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ON PLESSNER POINTS OF MEROMORPHIC FUNCTIONS

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The notion of a set of porosity was first formulated by Dolzhenko [4]. It was shown in [4], [5], [8] and [9] that this was useful for investigation of cluster sets. In this paper, we make use of this notion and sharpen some of Meier's results in [2], [3].

1. Notations and definitions. The geometry of the following is greatly simplified by considering meromorphic functions defined in the upper half plane, instead of the unit disc. Therefore unless otherwise stated, we denote the upper half z-plane $\{z; \mathrm{Im}(z) > 0\}$ by G and the real axis by Γ, and let the function $w = f(z)$ be meromorphic in G and take values in the w-sphere Ω.

Let $\zeta \in \Gamma$. We denote by $\rho_\zeta(\psi)$ the chord of the upper half plane G terminating at ζ and making an angle ψ, $0 < \psi < \pi$, with the positive real axis, and by $\rho_\zeta(\psi; \delta)$, $0 < \delta < \infty$, the set of the points $z = x + iy$ on the chord $\rho_\zeta(\psi)$, which satisfy the condition $0 < y < \delta$. We denote by $\Delta_\zeta(\alpha, \beta)$, $0 < \alpha < \beta < \pi$, the open angular domain (the *Stolz angle* with the vertex ζ) bounded by two chords $\rho_\zeta(\alpha)$, $\rho_\zeta(\beta)$, and by $\Delta_\zeta(\alpha, \beta; \delta)$, $0 < \delta < \infty$, the set of the points $z = x + iy$ in the Stolz angle $\Delta_\zeta(\alpha, \beta)$, which satisfy the condition $0 < y < \delta$. In the case when no confusion occurs, we use the simple notation Δ_ζ without specifying α, β or α, β, δ.

We denote by $C_{\rho_\zeta(\psi)}(f)$ the cluster set of $f(z)$ on a chord $\rho_\zeta(\psi)$ and by $C_{\Delta_\zeta(\alpha,\beta)}(f)$ the cluster set of $f(z)$ on a Stolz angle $\Delta_\zeta(\alpha, \beta)$. We say a point $\zeta \in \Gamma$ is a *Plessner point* of $f(z)$, if $C_{\Delta_\zeta}(f) = \Omega$ for any Stolz angle Δ_ζ.

Suppose a set $T \subset \Gamma$ and a point $\zeta \in \Gamma$ are given. For an $\varepsilon > 0$, we denote a segment $\{\xi \in \Gamma; \zeta - \varepsilon < \xi < \zeta + \varepsilon\}$ by $\Gamma(\varepsilon, \zeta)$. Let $\gamma(\zeta, \varepsilon, T)$ be the largest of lengths of arcs contained in $\Gamma(\varepsilon, \zeta)$ and not intersecting with T. The set T is said to be *of porosity at* ζ, if

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{\varepsilon} \gamma(\zeta, \varepsilon, T) > 0 .$$

A set T is said to be *of porosity* if it is so at each $\zeta \in T$. A set which is a countable union of sets of porosity is said to be *of σ-porosity*. A set of σ-porosity is of the first Baire category. It is easily seen that a set which is of porosity has no points of density with respect to outer measure. Therefore every set of σ-porosity is of outer measure 0. But there exists a set, which is of measure 0 and not of σ-porosity (see [7]).

2. **Statement of results.** In this section, we state two theorems which are proved in this paper. Theorem 1 will be proved in the section 3 and Theorem 2 in the section 4.

THEOREM 1. *Suppose that $f(z)$ is meromorphic in the upper half plane G. Then, except for a set of σ-porosity, every Plessner point $\zeta \in \Gamma$ of $f(z)$ has either of the following two properties* A), B):

A) *every value on Ω, with at most two exceptions, is taken by $f(z)$ infinitely often in any Stolz angle Δ_ζ with the vertex ζ. Thus, for each exceptional value $a \in \Omega$, if any, there is a Stolz angle Δ_ζ with the vertex ζ where $f(z)$ takes a, a finite number of times,*

B) *every value on Ω is either a cluster value of $f(z)$ on all chords $\rho_\zeta(\psi)$ terminating at ζ in the upper half plane G, or it is taken by $f(z)$ infinitely often in any Stolz angle Δ_ζ with the vertex ζ.*

REMARK 1. By Meier [3], it was shown that the above exceptional set is a set of measure 0 and of the first Baire category.

THEOREM 2. *Suppose that $f(z)$ is meromorphic in the upper half plane G. Then, except for a set of σ-porosity, every Plessner point $\zeta \in \Gamma$ of $f(z)$ has the following property* B*):

B*) *every value on Ω is either a cluster value of $f(z)$ on all chords $\rho_\zeta(\psi)$ terminating at ζ except only one value of ψ, or it is taken by $f(z)$ infinitely often in any Stolz angle Δ_ζ with the vertex ζ.*

REMARK 2. Meier proved in [2] and [3] that the above exceptional set is a set of measure 0 and of the first Baire category. See Noshiro [6].

3. **Proof of Theorem 1.** In the sections 3 and 4, the following notations are used.

Let $\{c_t\}_{t=1}$ be a sequence consisting of all complex numbers whose real and imaginary parts are both rational. Let $\{r_t\}_{t=1}$ be a sequence of all rational numbers satisfying $0 < r_t < \pi$. We denote by $\overline{ab}(=\overline{ba})$ the Euclidean length of the interval (a, b) on the real axis Γ.

We denote by E the set of points of Γ, for which neither A) nor B) are satisfied. Then, for each point $\zeta \in E$, there exist three values a_ζ^1, a_ζ^2, $a_\zeta^3 \in \Omega$, a chord $\rho_\zeta(\psi_\zeta)$ and three Stolz angles Δ_ζ^1, Δ_ζ^2, Δ_ζ^3, such that $a_\zeta^1 \notin C_{\rho_\zeta(\psi_\zeta)}(f)$ and $f(z) \neq a_\zeta^\lambda$ for $z \in \Delta_\zeta^\lambda$ ($\lambda = 1, 2, 3$).

We denote by P the set of points of E, at which the above three values a_ζ^1, a_ζ^2, a_ζ^3 are finite.

Now we have to prove that no point of the set P is a Plessner point of $f(z)$ except for a set of σ-porosity. For the set $E - P$, we can see the same conclusion without difficulty.

For positive integers $m_1, m_2, m_3, n_1, n_2, n_3, p$ satisfying inequalities

$$(1) \qquad \frac{2}{p} < r_{n_\lambda} < \pi - \frac{2}{p} \qquad (\lambda = 1, 2, 3)\,,$$

$$|c_{m_i} - c_{m_j}| < \frac{100}{p} \qquad (i \neq j,\ i, j = 1, 2, 3)\,,$$

we define $P_{m_1,m_2,m_3,n_1,n_2,n_3,p}$ as the set of points ζ of P, at which the following conditions are satisfied:

$$|f(z) - c_{m_1}| > \frac{10}{p} \quad \text{for} \quad z \in \rho_\zeta\left(\psi_\zeta; \frac{1}{p}\right)\,,$$

$$f(z) \neq a_\zeta^\lambda \quad \text{for} \quad z \in \Delta_\zeta\left(r_{n_\lambda} - \frac{1}{p}, r_{n_\lambda} + \frac{1}{p}, \frac{1}{p}\right) \qquad (\lambda = 1, 2, 3)\,,$$

$$|a_\zeta^\lambda - c_{m_\lambda}| < \frac{1}{p} \qquad (\lambda = 1, 2, 3)\,.$$

Then we have (see Meier [3])

$$P = \bigcup_{m_1,m_2,m_3,n_1,n_2,n_3,p} P_{m_1,m_2,m_3,n_1,n_2,n_3,p}\,.$$

We define $P^*_{m_1,m_2,m_3,n_1,n_2,n_3,p}$ as the set of points of $P_{m_1,m_2,m_3,n_1,n_2,n_3,p}$, at which $P_{m_1,m_2,m_3,n_1,n_2,n_3,p}$ is not of porosity. Then at every point of the set

$$P_{m_1,m_2,m_3,n_1,n_2,n_3,p} - P^*_{m_1,m_2,m_3,n_1,n_2,n_3,p}\,,$$

the set $P_{m_1,m_2,m_3,n_1,n_2,n_3,p}$ is of porosity. Therefore the set

$$P_{m_1,m_2,m_3,n_1,n_2,n_3,p} - P^*_{m_1,m_2,m_3,n_1,n_2,n_3,p}$$

is a set of porosity.

We will show that no point of the set $P^*_{m_1,m_2,m_3,n_1,n_2,n_3,p}$ is a Plessner point of $f(z)$. In the following, we set

$$P_{m_1,m_2,m_3,n_1,n_2,n_3,p} = P \quad \text{and} \quad P^*_{m_1,m_2,m_3,n_1,n_2,n_3,p} = P^*\,.$$

Let $\zeta \in P^*$. Since $\zeta \in P$, we can choose a positive number δ_ζ which satisfies

$$(2) \qquad \sin(\delta_\zeta) < \sin(\psi_\zeta) \cdot \sin\left(\frac{1}{p}\right)\,.$$

Now, we need the following lemma which will be verified at the end of this proof of Theorem 1.

LEMMA 1. *Let* $\zeta \in P^*$ *and* $z_0 \in \rho_\zeta(\psi_\zeta)$, $z_0 = \zeta + Re^{i\psi_\zeta}$. *We denote by* C_ζ^R

the disc $\{z; |z - z_0| < R \cdot \sin(\delta_\zeta)\}$. *Then, if* R *is sufficiently small, there exist* $\zeta_\lambda(z_0) \in P$ $(\lambda = 1, 2, 3)$ *such that*

$$\Delta_{\zeta_\lambda(z_0)}\left(r_{n_\lambda} - \frac{1}{p}, r_{n_\lambda} + \frac{1}{p}\right) \supset C_\zeta^R .$$

From this Lemma 1, if R is sufficiently small, three different complex numbers $a^1_{\zeta_1(z_0)}, a^2_{\zeta_2(z_0)}, a^3_{\zeta_3(z_0)} \in \Omega$ are not taken by $f(z)$ in C_ζ^R. The remaining part of the proof proceeds entirely analogously to Meier's [3, Satz 1], so may be omitted.

Thus, we come to the conclusion that except for the set

$$\bigcup_{m_1, m_2, m_3, n_1, n_2, n_3, p} (P_{m_1, m_2, m_3, n_1, n_2, n_3, p} - P^*_{m_1, m_2, m_3, n_1, n_2, n_3, p})$$

of σ-porosity, no point of the set P is a Plessner point of $f(z)$.

Now, to complete the proof of the theorem, we need only prove Lemma 1. Here, we will only prove the existence of $\zeta_1(z_0) \in P$, since the existence of $\zeta_2(z_0), \zeta_3(z_0) \in P$ is proved by the analogous method.

Suppose that there exists a sequence $\{z_0^n = \zeta + R_n e^{i\psi_\zeta}\}$, $z_0^n \to \zeta$ $(n \to \infty)$ such that $(\xi_1(z_0^n), \xi_2(z_0^n)) \cap P = \emptyset$, where we denote by $\xi_1(z_0^n)$ (or $\xi_2(z_0^n)$) the point on Γ at which the chord, making the angle $r_{n_1} - 1/p$ (or $r_{n_1} + 1/p$) with the positive real axis and tangent to the disc $C_\zeta^{R_n}$ from right (or left), terminates. If we set

$$\varepsilon_n = \max\{\overline{\xi_1(z_0^n)\zeta}, \overline{\xi_2(z_0^n)\zeta}\} ,$$

we have

$$\varepsilon_n \leqq R_n\left(\frac{3}{\tan\left(\frac{1}{p}\right)} + 1\right)$$

from (1). Since $\overline{\xi_1(z_0^n)\xi_2(z_0^n)}$ is larger than

$$2R_n\left(\sin(\psi_\zeta) \cdot \sin\left(\frac{1}{p}\right) - \sin(\delta_\zeta)\right) ,$$

which is positive from (2), we have

$$\gamma(\zeta, \varepsilon_n, P) \geqq 2R_n\left(\sin(\psi_\zeta) \cdot \sin\left(\frac{1}{p}\right) - \sin(\delta_\zeta)\right) .$$

Therefore, we have

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{\varepsilon}\gamma(\zeta, \varepsilon, P) \geqq \overline{\lim_{n\to\infty}} \frac{1}{\varepsilon_n}\gamma(\zeta, \varepsilon_n, P) > 0 .$$

So the set P is of porosity at ζ, which contradicts the hypothesis $\zeta \in P^*$.

This completes the proof of Lemma 1 and hence of the theorem. q.e.d.

4. **Proof of Theorem 2.** We denote by E the set of points of Γ, at which B^*) is not satisfied. Then for every point $\zeta \in E$, there exist a value $a_\zeta \in \Omega$, two chords $\rho_\zeta(\alpha_\zeta)$, $\rho_\zeta(\beta_\zeta)$, $\alpha_\zeta < \beta_\zeta$ and a Stolz angle Δ_ζ such that

$$a_\zeta \notin \{C_{\rho_\zeta(\alpha_\zeta)}(f) \cup C_{\rho_\zeta(\beta_\zeta)}(f)\} \quad \text{and} \quad f(z) \neq a_\zeta \quad \text{for} \quad z \in \Delta_\zeta .$$

We denote by P the set of points of E, at which the above values a_ζ's are finite.

Now we will prove that no point of the set P is a Plessner point of $f(z)$ except for a set of σ-porosity. For the set $E - P$, we can prove the same conclusion without difficulty.

For positive integers r, s, t, we define $P_{r,s,t}$ as the set of points of P, for which the following conditions are satisfied:

$$(1) \qquad r_t > \frac{2}{s}, \quad r_t + \frac{2}{s} < \pi ,$$

$$(2) \qquad f(z) \neq a_\zeta \quad \text{for} \quad z \in \Delta_\zeta\left(r_t - \frac{1}{s}, r_t + \frac{1}{s}, \frac{1}{s}\right),$$

$$\frac{1}{s} < \alpha_\zeta < \beta_\zeta < \pi - \frac{1}{s} ,$$

$$(3) \quad |a_\zeta - c_r| < \frac{1}{s}, \ |f(z) - c_r| \geqq \frac{2}{s} \ \text{ for } \ z \in \left\{\rho_\zeta\left(\alpha_\zeta; \frac{1}{s}\right) \cup \rho_\zeta\left(\beta_\zeta; \frac{1}{s}\right)\right\} .$$

Then we have (see Meier [2])

$$P = \bigcup_{r,s,t} P_{r,s,t} .$$

Next, for positive integers l, m_1, m_2 satisfying the inequality

$$\frac{1}{s} < r_{m_1} - \frac{1}{l} < r_{m_1} + \frac{1}{l} < r_{m_2} - \frac{1}{l} < r_{m_2} + \frac{1}{l} < \pi - \frac{1}{s} ,$$

we define T_{l,m_1,m_2} as the set of points $\zeta \in P_{r,s,t}$, for which the following conditions are satisfied:

$$(4) \qquad \frac{\beta_\zeta - \alpha_\zeta}{4} > \frac{2}{l}, \quad \frac{\pi}{4} - \frac{\beta_\zeta - \alpha_\zeta}{4} > \frac{1}{l} ,$$

$$(5) \qquad |\alpha_\zeta - r_{m_1}| < \frac{1}{l}, \quad |\beta_\zeta - r_{m_2}| < \frac{1}{l} ,$$

$$(6) \qquad \sin\left(\delta_\zeta - \frac{1}{l}\right) \cdot \sin\left(\frac{1}{s}\right) > \sin\left(\frac{2}{l}\right)$$

for a positive number δ_ζ satisfying

$$(7) \qquad \sin^2\left(\frac{1}{s}\right)\cdot \min\left\{\sin\left(\frac{\beta_\zeta-\alpha_\zeta}{4}\right), \cos\left(\frac{\beta_\zeta-\alpha_\zeta}{4}+\frac{\pi}{4}\right)\right\} > \sin(\delta_\zeta)\,.$$

Then we have

$$P_{r,s,t} = \bigcup_{l,m_1,m_2} T_{l,m_1,m_2}\,.$$

In fact, if $\zeta \in P_{r,s,t}$, we choose a δ_ζ satisfying (7) and an l satisfying (4), (6) and then we can choose r_{m_1}, r_{m_2} satisfying (5). Thus we have $\zeta \in T_{l,m_1,m_2}$.

We define T^*_{l,m_1,m_2} as the set of points $\zeta \in T_{l,m_1,m_2}$, at which T_{l,m_1,m_2} is not of porosity. Then at every point of the set $T_{l,m_1,m_2} - T^*_{l,m_1,m_2}$, the set T_{l,m_1,m_2} is of porosity. Therefore the set $T_{l,m_1,m_2} - T^*_{l,m_1,m_2}$ is a set of porosity.

We will show that no point of the set T^*_{l,m_1,m_2} is a Plessner point of $f(z)$. In the following, we set $T_{l,m_1,m_2} = T$ and $T^*_{l,m_1,m_2} = T^*$. We put

$$\psi_1 = r_{m_1} - \frac{1}{l},\ \psi_2 = r_{m_1} + \frac{1}{l},\ \psi_3 = r_{m_2} - \frac{1}{l},\ \psi_4 = r_{m_2} + \frac{1}{l}\,.$$

Then we have

$$(8) \qquad \frac{1}{s} < \psi_1 < \alpha_\zeta < \psi_2 < \psi_3 < \beta_\zeta < \psi_4 < \pi - \frac{1}{s}\,.$$

Now, we need the following lemma whose proof will be given later.

LEMMA 2. *Let* $\zeta \in T^*$ *and* $z_0 = \zeta + Re^{i\psi} \in \Delta_\zeta(\psi^* - 1/l, \psi^* + 1/l)$, $\psi^* = (\psi_2 + \psi_3)/2$. *We denote by* C_ζ^R *the* disc $\{z; |z - z_0| < R\cdot\sin(\delta_\zeta - 1/l)\}$. *We denote by* $\xi_\lambda(z_0)$ *the point on* Γ *at which the chord, passing through* z_0 *and making the angle* ψ_λ *with the positive real axis, terminates* ($\lambda = 1, 2, 3, 4$). *We denote by* $\xi'_\lambda(z_0)$ ($\lambda = 1, 2, 3, 4$) *the point on* Γ *at which the chord, making the angle* $\psi_{\lambda+1}$ ($\lambda = 1, 3$) *or* $\psi_{\lambda-1}$ ($\lambda = 2, 4$) *with the positive real axis and tangent to the* disc C_ζ^R *from left* ($\lambda = 1, 3$) *or right* ($\lambda = 2, 4$).

Then, for every $z_0 \in \Delta_\zeta(\psi^* - 1/l, \psi^* + 1/l)$ *with sufficiently small* R, *there exist* $\zeta_\lambda(z_0) \in T$ ($\lambda = 1, 2, 3, 4$) *such that*

$$\xi'_\lambda(z_0) < \zeta_\lambda(z_0) < \xi_\lambda(z_0) \qquad (\lambda = 1, 3)$$

or

$$\xi_\lambda(z_0) < \zeta_\lambda(z_0) < \xi'_\lambda(z_0) \qquad (\lambda = 2, 4)\,.$$

From this Lemma 2 and (8), if R is sufficiently small, the chords

$$\rho_{\zeta_1(z_0)}(\alpha_{\zeta_1(z_0)}),\ \rho_{\zeta_2(z_0)}(\alpha_{\zeta_2(z_0)}),\ \rho_{\zeta_3(z_0)}(\beta_{\zeta_3(z_0)}) \quad \text{and} \quad \rho_{\zeta_4(z_0)}(\beta_{\zeta_4(z_0)})$$

intersect the disc C_ζ^R, and the quadrilateral Π bounded by these four chords is contained in a quadrilateral Π' bounded by four chords

$$\rho_{\xi_1'(z_0)}(\psi_2),\ \rho_{\xi_2'(z_0)}(\psi_1),\ \rho_{\xi_3'(z_0)}(\psi_4) \quad \text{and} \quad \rho_{\xi_4'(z_0)}(\psi_3)\ .$$

And also this quadrilateral Π' is contained in a disc $D_\zeta^R = \{z; |z - z_0| < \sigma_\zeta\}$, where

$$\sigma_\zeta = R \cdot \sin\left(\delta_\zeta - \frac{1}{l}\right) \cdot \max\left\{\sec\left(\frac{\psi_4 - \psi_2}{2}\right), \operatorname{cosec}\left(\frac{\psi_3 - \psi_2}{2}\right)\right\}.$$

Here, we use the following lemma whose proof will also be given later.

LEMMA 3. *Let $\zeta \in T^*$ and $z_0 = \zeta + Re^{i\psi} \in \Delta_\zeta(\psi^* - 1/l, \psi^* + 1/l)$, $\psi^* = (\psi_2 + \psi_3)/2$. Then, for every $z_0 \in \Delta_\zeta(\psi^* - 1/l, \psi^* + 1/l)$ with sufficiently small R, there exists an $\eta(z_0) \in T$ such that*

$$D_\zeta^R \subset \Delta_{\eta(z_0)}\left(r_t - \frac{1}{s}, r_t + \frac{1}{s}\right).$$

From this Lemma 3, the above quadrilateral Π is contained in the Stolz angle $\Delta_{\eta(z_0)}(r_t - 1/s, r_t + 1/s)$, $\eta(z_0) \in T$. Therefore we obtain

$$|f(z) - c_r| \geqq \frac{2}{s} \text{ on every side of the quadrilateral } \Pi$$

from (3) and

$$f(z) \neq a_{\eta(z_0)} \text{ in the quadrilateral } \Pi$$

from (2).

The remaining part of the proof proceeds quite similarly to Meier's [2], so may be omitted.

Thus, we come to the conclusion that except for the set

$$P'_{r,s,t} = \bigcup_{l, m_1, m_2} (T_{l,m_1,m_2} - T^*_{l,m_1,m_2})$$

of σ-porosity, no point of the set $P_{r,s,t}$ is a Plessner point of $f(z)$, hence, except for the set $\bigcup_{r,s,t} P'_{r,s,t}$ of σ-porosity, no point of the set P is a Plessner point of $f(z)$.

Now, to complete the proof of the theorem, we need only prove Lemma 2 and Lemma 3.

PROOF OF LEMMA 2. Here, we will only prove the existence of $\zeta_1(z_0) \in T$, since the existence of $\zeta_2(z_0), \zeta_3(z_0), \zeta_4(z_0) \in T$ is proved by the analogous method.

Suppose that there exists a sequence

$$\{z_0^n = \zeta + R_n e^{i\psi_n}\}_{n=1}^\infty,\ z_0^n \in \Delta_\zeta\left(\psi^* - \frac{1}{l}, \psi^* + \frac{1}{l}\right),\ z_0^n \to \zeta(n \to \infty)$$

such that

$$(\xi_1'(z_0^n),\ \xi_1(z_0^n)) \cap T = \emptyset\ .$$

If we set

$$\varepsilon_n = \max\{\overline{\xi_1(z_0^n)\zeta},\ \overline{\xi_1'(z_0^n)\zeta}\}\ ,$$

we have

$$\varepsilon_n \leqq R_n\left(\frac{2}{\tan\left(\dfrac{1}{s}\right)} + 2\right)$$

from (8). Since $\overline{\xi_1'(z_0^n)\xi_1(z_0^n)}$ is larger than

$$R_n\frac{\sin\left(\delta_\zeta - \dfrac{1}{l}\right)\cdot\sin\left(\dfrac{1}{s}\right) - \sin\left(\dfrac{2}{l}\right)}{\sin(\psi_1)\cdot\sin(\psi_2)}\ ,$$

which is positive from (6), we have

$$\gamma(\zeta,\ \varepsilon_n,\ T) \geqq R_n\frac{\sin\left(\delta_\zeta - \dfrac{1}{l}\right)\cdot\sin\left(\dfrac{1}{s}\right) - \sin\left(\dfrac{2}{l}\right)}{\sin(\psi_1)\cdot\sin(\psi_2)}\ .$$

Therefore, we have

$$\varlimsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\gamma(\zeta,\ \varepsilon,\ T) \geqq \varlimsup_{n\to\infty}\frac{1}{\varepsilon_n}\gamma(\zeta,\ \varepsilon_n,\ T) > 0\ .$$

So the set T is of porosity at ζ, which contradicts the hypothesis $\zeta \in T^*$. This completes the proof of Lemma 2. q.e.d.

PROOF OF LEMMA 3. We denote by $\phi_1(z_0)$ (or $\phi_2(z_0)$) the point on Γ at which the chord, making the angle $r_t - 1/s$ (or $r_t + 1/s$) with the positive real axis and tangent to the disc D_ζ^R from right (or left), terminates.

Suppose that there exists a sequence

$$\{z_0^n = \zeta + R_n e^{i\psi_n}\}_{n=1}^\infty,\ z_0^n \in \Delta_\zeta\left(\psi^* - \frac{1}{l},\ \psi^* + \frac{1}{l}\right),\ z_0^n \to \zeta \quad (n \to \infty)$$

such that $(\phi_1(z_0^n),\ \phi_2(z_0^n)) \cap T = \emptyset$. If we set

$$\varepsilon_n = \max\{\overline{\phi_1(z_0^n)\zeta},\ \overline{\phi_2(z_0^n)\zeta}\}\ ,$$

we have

$$\varepsilon_n \leqq R_n\left(1 + \frac{3}{\sin\left(\dfrac{1}{s}\right)}\right)$$

from (1). Since

$$\cos\left(\frac{\psi_4-\psi_2}{2}\right) > \cos\left(\frac{\pi}{4}+\frac{\beta_\zeta-\alpha_\zeta}{4}\right)$$

from (4),

$$\sin\left(\frac{\psi_3-\psi_2}{2}\right) > \sin\left(\frac{\beta_\zeta-\alpha_\zeta}{4}\right)$$

from (4) and $\sin(1/s) < \sin(\psi_n)$ from (8), $\overline{\phi_1(z_0^n)\phi_2(z_0^n)}$ is larger than

$$2R_n \frac{\sin^2\left(\frac{1}{s}\right)\cdot \min\left\{\sin\left(\frac{B_\zeta-\alpha_\zeta}{4}\right), \cos\left(\frac{\pi}{4}+\frac{\beta_\zeta-\alpha_\zeta}{4}\right)\right\} - \sin(\delta_\zeta)}{\min\left\{\sin\left(\frac{\beta_\zeta-\alpha_\zeta}{4}\right), \cos\left(\frac{\pi}{4}+\frac{\beta_\zeta-\alpha_\zeta}{4}\right)\right\}},$$

which is positive from (7). Clearly

$$\gamma(\zeta, \varepsilon_n, T) \geqq 2R_n \frac{\sin^2\left(\frac{1}{s}\right)\cdot \min\left\{\sin\left(\frac{\beta_\zeta-\alpha_\zeta}{4}\right), \cos\left(\frac{\pi}{4}+\frac{\beta_\zeta-\alpha_\zeta}{4}\right)\right\} - \sin(\delta_\zeta)}{\min\left\{\sin\left(\frac{\beta_\zeta-\alpha_\zeta}{4}\right), \cos\left(\frac{\pi}{4}+\frac{\beta_\zeta-\alpha_\zeta}{4}\right)\right\}}.$$

Therefore we have

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{\varepsilon}\gamma(\zeta, \varepsilon, T) \geqq \overline{\lim_{n\to\infty}} \frac{1}{\varepsilon_n}\gamma(\zeta, \varepsilon_n, T) > 0 .$$

So the set T is of porosity at ζ, which contradicts the hypothesis $\zeta \in T^*$. This completes the proof of Lemma 3. q.e.d.

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