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RELATIVE CLASS NUMBERS OF NORMAL *CM*-FIELDS

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An algebraic number field K of finite degree is called a CM-field [3], if it is totally imaginary and it contains a totally real subfield K_0 such that $[K: K_0] = 2$. If we put $L_1(s) = \zeta_K(s)/\zeta_{K_0}(s)$, it is known that we have the formula

$$(1) \qquad L_1(1) = \frac{(2\pi)^{N/2} h_1 \sqrt{D_0}}{qw\sqrt{D}},$$

where N is the degree of K over the rationals, h_1 is the relative class number, $q = 1$ or 2, w is the number of the roots of unity in K, and D and D_0 are absolute values of discriminants of K and K_0 respectively. For normal CM-fields, we have proved in [4] that h_1 goes to infinity when $N/\log D$ goes to 0. In this paper we will obtain effectiveness in this theorem, i.e.,

THEOREM. *Let ε be any positive number and let H be any positive integer. Then there exists an effectively determined positive number $D(H, \varepsilon)$ such that $D < D(H, \varepsilon)$ for any non-abelian normal CM-field which satisfies $h_1 < H$ and*

$$\frac{N}{\log D} < \mathrm{Min}\left(\frac{1-\varepsilon}{72c_6}, \frac{1}{4c_3}\right).$$

In the above inequality, c_3 and c_6 are absolute constants which can be effectively determined.

The existence of a suitable subfield of K_0 enables us to obtain a lower estimate of $L_1(1)$, and techniques of the estimate are due to Landau [2]. We note that effectiveness theorem for abelian case has been given in [5] except two cases.

1. Lemmas. Let F be an algebraic number field of degree n. Let d be the absolute value of its discriminant. Let χ be a character of an ideal class group of F. Let $\mathfrak{f} = \mathfrak{f}(\chi)$ denote the conductor of χ. We put $k = k(\chi) = d \cdot N_F\mathfrak{f}$ where N_F means the absolute norm. Let $L(s, \chi)$ be the L-function with character χ. We put

(2) $$\begin{aligned}\Phi(s, \chi) &= \varphi(s, \chi)L(s, \chi)\\ &= \frac{2^{r_1}Rh}{w}\frac{\delta}{s(s-1)} + \Psi_1(s, \chi) + \Psi_2(1-s, \chi)\end{aligned}$$

as usual, where

$$\varphi(s, \chi) = (2^{-r_2}\pi^{-n/2}\sqrt{k})^s \Gamma\left(\frac{s+1}{2}\right)^{\nu}\Gamma\left(\frac{s}{2}\right)^{r_1-\nu}\Gamma(s)^{r_2}$$

and $\delta = 1$ for $\chi = \chi_0 =$ principal character and $\delta = 0$ otherwise (See [1] and [2]). In the following we consider s as a real variable. We put $\Phi_0(s, \chi) = \varphi(s, \chi)\zeta_F(s)$. If $s \leqq s_1$ and if $s_1 > 1$, it is seen as in [2] that

(3) $$|\Psi_i(s, \chi)| \leqq \Phi_0(s_1, \chi) .$$

In the following lemmas, constants $c_1, c_2, \cdots$ mean absolute constants which can be effectively determined.

LEMMA 1. *Let κ be the residue of $\zeta_F(s)$ at $s = 1$. Then it holds*

$$\kappa < c_1^n d^{1/12} .$$

PROOF. In the formula (2) for $\chi = \chi_0$, $\Psi_i(s, \chi_0)$ take positive real values. Hence

$$\begin{aligned}\frac{36}{7}\frac{2^{r_1}Rh}{w} < \Phi\left(\frac{7}{6}, \chi_0\right) &< d^{7/12}\zeta_F\left(\frac{7}{6}\right)\\ &< d^{7/12}\zeta\left(\frac{7}{6}\right)^n < 7^n d^{7/12} .\end{aligned}$$

Therefore

$$\kappa = \frac{2^{r_1+r_2}\pi^{r_2}Rh}{w\sqrt{d}} < c_1^n d^{1/12} .$$

LEMMA 2. *If $1 < s < 7/6$,*

$$|L(s, \chi)| < \left(\frac{\delta}{s-1} + 1\right)c_2^n k^{1/12} .$$

PROOF. Above remark (3) shows

$$|\Psi_1(s, \chi)| \leqq \Phi_0\left(\frac{7}{6}, \chi\right) < 7^n k^{7/12}$$

and

$$|\Psi_2(1-s, \chi)| < 7^n k^{7/12}$$

as in the proof of Lemma 1. Then

$$\begin{aligned}|L(s,\chi)| &= |\Phi(s,\chi)|/|\varphi(s,\chi)| \\ &< \left(\frac{2^{r_1}Rh}{w}\frac{\delta}{s(s-1)} + 2\cdot 7^n k^{7/12}\right)\Big/ c_{21}^n\sqrt{k} \\ &< \frac{\kappa\delta c_{22}^n}{s(s-1)} + c_{23}^n k^{1/12} \\ &< \left(\frac{\delta}{s-1}+1\right)c_2^n k^{1/12} .\end{aligned}$$

LEMMA 3. *Let* $1 < s < 7/6$. *Then it holds*

$$\left|\sum\Re\frac{1}{s-\rho} - \Re\frac{L'}{L}(s,\chi) - \frac{\delta}{s-1} - \frac{1}{2}\log k\right| < nc_3 ,$$

where ρ *runs over the zeros of* $L(s,\chi)$ *such that* $0 < \Re\rho < 1$.

PROOF. As $\Phi(s,\chi) \neq 0$ for any s in this interval, it is seen from the equality (2) that

$$\left|\frac{\Phi'}{\Phi}(s,\chi) - \frac{L'}{L}(s,\chi) - \frac{1}{2}\log k\right| < nc_{31} .$$

We have the assertion by taking real part and substituting Landau's equation [2, (7)]

$$\Re\frac{\Phi'}{\Phi}(s,\chi) = -\frac{\delta}{s-1} - \frac{\delta}{s} + \sum\Re\frac{1}{s-\rho} .$$

2. In this section we will prove our theorem.

LEMMA 4. *Let* K *be a non-abelian normal CM-field, and let* K_0 *be its maximal real subfield. Then there exists a subfield* F *of* K_0 *satisfying one of the following conditions*:

(i) K/F *is a cyclic extension of degree* 4.

(ii) K/F *is a cyclic extension of degree* $2p$ *for some odd prime number* p.

(iii) K/F *is an abelian extension of degree* 8.

PROOF. Let G be the Galois group of K over the rationals. Let G_0 be the subgroup corresponding to K_0. Then G_0 is a central subgroup of order 2. If the order of G is not a power of 2, there exists a subgroup H of order p. Then the subfield corresponding to G_0H satisfies (ii). If the order of G is a power of 2, there exists an element x of order 4 because G is not abelian. Let H be a subgroup generated by x. If H includes G_0, the corresponding subfield satisfies (i). If it is not the case, G_0H is an abelian subgroup of order 8. Then the corresponding subfield satisfies (iii).

Let F be a subfield satisfying one of conditions in Lemma 4. As K is an abelian extension of F, $\zeta_K(s)$ is a product of L-functions over F, i.e.,

$$\begin{aligned}\zeta_K(s) &= L_1(s)\zeta_{K_0}(s)\\ &= L_1(s)L_2(s)\zeta_F(s)\\ &= \textstyle\prod_1 L(s,\chi)\prod_2 L(s,\chi)\cdot\zeta_F(s)\,.\end{aligned}$$

In the above, $\prod_1$ and $\prod_2$ mean products over χ corresponding to L_1 and L_2, respectively. Let $m=[K\colon F]$, i.e., $N=mn$. Hasse's conductor-discriminant formula shows

$$D = N_F\Big(\prod_\chi \mathfrak{f}(\chi)\Big)d^m = \prod_\chi k(\chi)\,.$$

Taking logarithm,

LEMMA 5. *Let* $\sum_1$ *and* $\sum_2$ *denote sums over* χ *corresponding to* L_1 *and* L_2 *respectively. Then*

$$\log D = \textstyle\sum_1 \log k(\chi) + \sum_2 \log k(\chi) + \log d$$

and

$$\log D_0 = \textstyle\sum_2 \log k(\chi) + \log d$$

hold.

Let $s_0 = 1 + (\log D)^{-1}$. We may assume $s_0 < 7/6$. We will find an upper estimate of $\sum \Re(1/(s_0-\rho))$, where the sum is taken over all zeros of $L_1(s)$ such that $0<\Re\rho<1$. As $\zeta_K(s)$ is decreasing for $s>1$,

$$\frac{L_1'}{L_1}(s_0) + \frac{L_2'}{L_2}(s_0) + \frac{\zeta_F'}{\zeta_F}(s_0) \leqq 0\,.$$

Then Lemmas 3 and 5 show the following inequalities:

$$\begin{aligned}\sum\Re\frac{1}{s_0-\rho} &< \frac{L_1'}{L_1}(s_0) + \frac{1}{2}\textstyle\sum_1\log k(\chi) + \frac{N}{2}c_3\\ &< -\frac{L_2'}{L_2}(s_0) - \frac{\zeta_F'}{\zeta_F}(s_0) + \frac{1}{2}\textstyle\sum_1\log k(\chi) + \frac{Nc_3}{2}\\ &< \frac{1}{2}\textstyle\sum_2\log k(\chi) + \frac{1}{2}\log d + \frac{1}{s_0-1} + \frac{1}{2}\sum_1\log k(\chi) + Nc_3\\ &= \frac{3}{2}\log D + Nc_3\,.\end{aligned}$$

Now we assume that $4Nc_3 < \log D$. Then it holds $\sum\Re(1/(s_0-\rho)) < (7/4)\log D$.

We will write $L_1(s)$ as a product of $L_3(s)$ and $L_4(s)$. Let $L_3(s) = L_1(s)$ and $L_4(s) = 1$, if $\Re(1/(s_0 - \rho)) \leqq (7/8) \log D$ for every ρ. If there exists a ρ not satisfying this inequality, we take $L_4(s)$ to be the corresponding L-function $L(s, \chi_4)$ and $L_3(s)$ the product of other L-functions. If $L(\rho, \chi) = 0$, it holds $L(\bar{\rho}, \bar{\chi}) = 0$. If $\chi \neq \bar{\chi}$, it follows ρ and $\bar{\rho}$ are zeros of $L_1(s)$ (a multiple zero if ρ is real). As $\Re(1/(s_0 - \rho)) = \Re(1/(s_0 - \bar{\rho}))$, it holds $\Re(1/(s_0 - \rho)) \leqq (7/8) \log D$ for such ρ. Hence it should be $\chi_4^2 = \chi_0$, and $L_4(s)$ is an L-function corresponding to an imaginary quadratic extension F_1 of F. As $K = K_0F_1$, F_1 is totally imaginary, i.e., F_1 is a *CM*-field. We note that $L_4(s) = 1$ if F satisfies condition (i) of Lemma 4. Hence $[K: F_1] \geqq 3$ and $k(\chi_4) \leqq D^{1/3}$ hold. When ρ runs over the zeros of $L_3(s)$ such that $0 < \Re\rho < 1$,

$$\sum_3 \Re \frac{1}{s_0 - \rho} < \frac{7}{4} \log D$$

and

$$\Re \frac{1}{s_0 - \rho} < \frac{7}{8} \log D$$

hold. Hence it holds

$$\sum_3 \Re \frac{1}{s - \rho} < \frac{14 \log D}{8 + 7(s - s_0) \log D}$$

for every s such that $1 \leqq s \leqq s_0$ [5, p. 342]. If we put

$$s = 1 + \frac{x}{\log D}\,, \qquad 0 \leqq x \leqq 1\,,$$

Lemma 3 shows

$$\begin{aligned} \frac{L_3'}{L_3}(s) &< \sum_3 \Re \frac{1}{s - \rho} + \frac{Nc_3}{2} \\ &< \frac{14 \log D}{7x + 1} + \frac{Nc_3}{2} \end{aligned}$$

and

$$\int_1^{s_0} \frac{L_3'}{L_3}(s)ds < \int_0^1 \frac{14\, dx}{7x + 1} + \frac{Nc_3}{2 \log D} < \log c_4\,.$$

Therefore

$$\begin{aligned} -\log L_3(1) &= -\log L_3(s_0) + \int_1^{s_0} \frac{L_3'}{L_3}(s)ds \\ &< \log \zeta_F(s_0) + \log L_2(s_0) + \log L_4(s_0) + \log c_4 \end{aligned}$$

and

$$L_3(1)^{-1} < c_4(\log D + 1)c_2^N D_0^{1/12} k(\chi_4)^{1/12}$$
$$< c_5^N D^{5/72} \log D$$

hold. As F_1 is a CM-field, (1) shows

$$L_4(1) \geqq \frac{(2\pi)^n}{qw\sqrt{k\,(\chi_4)}} \geqq \frac{1}{\sqrt{k\,(\chi_4)}} \geqq D^{-1/6}\,.$$

Hence

$$L_1(1)^{-1} = L_3(1)^{-1} L_4(1)^{-1} < c_5^N D^{17/72} \log D\,.$$

Therefore the formula (1) shows

$$h_1 > \frac{D^{1/4}}{(2\pi)^{N/2}} L_1(1) > \frac{D^{1/72}}{e^{c_6 N} \log D}\,.$$

This proves the theorem.

COROLLARY. *Let K be a non-abelian CM-field of degree 4. Let H be any positive integer, and let h_1 be the relative class number of K. If $h_1 < H$, the discriminant of K is smaller than some effectively determined value $D(H)$.*

PROOF. K is contained in the normal CM-field E of degree 8. Let K_0 and E_0 be maximal real subfields of K and E, respectively. Let K' be the conjugate of K. As E/K_0 is abelian, it holds

$$L_{1,E}(s) = L_{1,K}(s) L_{1,K'}(s) = L_{1,K}(s)^2\,.$$

Then the formula (1) gives

$$h_{1,K}^2 = \frac{2q_K^2}{q_E} h_{1,E} \sqrt{\frac{d_{E_0}}{d_E} \cdot \frac{d_K}{d_{K_0}}}\,,$$

where d_K, d_E, $\cdots$ are absolute values of discriminants of K, E, $\cdots$. Hasse's conductor-discriminant formula gives

$$d_E = N_{K_0} d_{K/K_0}^2 N_{K_0} d_{E_0/K_0} d_{K_0}^4$$
$$= N_{K_0} d_{K/K_0}^2 d_{K_0}^2 d_{E_0}$$
$$= d_K^2 d_{E_0} / d_{K_0}^2\,.$$

Hence it holds

$$h_{1,K}^2 = \frac{2q_K^2}{q_E} h_{1,E}\,.$$

Therefore our theorem shows the assertion.

REMARK. The case (i) of Lemma 4 occurs for E, and the corresponding subfield is a quadratic field. This facts enable us to obtain much better estimate than the general case.

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