t\to\infty} x(t)$. A set $K \subset C_{t_0}^l$ is compact if and only if it is uniformly bounded, equicontinuous and "uniformly convergent" in the following sense: for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $||x(t) - l_x|| < \varepsilon$, for all $t > \delta(\varepsilon)$ and all $x \in K$. For a proof of this statement the reader is referred to Avramescu [2]. For a matrix $A(t, x) = (a_{ij}(t, x))$ on $[t_0, +\infty) \times R^n$, $i, j = 1, 2, \cdots, n$, we put $||A(t, x)|| = \max_{i,j} |a_{ij}(t, x)|$. By $S_{t_0}^r$, $r > 0$ we denote the ball $\{f; f \in C_{t_0}^b, ||f||_b \leqq r\}$. We also make use of Tychonov's fixed point theorem as quoted in Hartman's book [4]:

"Let L be a linear, locally convex, topological, complete Hausdorff space. Let M be a closed, convex subset of L and $T: M \to M$ be a continuous operator such that the closure of TM is compact. Then T has a fixed point in M."

For the system (III) we suppose that A, F are n-vector-valued functions, which are defined and continuous on $R_+ \times R^n$, where $R_+ = [0, +\infty)$. By a solution of a system of the form (III) we mean any function $x \in C'_{t_0}$ (= the space of all continuously differentiable $f \in C_{t_0}$), which satisfies (III) on the interval $[t_0, +\infty)$. The number t_0 will depend on the particular solution under consideration. By $x(t, t_0, x_0)$ we denote a solution of (III), which passes through the point (t_0, x_0) at time t_0. A solution of the system (II) will always be denoted by $y = y(t, t_0, y_0)$.

The following definitions of convergence are given by Avramescu in [2].

(i) System (III) is said to be "convergent" if $\lim_{t\to\infty} x(t, t_0, x_0) = l_x(t_0, x_0)$ exists and is finite for each $(t_0, x_0) \in R_+ \times R^n$.

(ii) System (III) is said to be "equi-convergent" if it is convergent and to each triple $\varepsilon > 0$, $\alpha \geqq 0$, $t_0 \geqq 0$ there corresponds a function $T(t_0, \alpha, \varepsilon)$ such that

$$||x(t, t_0, x_0) - l_x(t_0, x_0)|| < \varepsilon$$

for every $t > T(t_0, \alpha, \varepsilon) + t_0$, and every x_0 with $||x_0|| \leqq \alpha$.

(iii) System (III) is said to be "equi-uniformly convergent" if it is equi-convergent and T does not depend on t_0.

(iv) System (III) is said to be "coalescent" if it is convergent and $l_x(0, x_0)$ is a constant.

(v) The solutions of (III) are said to be "uniformly bounded" if for each $\alpha \geqq 0$, $t_0 \geqq 0$ there exists a function $\beta(\alpha) \geqq 0$ such that $\|x(t, t_0, x_0)\| \leqq \beta(\alpha)$ whenever $\|x_0\| \leqq \alpha$ and $t \geqq t_0$.

2. Our first result guarantees the existence of convergent solutions $x(t, t_0, x_0)$ of System (III) for any $(t_0, x_0) \in [0, +\infty) \times R^n$ provided that this is true for the system (II).

THEOREM 1. *Assume that* $y = y(t, t_0, y_0)$ *is a solution of* (II) *and*

$$\text{(i)} \qquad \|A(t, v_1) - A(t, v_2)\| \leqq q(t, \|v_1 - v_2\|)$$

for every $t \geqq t_0$ *and every* $v_1, v_2 \in R^n$, *where* $q: [t_0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *is continuous such that*

$$\liminf_{n\to\infty} (1/n) \int_{t_0}^{\infty} \sup_{\|u\| \leqq n} q(t, \|u\|)dt = 0 ;$$

$$\text{(ii)} \qquad \lim_{n\to\infty} (1/n) \int_{t_0}^{\infty} \sup_{\|u\| \leqq n} \|F(t, y + u)\|dt = 0 .$$

Then there exists a solution $x(t, \tilde{t}_0, x_0)$ *of the system* (III), *for any* $(\tilde{t}_0, x_0) \in [t_0, +\infty) \times R^n$.

PROOF. Given $(\tilde{t}_0, x_0) \in [t_0, +\infty) \times R^n$, the conditions (i), (ii) imply the existence of an n_0 such that

$$(1) \qquad \|x_0 - y_0\| + \int_{\tilde{t}_0}^{\infty} \|A(t, y + f) - A(t, y)\|dt + \int_{\tilde{t}_0}^{\infty} \|F(t, y + f)\|dt \leqq n_0$$

for any function $f \in S_{\tilde{t}_0}^{n_0}$. Now, consider the operator $T: S_{\tilde{t}_0}^{n_0} \to S_{\tilde{t}_0}^{n_0}$ with

$$(2) \qquad v(t) = (Tf)(t) = x_0 - y_0 + \int_{\tilde{t}_0}^{t} [A(s, y(s) + f(s)) - A(s, y(s))]ds + \int_{\tilde{t}_0}^{t} F(s, y(s) + f(s))ds .$$

To show that the ball $S_{\tilde{t}_0}^{n_0}$ is closed w.r.t. the topology of uniform convergence on compact subintervals of $[\tilde{t}_0, +\infty)$, let $f_n \in S_{\tilde{t}_0}^{n_0}$ be such that $f_n \to f \in C_{\tilde{t}_0}^{b}$ uniformly on every compact subinterval of $[\tilde{t}_0, +\infty)$. Then, since $\lim_{n\to\infty} \|f_n(t)\| = \|f(t)\|$ and $\|f_n(t)\| \leqq n_0$, it follows that $\|f(t)\|_b \leqq n_0$, which shows our assertion. Now let $f_n, f \in S_{\tilde{t}_0}^{n_0}$ be as above. Then from (2) we obtain

$$(3) \qquad \|Tf_n - Tf\|_b \leqq \int_{\tilde{t}_0}^{\infty} \|A(s, y(s) + f_n(s)) - A(s, y(s) + f(s))\|ds + \int_{\tilde{t}_0}^{\infty} \|F(s, y(s) + f_n(s)) - F(s, y(s) + f(s))\|ds .$$

Since the integrands in the right-hand member of (3) are bounded by the integrable functions

$$\sup_{\|u\|\leqq 2n_0} q(t, \|u\|) \text{ and } \sup_{\|u\|\leqq n_0} \|F(t, y + u)\|$$

respectively, it follows from Lebesgue's dominated convergence theorem that $\lim_{n\to\infty} \|Tf_n - Tf\|_b = 0$. The rest of the proof of the fact that T has a fixed point in $S_{\tilde{t}_0}^{n_0}$ follows as in Kartsatos [5] and we omit it here. Let $(Tv)(t) = v(t)$, $t \in [\tilde{t}_0, +\infty)$. Then putting $x(t) = v(t) + y(t)$, we obtain $x(\tilde{t}_0) = x_0$ and the theorem is proved.

COROLLARY 1. *Assume that System* (II) *has a solution* $y(t, t_0, x_0)$ *for every* $(t_0, x_0) \in [0, +\infty) \times R^n$ *such that* $\lim_{t\to\infty} y(t, t_0, x_0) = l_y(t_0, x_0)$ *and it is known "a priori" that if* $x(t, t_0, x_0)$ *is a solution of System* (III), *then it is unique with respect to the initial condition* $x(t_0) = x_0$. *Then, provided that the hypotheses of Th.* 1 *are satisfied for every solution* $y(t, t_0, x_0)$ *of the above type, System* (III) *is convergent.*

PROOF. It is evident that the solution $x(t, t_0, x_0)$, guaranteed by Th. 1, has a finite limit as $t \to +\infty$, because $\lim_{t\to\infty} v(t, t_0, 0)$ exists and is finite, where $v(t, t_0, 0) \equiv x(t, t_0, x_0) - y(t, t_0, x_0)$.

In what follows in this section, the systems (II), (III) will be supposed to have unique solutions with respect to any initial conditions $(t_0, x_0) \in [0, +\infty) \times R^n$. The next theorem ensures equi-convergence for System (III).

THEOREM 2. *Under the hypotheses of Corollary* 1, *assume further that the systems* (II), (III) *are uniformly bounded and that System* (II) *is equi-convergent. Then System* (III) *is equi-convergent.*

PROOF. Let $h_k(t) = \max_{\|u\|\leqq k} \|F(t, u)\|$, $t \geqq 0$. Since the Systems (II), (III) are uniformly bounded, it follows that for every $\alpha > 0$ there exists a function $\beta_1(\alpha) \geqq 0$, $(\beta_2(\alpha) \geqq 0)$ such that

$$(4) \qquad \|x(t, t_0, x_0)\| \leqq \beta_1(\alpha) \text{ for } \|x_0\| \leqq \alpha \text{ and } t \geqq t_0\,,$$
$$(\|y(t, t_0, y_0)\| \leqq \beta_2(\alpha) \text{ for } \|y_0\| \leqq \alpha \text{ and } t \geqq t_0)\,.$$

Let $x(t, t_0, x_0)$, $y(t, t_0, x_0)$ be two fixed solutions of (III), (II) respectively, which satisfy (4). Then for $\beta(\alpha) = \beta_1(\alpha) + \beta_2(\alpha)$ we have

$$(5) \qquad \|x(t, t_0, x_0) - y(t, t_0, x_0)\| \leqq \beta(\alpha) \text{ for } t \geqq t_0\,.$$

Now let $q(t, \alpha) = \sup_{\|u\|\leqq\beta(\alpha)} q(t, \|u\|)$, $t \geqq 0$. Then it follows from (i) of Th. 1 that

$$(6) \qquad \int_0^\infty q(t, \alpha)dt < +\infty\,.$$

Since System (II) is equi-convergent, for every $\varepsilon > 0$, $\alpha \geqq 0$, $t_0 \geqq 0$, there exists a function $T_1(t_0, \alpha, \varepsilon)$ such that $\|y(t, t_0, x_0) - l_y(t_0, x_0)\| < \varepsilon/3$ for every $t > T_1(t_0, \alpha, \varepsilon) + t_0$ and every x_0 with $\|x_0\| \leqq \alpha$. Let $\varepsilon > 0$ and fix α as above. Since by Corollary 1 System (III) is convergent, $\lim_{t\to\infty} x(t, t_0, x_0) = l_x(t_0, x_0)$ exists and is finite (the limit $l_x(t_0, x_0)$ does not depend on $x(t)$ but we use this notation in order to distinguish from the limit of $y(t, t_0, x_0)$). Moreover, for $t \geqq t_0$

$$(7) \qquad x(t, t_0, x_0) - y(t, t_0, x_0) = \int_{t_0}^{t} [A(s, x(s)) - A(s, y(s))]ds + \int_{t_0}^{t} F(s, x(s))ds \,.$$

Taking the limit as $t \to +\infty$ in both sides of (7), we obtain

$$(8) \quad l_x(t_0, x_0) - l_y(t_0, x_0) = \int_{t_0}^{\infty}[A(s, x(s)) - A(s, y(s))]ds + \int_{t_0}^{\infty} F(s, x(s))ds$$

which, combined with (6) and (7), yields

$$(9) \qquad \Big| \|x(t, t_0, x_0) - l_x(t, t_0, x_0)\| - \|y(t, t_0, x_0) - l_y(t_0, x_0)\| \Big| \leqq \int_t^{\infty} \|A(s, x(s)) - A(s, y(s))\|ds + \int_t^{\infty} \|F(s, x(s))\|ds \leqq \int_t^{\infty} q(s, \alpha)ds + \int_t^{\infty} h_{\beta_1(\alpha)}(s)ds \,,$$

where $t \geqq t_0$. Let $T(t_0, \alpha, \varepsilon) \geqq T_1(t_0, \alpha, \varepsilon)$ be such that

$$\int_t^{\infty} q(s, \alpha)ds < \varepsilon/3 \,, \quad \int_t^{\infty} h_{\beta_1(\alpha)}(s)ds < \varepsilon/3$$

for every $t \geqq T(t_0, \alpha, \varepsilon) + t_0$. Then, for $t > T(t_0, \alpha, \varepsilon) + t_0$, (9) implies

$$(10) \qquad \|x(t, t_0, x_0) - l_x(t_0, x_0)\| < \|y(t, t_0, x_0) - l_y(t_0, x_0)\| + 2\varepsilon/3 < \varepsilon \,,$$

which proves the equi-convergence of System (III).

COROLLARY 2. *Under the hypotheses of Corollary* 1, *assume further that the systems* (II), (III) *are uniformly bounded and that the System* (II) *is equi-uniformly convergent. Then the System* (III) *is equi-uniformly convergent.*

The proof is the same as that of Th. 2. T is now independent of t_0 since so is T_1.

We show now that the conditions on A in Th. 1 prevent System (II) from being coalescent.

THEOREM 3. *If A satisfies* (i) *of Th.* 1, *then System* (II) *cannot be coalescent.*

PROOF. Suppose that System (II) coalesces at the point y_∞, and consider the integral equation

$$v(t) = \xi - \int_t^\infty [A(s, v(s) + y(s)) - A(s, y(s))]ds , \tag{11}$$

where $y(t, 0, y_0)$ is a fixed solution of (II) and $\|\xi\| > 0$. By the method used in Th. 1 (cf. also Kartsatos [6]) it can be shown that (11) has a solution $v = v(t, t_0, v_0)$ defined on $[0, +\infty)$ and such that $\lim_{t\to\infty} v(t) = \xi$. Letting $z(t, 0, z_0) = v(t, 0, v_0) + y(t, 0, y_0)$, we obtain $\lim_{t\to\infty} z(t, 0, z_0) = \xi + y_\infty$, a contradiction to coalescence.

3. In this section we study systems of the form

$$x' = A(t, x) + G(t, x)x , \tag{IV}$$

where the $n \times n$ matrix G is defined and continuous on $[0, +\infty) \times R^n$. We first give a theorem concerning the existence of solutions of (IV) in $C^l_{t_0}$. By $S^{l,r}_{t_0}$ we denote the ball $\{f; f \in C^l_{t_0}$ and $\|f\|_b \leqq r\}$.

THEOREM 4. *Assume that for each* $f \in C^l_{t_0}$, *the system*

$$u' = G(t, f)u + A(t, f) \tag{IV$_f$}$$

has a unique solution $u(t, t_0, u_0) \in C^l_{t_0}$, *where* u_0 *is a fixed vector in* R^n. *Moreover, assume that*

(i) $\|G(t, f)\| \leqq p(t)$ *for every* $(t, f) \in [t_0, +\infty) \times C^l_{t_0}$, *where* p *is continuous and such that* $\int_{t_0}^\infty p(t)dt < +\infty$;

(ii) $\liminf_{n\to\infty} (1/n)\int_{t_0}^\infty \sup_{\|u\|\leqq n} \|A(t, u)\|dt = 0$

Then, there exists a solution $x(t)$ *of the system* (IV) *which belongs to the space* $C^l_{t_0}$.

PROOF. Let T be the operator which assigns to each function $f \in C^l_{t_0}$ the unique solution $u \in C^l_{t_0}(u(t_0) = u_0)$, of the system (IV$_f$). We first show that there exists a ball $S^{l,n_0}_{t_0}$ such that $T(S^{l,n_0}_{t_0}) \subset S^{l,n_0}_{t_0}$. In fact, assume that this is not true. Then there exists a sequence $\{f_n\}$, $n = 1, 2, \cdots$ such that $f_n \in S^{l,n}_{t_0}$ and $\|Tf_n\|_b > n$. Putting $u_n = Tf_n$ we obtain

$$u_n(t) = u_0 + \int_{t_0}^t G(s, f_n(s))u_n(s)ds + \int_{t_0}^t A(s, f_n(s))ds , \tag{12}$$

which implies

$$\|u_n(t)\| \leqq \|u_0\| + \int_{t_0}^t \|G(s, f_n(s))\|\|u_n(s)\|ds + \int_{t_0}^t \|A(s, f_n(s))\|ds . \tag{13}$$

An application of Gronwall's inequality in (13) implies

$$\|u_n(t)\| \leqq (\|u_0\| + \int_{t_0}^{\infty} \|A(s, f_n(s))\| ds) \exp\left[\int_{t_0}^{\infty} p(t) dt\right], \tag{14}$$

or

$$\frac{\|u_n\|}{n} \leqq \left[\frac{\|u_0\|}{n} + \frac{1}{n}\int_{t_0}^{\infty} \sup_{\|u\| \leqq n} \|A(s, u)\| ds\right] \exp\left[\int_{t_0}^{\infty} p(t) dt\right].$$

From inequality (14) we obtain $\liminf_{n\to\infty} \|u_n\|/n = 0$, a contradiction. To show that the set $TB(B = S_{t_0}^{l, n_0})$ is equi-continuous, let t', t'' be two points in $[t_0, +\infty)$ with $t'' \geqq t'$. Then we obtain

$$\begin{aligned} \|u_n(t'') - u_n(t')\| &\leqq \int_{t'}^{t''} \|G(s, f_n(s))\| \|u_n(s)\| ds + \int_{t'}^{t''} \|A(s, f_n(s))\| ds \\ &\leqq n_0 \int_{t'}^{t''} P(t) dt + \int_{t'}^{t''} \sup_{\|u\| \leqq n_0} \|A(s, u)\| ds . \end{aligned} \tag{15}$$

The rest follows as in [5] and we omit it here. Now let $\lambda_n = \lim_{t\to\infty} u_n(t)$. Then we have

$$\lambda_n = u_0 + \int_{t_0}^{\infty} G(t, f_n(t)) u_n(t) dt + \int_{t_0}^{\infty} A(t, f_n(t)) dt , \tag{16}$$

and, consequently,

$$\begin{aligned} \|u_n(t) - \lambda_n\| &\leqq n_0 \int_{t}^{\infty} \|G(s, f_n(s))\| ds + \int_{t}^{\infty} \|A(s, f_n(s))\| ds \\ &\leqq n_0 \int_{t}^{\infty} P(s) ds + \int_{t}^{\infty} \sup_{\|u\| \leqq n_0} \|A(s, u)\| ds . \end{aligned} \tag{17}$$

It follows from (17) that TB is a uniformly convergent family. Since TB is bounded, equicontinuous and uniformly convergent, it is compact in $C_{t_0}^{l}$. To show that T is continuous, let $\lim_{n\to\infty} \|f_n - f\|_b = 0, f_n, f \in B$. Since the set TB is compact, there exists a subsequence $\{u_{k_n}\}$ of $\{u_n = Tf_n\}$ such that $\lim_{n\to\infty} \|u_{k_n} - u\|_b = 0$, where u is an element in TB. Now since the sequence $G(t, f_{k_n}(t)) u_{k_n}(t)$, $A(t, f_{k_n}(t))$ converge pointwise to $G(t, f) u(t)$ and $A(t, f(t))$ respectively, and

$$\begin{aligned} \|G(t, f_{k_n}(t)) u_{k_n}(t) - G(t, f(t)) u(t)\| &\leqq 2 n_0 p(t) , \\ \|A(t, f_{k_n}(t)) - A(t, f(t))\| &\leqq 2 \sup_{\|u\| \leqq n_0} \|A(t, u)\| , \end{aligned} \tag{18}$$

and application of Lebesgue's dominated convergence theorem shows that $Tf = u$. Since we could have started with any subsequence of $\{Tf_n\}$ instead of $\{Tf_n\}$ itself, we have actually shown that every subsequence of $\{Tf_n\}$ contains a subsequence converging to Tf. This proves the continuity of the operator T. By Tychonov's theorem, T has a fixed point in $S_{t_0}^{l, n_0}$, and this proves the theorem.

REMARK. Assume that the perturbation $F(t, x)$ in System (III) is continuously differentiable with respect to x. Then this function satisfies

$$F(t, x) = G^0(t, x)x + F^0(t, x) \tag{19}$$

where $F_i^0(t, x) = F_i(t, x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)$ and G^0 is a diagonal $n \times n$ matrix, whose diagonal elements are given by

$$G_{ii}^0(t, x) = \int_0^1 \frac{\partial F_i(t, x_1, x_2, \cdots, \tau x_i, \cdots, x_n)}{\partial x_i} d\tau . \tag{20}$$

Thus, Theorem 4 holds for systems of the type (III) with perturbations like (19), under suitable assumptions on the functions $G_{ii}^0(t, x)$, $F_i^0(t, x)$. Th. 1 holds if we interchange the limit conditions on the integrals in (i) and (ii). However, the conditions were imposed only in order to guarantee that for some n_0, $TS_{\tilde{t}_0}^{n_0} \subset S_{\tilde{t}_0}^{n_0}$. It is evident that they can be avoided if we are only interested in the existence of solutions for all large t, provided of course that the functions $q(t, \|u\|)$, $F(t, y + v)$ are eventually uniformly bounded by integrable functions depending only on t and (t, y) respectively.

Analogous remarks can be made for Theorems 4 and 5.

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