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RELATIVE BOUNDEDNESS AND SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. Given two positive continuous functions α and β, necessary and sufficient conditions are given for the system $u''(t) = f(t) + A(t)u(t)$ to have an α-bounded solution u for each β-bounded forcing function f. Applications are given to a nonlinear perturbation problem: $u''(t) = A(t)u(t) + F(t, u(t))$. Indications are given on how to extend these ideas to n^{th} order equations.

I. Introduction. Let Y be a finite-dimensional normed linear space with norm $|\ |$, let R^+ be the set of all nonnegative real numbers, and let $\mathscr{A}[Y]$ be the algebra of linear functions from Y to Y, with induced norm $\|\ \|$. Let A be a continuous function from R^+ to $\mathscr{A}[Y]$. In [2] and [3, Chapter V], W. A. Coppel has obtained necessary and sufficient conditions for it to be true that if f is a bounded continuous function from R^+ to Y then there is a bounded solution u on R^+ of

$$u'(t) = f(t) + A(t)u(t) \,. \tag{1}$$

Coppel's ideas have been amplified and extended by several authors, usually in the direction of determining conditions which ensure that if f is in one of two given function spaces then there is a solution u of (1) in the other. For some recent results in this connection and an excellent discussion of this problem, we refer the reader to T. G. Hallam [4].

In the present work we shall conduct the same kind of study for the second-order problem

$$u''(t) = f(t) + A(t)u(t) \,. \tag{2}$$

If one rewrites (2) as a first-order equation over Y^2 and then invokes known results, one's hypotheses require extending the class of forcing functions in a way unnatural to our purposes, and one's conclusions give boundedness properties not only for u but also for u'. (Compare the discussion of J. L. Massera and J. J. Schäffer [6, Chapter 12, § 120].) Thus we see the rationale for studying (2) as is.

II. Relatively bounded solutions. If γ is a positive continuous function on R^+ then $\mathscr{B}_\gamma\mathscr{C}$ will denote the linear space to which f belongs only in case f is a continuous function from R^+ to Y and there is a number b such that $|f(t)| \leqq b\gamma(t)$ whenever t is in R^+. If f is in $\mathscr{B}_\gamma\mathscr{C}$ let $\|f\|_\gamma$ be the least number b such that $|f(t)| \leqq b\gamma(t)$ whenever t is in R^+. Now $\|\ \|_\gamma$ is a norm $\mathscr{B}_\gamma\mathscr{C}$ and $\mathscr{B}_\gamma\mathscr{C}$ is a Banach space with respect to $\|\ \|_\gamma$.

Let each of α and β be a continuous positive function on R^+, and let each of Φ_1, Φ_2, Ψ_1, and Ψ_2 be a twice continuously differentiable function from R^+ to $\mathscr{A}[Y]$. We will assume throughout that conditions (C1) and (C2) are satisfied.

(C1): If t is in $(0, \infty)$ and i is in $\{1, 2\}$ then

$$\Phi_i''(t) = A(t)\Phi_i(t)$$

and

$$\Psi_i''(t) = \Psi_i(t)A(t)\ .$$

(C2):

$$\begin{aligned} \Phi_1(0) &= I\ , & \Psi_1(0) &= I\ ,\\ \Phi_1'(0) &= 0\ , & \Psi_1'(0) &= 0\ ,\\ \Phi_2(0) &= 0\ , & \Psi_2(0) &= 0\ ,\\ \Phi_2'(0) &= I\ , & \Psi_2'(0) &= -I\ . \end{aligned}$$

Note that in the scalar case, or, more generally, if the set $\{A(t): t$ is in $R^+\}$ is a commutative subset of $\mathscr{A}[Y]$, that $\Phi_1 = \Psi_1$ and $\Phi_2 = -\Psi_2$. The following theorem is our main result.

THEOREM 1. *There are four members P_1, P_2, P_3, and P_4 (in general not projections) of $\mathscr{A}[Y]$ such that if Γ_1 and Γ_2 are continuous functions from $R^+ \times R^+$ to $\mathscr{A}[Y]$ given by*

$$\begin{aligned}\Gamma_1(t, s) = \Phi_1(t)P_1\Psi_2(s) &+ \Phi_1(t)P_2\Psi_1(s)\\ &+ \Phi_2(t)P_3\Psi_2(s) + \Phi_2(t)P_4\Psi_1(s)\end{aligned}$$

and

$$\begin{aligned}\Gamma_2(t, s) = \Phi_1(t)(I - P_1)\Psi_2(s) &- \Phi_1(t)P_2\Psi_1(s)\\ &- \Phi_2(t)P_3\Psi_2(s) + \Phi_2(t)(I - P_4)\Psi_1(s)\ ,\end{aligned}$$

then (i) *and* (ii) *are equivalent.*

(i) *If f is in $\mathscr{B}_\beta\mathscr{C}$ then there is a solution u of* (2) *in $\mathscr{B}_\alpha\mathscr{C}$.*

(ii) *The integral*

$$\int_0^\infty \| -P_3\Psi_2(s) + (I - P_4)\Psi_1(s)\| \beta(s)ds$$

is finite and there is a positive number K such that

$$\int_0^t \|\Gamma_1(t, s)\| \beta(s)ds + \int_t^\infty \|\Gamma_2(t, s)\| \beta(s)ds \leqq K\alpha(t)$$

whenever t *is in* R^+.

In the proof of Theorem 1 we shall need the following lemma.

LEMMA. *If* t *is in* R^+ *then*

$$\Phi_1(t)\Psi_2(t) + \Phi_2(t)\Psi_1(t) = 0$$

and

$$\Phi_1'(t)\Psi_2(t) + \Phi_2'(t)\Psi_1(t) = I .$$

PROOF. For the proof of this lemma we shall use systems. Let $\tilde{A}$ from R^+ to $\mathscr{A}[Y^2]$ be given by

$$\tilde{A}(t) = \begin{bmatrix} 0 & I \\ A(t) & 0 \end{bmatrix}.$$

Let $\tilde{\Phi}$ be the fundamental solution solving

$$\tilde{\Phi}'(t) = \tilde{A}(t)\tilde{\Phi}(t);\ \tilde{\Phi}(0) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Now $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(t) = \begin{bmatrix} \Phi_1(t) & \Phi_2(t) \\ \Phi_1'(t) & \Phi_2'(t) \end{bmatrix}.$$

Let $\tilde{\Psi}$ be given by $\tilde{\Psi}(t) = \tilde{\Phi}(t)^{-1}$. Now

$$\tilde{\Psi}(0) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$\tilde{\Psi}'(t) = -\tilde{\Psi}(t)\tilde{A}(t)$$

whenever t is in R^+. Thus $\tilde{\Psi}$ is given by

$$\tilde{\Psi}(t) = \begin{bmatrix} -\Psi_2'(t) & \Psi_2(t) \\ -\Psi_1'(t) & \Psi_1(t) \end{bmatrix}.$$

Now the conclusions of the lemma follow immediately from the relation

$$\tilde{\Phi}(t)\tilde{\Psi}(t) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

PROOF OF THEOREM 1. First assume (ii). Let f be in $\mathscr{B}_\beta\mathscr{C}$ and let u be given by

$$u(t) = \int_0^t \Gamma_1(t, s)f(s)ds - \int_t^\infty \Gamma_2(t, s)f(s)ds .$$

Clearly u is in $\mathscr{B}_\alpha\mathscr{C}$. Also, (ii) tells us that

$$\int_0^\infty ||(I - P_1)\Psi_2(s) - P_2\Psi_1(s)||\beta(s)ds \leqq K\alpha(0) .$$

This, with the first part of (ii), tells us that u is twice differentiable and satisfies (2). Thus (ii) implies (i).

Now assume (i). Let M be that linear subset of Y^2 to which (x, y) belongs if and only if the solution v of

$$v''(t) = A(t)v(t);\ v(0) = x,\ v'(0) = y \tag{3}$$

is in $\mathscr{B}_\alpha\mathscr{C}$. Let $\tilde{P}$ be a projection from Y^2 to M. If f is in $\mathscr{B}_\beta\mathscr{C}$ let u_f be that member of $\mathscr{B}_\alpha\mathscr{C}$ such that u_f solves (2) and $\tilde{P}(u_f(0), u'_f(0)) = (0, 0)$. We shall first show that there is a number L such that

$$||u_f||_\alpha \leqq L||f||_\beta$$

and

$$|u'_f(0)| \leqq L||f||_\beta$$

whenever f is in $\mathscr{B}_\beta\mathscr{C}$.

If u is a twice continuously differentiable function from R^+ to Y, let $\hat{u}$ from R^+ to Y be given by $\hat{u}(t) = u''(t) - A(t)u(t)$. Let $\mathscr{D}$ be that subset of $\mathscr{B}_\alpha\mathscr{C}$ to which u belongs if and only if u is twice continuously differentiable and $\hat{u}$ is in $\mathscr{B}_\beta\mathscr{C}$. If u is in $\mathscr{D}$ let $||u||_D = ||u||_\alpha + |u'(0)| + ||\hat{u}||_\beta$. Now $\mathscr{D}$ is a linear space and $||\ ||_D$ is a norm on $\mathscr{D}$. Suppose that $\{u_n\}_{n=1}^\infty$ is a $||\ ||_D$-Cauchy $\mathscr{D}$-valued sequence. Now we can find (u_0, z, f) in $\mathscr{B}_\alpha\mathscr{C} \times Y \times \mathscr{B}_\beta\mathscr{C}$ such shat $||u_n - u_0||_\alpha \to 0$, $|u'_n(0) - z| \to 0$, and $||\hat{u}_n - f||_\beta \to 0$ as $n \to \infty$. If t is in R^+ then

$$\begin{aligned} u_0(t) &= \lim_{n\to\infty} u_n(t) \\ &= \lim_{n\to\infty} \left(u_n(0) + tu'_n(0) + \int_0^t (t - s)u''_n(s)ds\right) \\ &= u_0(0) + tz + \lim_{n\to\infty} \int_0^t (t - s)(\hat{u}_n(s) + A(s)u_n(s))ds \\ &= u_0(0) + tz + \int_0^t (t - s)(f(s) + A(s)u_0(s))ds . \end{aligned}$$

Thus u_0 is twice differentiable, $u'_0(0) = z$, and $\hat{u}_0 = f$. So u_0 is in $\mathscr{D}$ and $||u_n - u_0||_D \to 0$ as $n \to \infty$. Thus $\mathscr{D}$ is a Banach space with respect to $||\ ||_D$.

Let T be the linear function from $\mathscr{D}$ to $\mathscr{B}_\beta\mathscr{C}$ given by $Tu = \hat{u}$. Clearly T is continuous with operator norm not exceeding 1. Suppose u is in $\mathscr{D}$ and $Tu = 0$. Now $u''(t) = A(t)u(t)$ whenever t is in R^+, u is in

$\mathscr{B}_\alpha\mathscr{C}$ (since $\mathscr{D}$ is a subset of $\mathscr{B}_\alpha\mathscr{C}$), and $\tilde{P}(u(0), u'(0)) = (0, 0)$. Thus $u = 0$ and T is one-to-one. That T is onto follows from our hypothesis. Thus according to [7, Theorem 4.1, p. 63], T^{-1} is a continuous function from $\mathscr{B}_\beta\mathscr{C}$ to $\mathscr{D}$, and there is a number L such that

$$||u_f||_D \leqq L||f||_\beta$$

whenever f is in $\mathscr{B}_\beta\mathscr{C}$. Since $||u||_\alpha \leqq ||u||_D$ and $|u'(0)| \leqq ||u||_D$ whenever u is in $\mathscr{D}$, this completes the first part of the proof.

Since $\tilde{P}$ is a linear function from Y^2 into Y^2 there are four members P_1, P_2, P_3, and P_4 of $\mathscr{A}[Y]$ such that

$$\tilde{P} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}.$$

Thus $(P_1x + P_2y, P_3x + P_4y)$ is in M whenever (x, y) is in Y^2. But if (x, y) is in Y^2 then the solution v of (3) is given by $v(t) = \Phi_1(t)x + \Phi_2(t)y$. Thus it is clear that if (x, y) is in Y^2 then the function described by

$$t \to \Phi_1(t)(P_1x + P_2y) + \Phi_2(t)(P_3x + P_4y)$$

is in $\mathscr{B}_\alpha\mathscr{C}$.

Suppose that f is in $\mathscr{B}_\beta\mathscr{C}$, f has compact support, and w_f from R^+ to Y is given by

$$w_f(t) = \int_0^t \Gamma_1(t, s)f(s)ds - \int_t^\infty \Gamma_2(t, s)f(s)ds .$$

(This second integral obviously exists since f has compact support.) Find c in R^+ such that $f(t) = 0$ if $t \geqq c$. Now, if $t \geqq c$, then

$$\begin{aligned} w_f(t) = \Phi_1(t)\Big(P_1\int_0^c \Psi_2(s)f(s)ds + P_2\int_0^c \Psi_1(s)f(s)ds\Big) \\ + \Phi_2(t)\Big(P_3\int_0^c \Psi_2(s)f(s)ds + P_4\int_0^c \Psi_1(s)f(s)ds\Big) , \end{aligned}$$

so the above observations tell us that w_f is in $\mathscr{B}_\alpha\mathscr{C}$. Routine computations and the identities of our lemma tell us that $w_f''(t) = f(t) + A(t)w_f(t)$ whenever t is in R^+. Now

$$w_f(0) = -\int_0^c [(I - P_1)\Psi_2(s) - P_2\Psi_1(s)]f(s)ds$$

and

$$w_f'(0) = -\int_0^c [-P_3\Psi_2(s) + (I - P_4)\Psi_1(s)]f(s)ds . \tag{4}$$

Since $\tilde{P}$ is a projection and $\tilde{P}^2 = \tilde{P}$, we have the identities

$$P_1 = P_1^2 + P_2P_3\,,$$
$$P_2 = P_1P_2 + P_2P_4\,,$$
$$P_3 = P_3P_1 + P_4P_3\,,$$
$$P_4 = P_3P_2 + P_4^2\,.$$

These identities make it clear that $\tilde{P}(w_f(0), w'_f(0)) = (0, 0)$. Thus we see that if f is in $\mathscr{B}_\beta\mathscr{C}$ and has compact support, then $w_f = u_f$.

Since $\mathscr{A}[Y]$ is a finite-dimensional vector space, all norms on $\mathscr{A}[Y]$ are equivalent, and (ii) is true with respect to one norm on $\mathscr{A}[Y]$ if and only if (ii) is true with respect to every norm on $A[Y]$. Thus, to complete the proof, we need only show that (ii) is true with respect to a conveniently chosen norm.

Let $m = \dim(Y)$, and let $\{z_1, \cdots, z_m\}$ be a basis for Y. Let $\{\xi_1, \cdots, \xi_m\}$ be linear scalar-valued functions on Y such that if x is in Y then $x = \xi_1(x)z_1 + \cdots + \xi_m(x)z_m$. Now assume that $|\ |$ is given by $|x| = \max\{|\xi_1(x)|, \cdots, |\xi_m(x)|\}$. With this choice of $|\ |$, and the associated $||\ ||$, we see that the formula given for w_f and the formula (4) given for $w'_f(0)$, together with an argument similar to that of [2, pp. 133–134], show that (ii) holds with $K = mL$. This completes the proof of Theorem 1.

All of our results for continuous forcing functions satisfying boundedness conditions have straightforward analogues for integrability conditions. For first-order equations the $\mathscr{L}^1$ case has been considered by Coppel [3, Theorem 2, p. 131] and the $\mathscr{L}^p$ case ($p > 1$) has been considered by R. Conti [1]. The aforementioned work of Hallam [4] extends both of these. If f is locally integrable, but not necessarily continuous, then solutions of (2) will be thought of in the sense of Carathéodory, i.e., u continuously differentiable, u' absolutely continuous, and (2) satisfied almost everywhere. If p is in $[1, \infty)$ we take $\mathscr{L}_\beta^p$ to be the space of all locally integrable functions f from R^+ to Y such that

$$\int_0^\infty |f(t)/\beta(t)|^p\,dt$$

is finite.

THEOREM 2. *These are equivalent*:

(i) *If f is in $\mathscr{L}_\beta^1$ then there is a solution u of* (2) *in $\mathscr{B}_\alpha\mathscr{C}$.*

(ii) *There is a number K such that $||-P_3\Psi_2(s) + (I - P_4)\Psi_1(s)||\beta(s) \leqq K$ whenever s is in R^+, such that $||\Gamma_1(t, s)||\beta(s) \leqq K\alpha(t)$ whenever $0 \leqq s \leqq t$, and such that $||\Gamma_2(t, s)||\beta(s) \leqq K\alpha(t)$ whenever $0 \leqq t \leqq s$.*

THEOREM 3. *Let p and q be in $(1, \infty)$ with $p + q = pq$. These are equivalent*:

(i) *If f is in $\mathscr{L}_\beta^p$ then there is a solution u of* (2) *in $\mathscr{B}_\alpha\mathscr{C}$.*

(ii) *The integral*

$$\int_0^\infty || -P_3\Psi_2(s) + (I - P_4)\Psi_1(s) ||^q \beta(s)^q ds$$

is finite and there is a number K such that

$$\int_0^t || \Gamma_1(t, s) ||^q \beta(s)^q ds + \int_t^\infty || \Gamma_2(t, s) ||^q \beta(s)^q ds \leqq K^q \alpha(t)^q$$

whenever t is in R^+.

III. A perturbation problem. In this section we shall consider the equation

$$u''(t) = A(t)u(t) + F(t, u(t)) \,, \tag{5}$$

in which F is a continuous function from $R^+ \times Y$ to Y. Our work is related to, but independent of, work of Hallam [5]. Also, our treatment of (5) can be thought of as a second-order analogue of the first-order work of Coppel [2], [3, Chapter III].

TEOREM 4. *Let ω be a continuous function from $R^+ \times R^+$ to R^+, suppose $\omega(r, s) \leqq \omega(r, t)$ whenever (r, s, t) is in $R^+ \times R^+ \times R^+$ and $s \leqq t$, suppose $\lim_{t\to\infty} \omega(t, c\alpha(t))/\beta(t) = 0$ whenever c is in R^+, and suppose $|F(t, x)| \leqq \omega(t, |x|)$ whenever (t, x) is in $R^+ \times Y$. Suppose also that* (ii) *of Theorem* 1 *holds. Then if γ is a positive number there is a member c of R^+ and a solution u of* (5) *on $[c, \infty)$ such that $|u(t)| \leqq \gamma\alpha(t)$ whenever t is in $[c, \infty)$.*

PROOF. Let γ be a positive number. Now $\lim_{t\to\infty} \omega(t, \gamma\alpha(t))/\beta(t) = 0$, so there is c in R^+ such that $\omega(t, \gamma\alpha(t)) \leqq (\gamma/K)\beta(t)$ whenever t is in $[c, \infty)$. Let $\mathscr{F}$ be the set to which f belongs only in case f is a continuous function from $[c, \infty)$ to Y and $|f(t)| \leqq \gamma\alpha(t)$ whenever t is in $[c, \infty)$. If f is in $\mathscr{F}$, let $T[f]$ be that continuous function g from $[c, \infty)$ to Y given by

$$g(t) = \int_c^t \Gamma_1(t, s)F(s, f(s))ds - \int_t^\infty \Gamma_2(t, s)F(s, f(s))ds \,.$$

Let f be in $\mathscr{F}$, and $g = T[f]$. Let t be in $[c, \infty)$. Now

$$|g(t)| \leqq \int_c^t || \Gamma_1(t, s) || \omega(s, |f(s)|)ds + \int_t^\infty || \Gamma_2(t, s) || \omega(s, |f(s)|)ds$$

$$\leqq \int_c^t \|\Gamma_1(t, s)\| \omega(s, \gamma\alpha(s))ds$$
$$+ \int_t^\infty \|\Gamma_2(t, s)\| \omega(s, \gamma\alpha(s))ds$$
$$\leqq (\gamma/K)\int_c^t \|\Gamma_1(t, s)\| \beta(s)ds$$
$$+ (\gamma/K)\int_t^\infty \|\Gamma_2(t, s)\| \beta(s)ds$$
$$\leqq \gamma\alpha(t) .$$

Thus T maps $\mathscr{F}$ into $\mathscr{F}$.

Suppose that $\{f_n\}_{n=0}^\infty$ is an $\mathscr{F}$-valued sequence, and $\lim_{n\to\infty} f_n(t) = f_0(t)$ whenever t is in $[c, \infty)$, the convergence being uniform on compact intervals. If n is a nonnegative integer let $g_n = T[f_n]$. Let ε be a positive number. Let $\eta = \varepsilon/(2K)$, and find d in $[c, \infty)$ such that $2\omega(t, \gamma\alpha(t)) \leqq \eta\beta(t)$ whenever t is in $[d, \infty)$. Let $\tilde{n}$ be a positive integer such that if $n \geqq \tilde{n}$ then $|F(t, f_n(t)) - F(t, f_0(t))| \leqq \eta\beta(t)$ whenever t is in $[c, d]$. Let $n \geqq \tilde{n}$. Now, if t is in $[c, d]$,

$$|g_n(t) - g_0(t)| \leqq \int_c^t \|\Gamma_1(t, s)\| \circ |F(s, f_n(s)) - F(s, f_0(s))| ds$$
$$+ \int_t^d \|\Gamma_2(t, s)\| \circ |F(s, f_n(s)) - F(s, f_0(s))| ds$$
$$+ \int_d^\infty \|\Gamma_2(t, s)\| 2\omega(s, \gamma\alpha(s))ds$$
$$\leqq \eta K\alpha(t) + \eta K\alpha(t) = \varepsilon\alpha(t) .$$

Similarly,

$$|g_n(t) - g_0(t)| \leqq \varepsilon\alpha(t)$$

whenever $t \geqq d$. It is now clear that $\lim_{n\to\infty} g_n(t) = g_0(t)$ whenever t is in $[c, \infty)$, and the convergence is uniform on compact intervals.

If f is in $\mathscr{F}$, $g = T[f]$, and t is in $[c, \infty)$, then

$$g'(t) = \int_c^t \frac{\partial}{\partial t}\Gamma_1(t, s)F(s, f(s))ds - \int_t^\infty \frac{\partial}{\partial t}\Gamma_2(t, s)F(s, f(s))ds , \tag{6}$$

since, according to Lemma 2, $\Gamma_1(t, t) + \Gamma_2(t, t) = 0$ for all t. But (6) and our hypotheses tell us that the family $\{g' : g \text{ is in } T[\mathscr{F}]\}$ is uniformly bounded on each compact interval of $[c, \infty)$. Thus the family $T[\mathscr{F}]$ is uniformly equicontinuous on each compact interval of $[c, \infty)$.

We have now shown that the mapping T from $\mathscr{F}$ to $\mathscr{F}$ satisfies all the hypotheses of the variant of the Schauder—Tychonoff Theorem given on [3, p. 9]. Thus there is a member u of $\mathscr{F}$ such that $u = T[u]$. This u fulfills of the claims of Theorem 4, so the proof is complete.

IV. Higher order systems. In this section we summarize briefly how one would go about extending our work to a problem of the form

$$\frac{d^n}{dt^n}u(t) = f(t) + A(t)u(t) , \tag{7}$$

where n is a positive integer. Let the family $\{\Phi_1, \cdots, \Phi_n\}$ be determined according to

$$\frac{d^n}{dt^n}\Phi_j(t) = A(t)\Phi_j(t) ,$$

$$\frac{d^k}{dt^k}\Phi_j(0) = \begin{cases} I \text{ if } j = k+1 \\ 0 \text{ if } j \neq k+1 , \end{cases}$$

where we take a function to be its own 0^{th} derivative. Let the family $\{\Psi_1, \cdots, \Psi_n\}$ be determined according to

$$\frac{d^n}{dt^n}\Psi_j(t) = (-1)^n\Psi_j(t)A(t) ,$$

$$\frac{d^k}{dt^k}\Psi_j(0) = \begin{cases} (-1)^{j+1}I & \text{if } j = k+1 \\ 0 & \text{if } j \neq k+1 . \end{cases}$$

Let M be the linear subset of Y^n determined just as M was in the second-order case for Y^2. Let $\tilde{P}$ be the projection from Y^n to M, and find a family $\{P_{ij}\}_{i=1,\, j=1}^{n}$ in $\mathscr{L}[Y]$ so that P_{ij} is in the i^{th} row and j^{th} column of the natural matrix representation of $\tilde{P}$. Let the family $\{Q_{ij}\}_{i=1,\, j=1}^{n}$ be given by $Q_{ij} = -P_{ij}$ if $i \neq j$ and $Q_{ii} = I - P_{ii}$.

Now let Γ_1 and Γ_2 be given by

$$\Gamma_1(t, s) = \sum_{k=1}^{n}\sum_{j=1}^{n}\Phi_k(t)P_{kj}\Psi_{n+1-j}(s)$$

and

$$\Gamma_2(t, s) = \sum_{k=1}^{n}\sum_{j=1}^{n}\Phi_k(t)Q_{kj}\Psi_{n+1-j}(s) .$$

Now the following theorem follows almost exactly as did Theorem 1.

THEOREM 5. *These are equivalent*:

(i) *If f is in $\mathscr{B}_\beta\mathscr{C}$ then there is a solution u of* (7) *in $\mathscr{B}_\alpha\mathscr{C}$.*

(ii) *If k is an integer and $1 \leqq k \leqq n-1$ then*

$$\int_0^\infty \left\| \frac{\partial^k}{\partial t^k} \Gamma_2(t, s) \Big|_{t=0} \right\| \beta(s) ds$$

is finite, and there is a number K such that

$$\int_0^t \| \Gamma_1(t, s) \| \beta(s) ds + \int_t^\infty \| \Gamma_2(t, s) \| \beta(s) ds \leqq K\alpha(t)$$

whenever t is in R^+.

There is also a natural analogue to Theorem 4 in the n^{th} order case. We leave the construction of this analogue to the reader.

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