Tôhoku Math. Journ.
25 (1973), 339-345.

ON THE BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS OF TYPE ρ, δ WITH $0 \leqq \rho = \delta < 1$

Dedicated to Professor Masanori Fukamiya on his 60th birthday

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(Received May 15, 1973)

1. This article presents the proof of the following result;

THEOREM. *Let $0 \leqq \rho = \delta < 1$ and let $P = p(X, D_x, X', D_{x'}, X'')$ be a pseudo-differential operator whose symbol belongs to $S^{0,0}_{\rho,\delta}$. Then P can be extended to a bounded map in L^2.*

When $0 \leqq \delta < \rho \leqq 1$ similar results are obtained by Hörmander [4] [5] and Kumano-go [6] and when $\rho = \delta = 0$, by Calderón and Vaillancourt [1]. In [2] they proved also the boundedness of $p(X, D_x, X') \in S^0_{\rho,\delta}$ in L^2 provided that $0 \leqq \rho = \delta < 1$ and that its symbol $p(x, \xi, x')$ has compact support in ξ. On the other hand, Hörmander proved in [5] that this need not be true if $0 < \rho < \delta < 1$, and Chin-Hung Ching, in [3], proved the result also fails if $\rho = \delta = 1$.

In the proof of our Theorem we shall use the result in [2] and the simplification theorem and the expansion formulas proved in [6].

2. Let $\mathscr{S}$ be the space of C^∞-functions defind in R^n whose derivatives decrease faster than any power of $|x|$ as $|x| \to \infty$. For $u \in \mathscr{S}$ we define the Fourier transform $\hat{u}(\xi)$ by

$$\hat{u}(\xi) = \int e^{i\langle x, \xi\rangle} u(x) dx \qquad \langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$$

and by H_s, $-\infty < s < \infty$, we denote the completion of $\mathscr{S}$ in the norm

$$||u||_s^2 = \int \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 d\!\!\bar{}\,\xi, \langle\xi\rangle = (1 + |\xi|^2)^{1/2}, d\!\!\bar{}\,\xi = (2\pi)^{-n} d\xi \,.$$

We shall use notation

$$\partial_{x_j} = \partial/\partial x_j, D_{x_j} = -i\partial_{x_j}; \partial_{\xi_j} = \partial/\partial\xi_j, D_{\xi_j} = -i\partial_{\xi_j}, \qquad j = 1, 2, \cdots, n \,.$$
$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \partial_{x,x',x''}^{\alpha,\alpha',\alpha''} \partial_{\xi,\xi'}^{\beta,\beta'} = \partial_x^\alpha \partial_{x'}^{\alpha'} \partial_{x''}^{\alpha''} \partial_\xi^\beta \partial_{\xi'}^{\beta'}$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

where x, x', x'', ξ, ξ' are points in R^n and $\alpha, \alpha', \alpha'', \beta, \beta'$ denote n-dimensional multi-indices of non-negative integers.

DEFINITION i). We say that a C^∞-function $p(x, \xi, x')$, defined in the whole (x, ξ, x')-space, belongs to $S^m_{\rho,\delta}$, $0 \leqq \rho \leqq 1$, $0 \leqq \delta < 1$, if and only if for any integer $j \geqq 0$

$$|p|^{(m)}_{(\rho,\delta),j} \equiv \operatorname*{Max}_{|\alpha+\alpha'+\beta|\leqq j} \operatorname*{Sup}_{(x,\xi,x')} \{|\partial^{\alpha,\alpha'}_{x,x'}\partial^\beta_\xi p(x, \xi, x')|\langle\xi\rangle^{-m-\delta|\alpha+\alpha'|+\rho|\beta|}\} < \infty$$

and define the corresponding operator $p(X, D_x, X')$ (we denote $p(X, D_x, X') \in S^m_{\rho,\delta}$) by

$$p(X, D_x, X')u(x) = \iint e^{i\langle x-x',\xi\rangle} p(x, \xi, x')u(x')dx'd\xi\,, \qquad u \in \mathcal{S}\,.$$

ii). We say that a C^∞-function $p(x, \xi, x', \xi', x'')$, defined in the whole (x, ξ, x', ξ', x'')-space, belongs to $S^{m,m'}_{\rho,\delta}$, $0 \leqq \rho \leqq 1$, $0 \leqq \delta < 1$, if and only if for any integer $j \geqq 0$

$$|p|^{(m,m')}_{(\rho,\delta),j} \equiv \operatorname*{Max}_{|\alpha+\alpha'+\alpha''+\beta+\beta'|\leqq j} \operatorname*{Sup}_{(x,\xi,x',\xi',x'')} \{|\partial^{\alpha,\alpha',\alpha''}_{x,x',x''}\partial^{\beta,\beta'}_{\xi,\xi'} p(x, \xi, x', \xi', x'')|$$
$$\cdot \langle\xi\rangle^{-m-\delta|\alpha|+\rho|\beta|}(\langle\xi\rangle + \langle\xi'\rangle)^{-\delta|\alpha'|}\langle\xi'\rangle^{-m'-\delta|\alpha''|+\rho|\beta'|}\} < \infty$$

and define the corresponding operator $P = p(X, D_x, X', D_{x'}, X'')$ (we denote $P \in S^{m,m'}_{\rho,\delta}$) by

$$Pu(x) = \iiiint e^{i\langle x-x',\xi\rangle + i\langle x'-x'',\xi'\rangle} p(x, \xi, x', \xi', x'')u(x'')dx''d\xi'dx'd\xi\,,$$
$$u \in \mathcal{S}\,.$$

We shall consider $S^m_{\rho,\delta}$ and $S^{m,m'}_{\rho,\delta}$ as the linear topological space with countable norms $|p|^{(m)}_{(\rho,\delta),j}$ and $|p|^{(m,m')}_{(\rho,\delta),j}$, $j = 0, 1, \cdots$, respectively.

iii). For $p(x, \xi, x', \xi', x'') \in S^{m,m'}_{\rho,\delta}$ we may define a new symbol $p_L(x, \xi, x')$ (we call it the left simplified symbol of $p(x, \xi, x', \xi', x'')$) by

$$p_L(x, \xi, x') = \iint e^{-i\langle\omega,\zeta\rangle}\langle\omega\rangle^{-n_0}\langle D_\zeta\rangle^{n_0} p(x, \xi+\zeta, x+\omega, \xi, x')d\omega d\zeta$$

where n_0 is an even integer $\geqq n+1$. By means of Theorem 1, 1 in [6], which is still true even if $\rho = 0$, $0 \leqq \delta < 1$, we can see that $p_L(X, D_x, X') = p(X, D_x, X', D_{x'}, X'')$ and $p_L(x, \xi, x') \in S^{m+m'+n\delta}_{\rho,\delta}$.

REMARK. By Theorem 1, 2 in [6] each operator $p(X, D_x) \in S^m_{\rho,\delta}$ is continuous map from $\mathcal{S}$ into itself and can be extended to a bounded map from $H_{s+m+n(\delta+1)+1}$ into H_s for any real s.

In particular

(2, 1) $\qquad \|p(X, D_x)u\|_0 \leqq C|p|^{(m)}_{(\rho,\delta),n+1}\|u\|_{m+n+1}$ for $u \in \mathcal{S}$

where C is a constant dependent on m and n but not on p. This remark means that each operator $p(X, D_x, X', D_{x'}, X'') \in S^{m,m'}_{\rho,\delta}$ is continuous map from $\mathcal{S}$ into itself and can be extended to a bounded map from $H_{s+m+m'+n(3\delta+1)+1}$ into H_s for any real s.

Then we have the following simplification theorem which is improved slightly on the Theorem 1, 1 in [6].

LEMMA 1. *Let* $0 \leqq \rho \leqq \delta < 1$ *and let* $p(x, \xi, x', \xi', x'') \in S^{m,m'}_{\rho,\delta}$. *Then its left simplified symbol* $p_L(x, \xi, x')$ *belongs to* $S^{m''}_{\rho,\delta}$, $m'' = m + m' + (\delta - \rho)n$ *and satisfies*

$$|p_L|^{(m'')}_{(\rho,\delta),j} \leqq C|p|^{(m,m')}_{(\rho,\delta),j+j'}, \qquad j = 0, 1, \cdots,$$

where $j' = \max(3n_0, 2n_0 + 2 + [(1-\delta)^{-1}(|m| + |m'| + \delta j + \rho n_0 + n + 1)])$ *and* C *is a constant dependent on* n, m, m', ρ, δ *and* j *but not on* p.

PROOF. For each indices α, α'', β set

$$J_{\alpha,\alpha'',\beta}(x, x', \omega, \xi, \zeta) = \langle\omega\rangle^{-n_0}\langle D_\zeta\rangle^{n_0}\partial^{\alpha,\alpha''}_{x,x'}\partial^{\beta}_{\xi}p(x, \xi+\zeta, x+\omega, \xi, x').$$

Then by integrating by parts we have

$$\begin{aligned}&\partial^{\alpha,\alpha''}_{x,x'}\partial^{\beta}_{\xi}p_L(x, \xi, x')\\ &= \sum_{j=1}^{3}\int_{A_j}\int e^{-i\langle\omega,\zeta\rangle}(1 + \langle\xi\rangle^{2\rho}|\omega|^2)^{-n_0/2}(1 + \langle\xi\rangle^{2\rho}(-\Delta_\zeta))^{n_0/2}J_{\alpha,\alpha'',\beta}d\omega d\zeta\\ &\equiv \sum_{j=1}^{3} I_j\end{aligned}$$

where $A_1 = \{\zeta; |\zeta| \leqq \langle\xi\rangle^{\delta}/2\}$, $A_2 = \{\zeta; \langle\xi\rangle^{\delta}/2 \leqq |\zeta| \leqq \langle\xi\rangle/2\}$, $A_3 = \{\zeta; \langle\xi\rangle/2 \leqq |\zeta|\}$ and $\Delta_\zeta = \sum_{j=1}^{n}\partial^2/\partial\zeta_j^2$. Since for some constant $C > 1$

$$\begin{aligned}&C^{-1}\langle\xi\rangle \leqq \langle\xi+\zeta\rangle \leqq C\langle\xi\rangle \quad \text{when} \quad |\zeta| \leqq \langle\xi\rangle/2\\ &\langle\xi+\zeta\rangle \leqq C\langle\zeta\rangle \qquad\qquad\quad \text{when} \quad |\zeta| \geqq \langle\xi\rangle/2\end{aligned}$$

we have

$$\begin{aligned}(2,2) \qquad &|\partial^{\alpha'}_{\omega}\partial^{\beta'}_{\zeta}J_{\alpha,\alpha'',\beta}(x, x', \omega, \xi, \zeta)|\\ &\leqq \text{const}\,|p|^{(m,m')}_{(\rho,\delta),|\alpha+\alpha'+\alpha''+\beta+\beta'|+n_0}\\ &\quad\cdot\begin{cases}\langle\xi\rangle^{m+m'+\delta|\alpha+\alpha'+\alpha''|-\rho|\beta+\beta'|} & \text{when } |\zeta| \leqq \langle\xi\rangle/2\\ \langle\zeta\rangle^{m_+ + m'_+ + \delta|\alpha+\alpha'+\alpha''|}\langle\omega\rangle^{-n_0} & \text{when } |\zeta| \geqq \langle\xi\rangle/2\end{cases}\end{aligned}$$

where $m_+ = \max(m, 0)$ and $m'_+ = \max(m', 0)$.

We shall estimate for each I_j. Since

$$I_1 = \int_{A_1}\int e^{-i\langle\omega,\zeta\rangle}(1 + \langle\xi\rangle^{2\rho}|\omega|^2)^{-n_0/2}(1 + \langle\xi\rangle^{2\rho}(-\Delta_\zeta))^{n_0/2}J_{\alpha,\alpha'',\beta}d\omega d\zeta$$

from (2,2) and $n_0 \geqq n+1$ we have

$$(2,3) \qquad |I_1| \leqq \text{const}\,|p|^{(m,m')}_{(\rho,\delta),|\alpha+\alpha''+\beta|+2n_0}\langle\xi\rangle^{m_0+(\delta-\rho)n}$$

where $m_0 = m + m' + \delta|\alpha + \alpha''| - \rho|\beta|$.

By integrating by parts we write

$$I_2 = \int_{A_2}\int e^{-i\langle\omega,\zeta\rangle}\langle\zeta\rangle^{-n_0}\langle D_\omega\rangle^{n_0}\{(1 + \langle\xi\rangle^{2\rho}|\omega|^2)^{-n_0/2} \cdot (1 + \langle\xi\rangle^{2\rho}(-\Delta_\zeta))^{n_0/2}J_{\alpha,\alpha'',\beta}\}d\omega d\zeta$$

so that from (2,4) and $\rho \leqq \delta$

$$(2,4)\quad |I_2| \leqq \text{const}\, |p|^{(m,m')}_{(\rho,\delta),|\alpha+\alpha''+\beta|+3n_0} \sum_{|\alpha_1'+\alpha_2'|\leqq n_0} \int_{|\zeta|\geqq\langle\xi\rangle^{\delta}/2} \langle\zeta\rangle^{-n_0}\langle\xi\rangle^{-\rho n+\rho|\alpha_1'|+\delta|\alpha_2'|+m_0}d\zeta$$
$$\leqq \text{const}\, |p|^{(m,m')}_{(\rho,\delta)|\alpha+\alpha''+\beta|+3n_0}\langle\xi\rangle^{m_0+(\delta-\rho)n}\,.$$

Let k be an integer such that

$$(2,5)\quad -2(1-\delta)k + m_+ + m'_+ + \delta|\alpha + \alpha''| + n + 1 + \rho n_0 \leqq m_- + m'_- - \rho|\beta|$$
$$m_- = \min(m, 0),\ m'_- = \text{mim}(m', 0)\,.$$

By integrating by parts we write

$$I_3 = \int_{A_3}\int e^{-i\langle\omega,\zeta\rangle}\langle\zeta\rangle^{-2k}\langle D_\omega\rangle^{2k}\{(1 + \langle\xi\rangle^{2\rho}|\omega|^2)^{-n_0/2} \cdot (1 + \langle\xi\rangle^{2\rho}(-\Delta_\zeta))^{n_0/2}J_{\alpha,\alpha'',\beta}\}d\omega d\zeta$$

so that from (2, 2) and (2, 5)

$$(2,6)\quad |I_3| \leqq \text{const}\, |p|^{(m,m')}_{(\rho,\delta),|\alpha+\alpha''+\beta|+2k+2n_0} \int_{A_3} \langle\zeta\rangle^{-2(1-\delta)k+\rho n_0+m_++m'_++\delta|\alpha+\alpha''|}d\zeta$$
$$\leqq \text{const}\, |p|^{(m,m')}_{(\rho,\delta),|\alpha+\alpha''+\beta|+2k+2n_0}\langle\xi\rangle^{m_0}\,.$$

Hence from (2, 3), (2, 4) and (2, 6) we have $p_L(x, \xi, x') \in S^{m''}_{\rho,\delta}$ and then completes the proof.

The following two Lemmas 2 and 3 are proved by Calderón and Vaillancourt [2] and by Kumano-go [6], respectively.

LEMMA 2. *Let $0 \leqq \rho = \delta < 1$ and let $p(x, \xi, x') \in S^0_{\rho,\delta}$. Suppose that $p(x, \xi, x')$ has compact support in ξ. Then the operator $p(X, D_x, X')$ can be extended to a bounded map in H_0 and its operator norm $\|p(X, D_x, X')\|$ satisfies*

$$\|p(X, D_x, X')\| \leqq C|p|^{(0)}_{(\rho,\delta),j_0}$$

where $j_0 = 4 + 2[n/2] + 2[5n/4(1-\delta)]$ and C is a constant dependent on δ and n but not on the support of p.

LEMMA 3. *Let $0 \leqq \rho \leqq 1$, $0 \leqq \delta < 1$, and let $p(x, \xi) \in S^m_{\rho,\delta}$, $q(x, \xi) \in S^{m'}_{1,0}$. Then the left simplified symbol $r(x, \xi)$ of $q(X, D_x)p(X', D_{x'})$ has the form*

$$r(x, \xi) = \sum_{|\alpha|<N} (-i)^{|\alpha|}/\alpha! \partial_\xi^\alpha q(x, \xi) \partial_x^\alpha p(x, \xi) + r_N(x, \xi)$$

$$r_N(x, \xi) \in S^{m''}_{\rho,\delta}, \quad m'' = m + m' + (\delta - 1)N, \qquad N = 1, 2, \cdots,$$

and satisfies

$$|r_N|^{(m'')}_{(\rho,\delta),j} \leqq C |q|^{(m')}_{(\rho,\delta),j+j'} |p|^{(m)}_{(\rho,\delta),j+j''}, \qquad j = 0, 1, \cdots,$$

where

$$j' = N + [n\delta] + 1 + n_0,$$
$$j'' = N + [n\delta] + 3 + [(1-\delta)^{-1}(|m| + |m'| + \delta[n\delta] + \delta N + \delta + j + n + 1)]$$

and C is a constant dependent on $n, m, m', \rho, \delta, N$ and j but not on p and q.

In [6] Kumano-go proved this result for the wider class when $0 \leqq \delta < 1, 0 < \rho \leqq 1$ but in this proof there is no difficulty even if $\rho = 0, 0 \leqq \delta < 1$.

3. By the simplification theorem in [6] and Lemma 1, in order to prove Theorem, it is enough to show the following;

THEOREM. *Let $0 \leqq \rho = \delta < 1$ and $p(x, \xi) \in S^0_{\rho,\delta}$. Then the map $p(X, D_x)$ can be extended to a bounded map from H_0 into itself and satisfies for $u \in \mathscr{S}$*

$$\|p(X, D_x)u\|_0 \leqq C_1 |p|^{(0)}_{(\rho,\delta),j_0} \|u\|_0 + C_2 |p|^{(0)}_{(\rho,\delta),j} \|u\|_{\delta-1}$$

where j_0 is the integer given in Lemma 2, j is some large integer and C_k, $k = 1, 2$, are constants dependent on δ and n but not on p.

COROLLARY. *Let $0 \leqq \rho = \delta < 1$ and $p(x, \xi, x', \xi', x'') \in S^{m;m'}_{\rho,\delta}$. Then the map $p(X, D_x, X', D_{x'}, X'')$ can be extended to a bounded map from $H_{m+m'+s}$ into H_s for any real s.*

PROOF OF THEOREM. At first we note that there is a partition $\{Q_j; j = 1, 2, \cdots, \}$ of R^n into closed cubes such that (i) for some constant $C > 1$,

$$C^{-1}\langle\xi\rangle \leqq \operatorname{diam}(Q_j) \leqq C\langle\xi\rangle \quad \text{for} \quad \xi \in Q_j^* \qquad j = 1, 2, \cdots,$$

(ii) there is a bound on the number of overlaps of Q_j^*. Here we denote by Q_j^* the double of Q_j. For example such partition can be constructed as follows. Let $a_\nu = (3/2)^\nu$, $\nu = 1, 2, \cdots$. Set $Q_1 = \{\xi; \max|\xi_k| \leqq a_1\}$. Suppose that $\{\xi; \max|\xi_k| \leqq a_\nu\}$ has the partition $\{Q_1, Q_{\lambda,j}; \lambda \leqq \nu, j = 1, 2, \cdots, 6^n - 4^n\}$. Then we may define the partition $\{Q_{\nu+1,j}; j = 1, 2, \cdots, 6^n - 4^n\}$ of $\bigcup_{r=1}^n \{\xi; a_\nu \leqq |\xi_r| \leqq a_{\nu+1}, |\xi_s| \leqq a_{\nu+1}, s \neq r\}$ by

$$\{\xi;\, a_\nu \leqq \xi_r \leqq a_{\nu+1},\, l_s a_\nu/2 \leqq \xi_s \leqq (1+l_s)a_\nu/2,\, s \neq r\}$$
$$\{\xi;\, -a_{\nu+1} \leqq \xi_r \leqq -a_\nu,\, l_s a_\nu/2 \leqq \xi_s \leqq (1+l_s)a_\nu/2,\, s \neq r\}$$
$$r, s = 1, 2, \cdots, n\,, \qquad l_s = -3, -2, \cdots, 2\,.$$

Then $\{Q_j\}$ is given by $\{Q_1, Q_{\nu,j};\, \nu = 2, 3, \cdots, j = 1, 2, \cdots, 6^n - 4^n\}$. It is easy to see that $\{Q_j\}$ has the properties (i) and (ii). Take $\phi \in C^\infty(R^n)$ such that $0 \leqq \phi \leqq 1$, $\phi(\xi) = 1$ on $\max|\xi_k| \leqq 1$ and the support of ϕ is contained in $\max|\xi_k| \leqq 3/2$. Set

$$\phi_j(\xi) = \phi((\xi - \xi_{(j)})/d_j),\ \psi_j(\xi) = \phi_j(\xi)/(\Sigma_{j'}\phi_{j'}(\xi)^2)^{1/2}\,.$$

Here $\xi_{(j)}$ is the center of Q_j and $d_j = \mathrm{diam}\,(Q_j)/2\sqrt{n}$. Then we have, from property (i) of $\{Q_j\}$, $\{\psi_j(\xi)\}$ is bounded in $S^0_{1,0}$ and

$$\limsup_{N\to\infty}\Big\|\sum_{j=1}^N \psi_j(D_x)u\Big\|_0 \geqq \|u\|_0 \quad \text{for} \quad u \in \mathscr{S}\,. \tag{3,1}$$

For

$$\Big\|\sum_{j=1}^N \psi_j(D_x)u\Big\|_0^2 = \int\Big(\sum_{j=1}^N \psi_j(\xi)\Big)^2|\hat{u}(\xi)|^2 d\xi$$
$$\geqq \int\sum_{j=1}^N \psi_j(\xi)^2|\hat{u}(\xi)|^2 d\xi\,.$$

Let $p_j(x, \xi) = p(x, \xi)\psi_i(\xi)$ and let $q_j(x, \xi)$ be the left simplified symbol of $\psi_j(D_x)p(X', D_{x'})$. From Lemma 3, we write

$$q_j(x, \xi) = p_j(x, \xi) + \sum_{1\leqq|\alpha|<k}(-i)^{|\alpha|}/\alpha!\, r_{j,\alpha}(x, \xi) + r_{j,k}(x, \xi)$$
$$r_{j,\alpha}(x, \xi) = \partial_\xi^\alpha\psi_j(\xi)\partial_x^\alpha p(x, \xi) \in S_{\rho,\delta}^{(\delta-1)|\alpha|},\ r_{j,k}(x, \xi) \in S_{\rho,\delta}^{(\delta-1)k}\,.$$

From properties (i) and (ii) of $\{Q_j\}$ and $\mathrm{supp}\,\psi_j \subset Q_j^*$, for each $k \geqq 2$ $\{\sum_{j=1}^N p_j(x, \xi);\, N \geqq 1\}$, $\{\sum_{j=1}^N r_{j,\alpha}(x, \xi)\langle\xi\rangle^{(1-\delta)|\alpha|};\, N \geqq 1\}$ and $\{\sum_{j=1}^N r_{j,k}(x, \xi);\, N \geqq 1\}$ are bounded in $S^0_{\rho,\delta}$, $S^0_{\rho,\delta}$ and $S_{\rho,\delta}^{(\delta-1)k}$, respectively. So that from Lemma 2

$$\Big\|\sum_{j=1}^N p_j(X, D_x)u\Big\|_0 \leqq \mathrm{const}\,|p|^{(0)}_{(\rho,\delta),j_0}\|u\|_0\,, \qquad N = 1, 2, \cdots, \tag{3,2}$$

$$\Big\|\sum_{j=1}^N r_{j,\alpha}(X, D_x)u\Big\|_0 \leqq \mathrm{const}\,|p|^{(0)}_{(\rho,\delta),j_0+|\alpha|}\|u\|_{\delta-1}\,, \qquad N = 1, 2, \cdots, \quad 1 \leqq |\alpha| < k \tag{3,3}$$

for $u \in \mathscr{S}$. On the other hand, if we take k so large that

$$(\delta - 1)k + n + 1 \leqq \delta - 1$$

from the inequality (2,1) and Lemma 3 we have for $u \in \mathscr{S}$

$$\Big\|\sum_{j=1}^N r_{j,k}(X, D_x)u\Big\|_0 \leqq \mathrm{const}\,|p|^{(0)}_{(\rho,\delta),j}\|u\|_{\delta-1}\,, \qquad N = 1, 2, \cdots, \tag{3,4}$$

for some large j. Hence from (3, 1), (3, 2), (3, 3) and (3, 4)

$$\|p(X, D_x)u\|_0 \leqq \limsup \left\| \sum_{j=1}^{N} \psi_j(D_x)p(X', D_{x'})u \right\|_0$$
$$\leqq \text{const}\, |p|^{(0)}_{(\rho,\delta),j_0} \|u\|_0 + \text{const}\, |p|^{(0)}_{(\rho,\delta),j} \|u\|_{\delta-1} \,.$$

This completes the proof.

PROOF OF COROLLARY. Let $p_1(x, \xi)$ be the left simplified symbol of the left simplified symbol $p_L(x, \xi, x')$ of $p(x, \xi, x', \xi', x'')$. By Lemma 1 we have $p_1(x, \xi) \in S^{m+m'}_{\rho,\delta}$. Let $p_2(x, \xi)$ be the left simplified symbol of $\langle D_x \rangle^s p_1(X', D_{x'})$. Then by Lemma 3 we can write

$$p_2(x, \xi) = p_3(x, \xi)\langle \xi \rangle^{m+m'+s} + p_4(x, \xi)\langle \xi \rangle^{m+m'+s+\delta-1}$$
$$p_k(x, \xi) \in S^0_{\rho,\delta} \,, \qquad k = 3, 4 \,.$$

So that the result follows from Theorem.

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