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THE GROWTH OF THE RESOLVENT AND HYPERINVARIANT SUBSPACES

Dedicated to Professor Masanori Fukamiya on his 60th birthday

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1. Introduction. Let $\mathfrak{X}$ (respectively, $\mathfrak{H}$) be a separable, infinite dimensional complex Banach (respectively, Hilbert) space. We denote by $\mathfrak{B}(\mathfrak{X})$ the algebra of all bounded linear operators acting in $\mathfrak{X}$. Let T be a bounded linear operator on $\mathfrak{X}$. We call the bi-commutant of T the set of operators commuting with all operators which commute with T. A closed subspace $\mathfrak{Y}$ of $\mathfrak{X}$ is said to be hyperinvariant for T if $S\mathfrak{Y} \subset \mathfrak{Y}$ for any operator S that commutes with T.

The purpose of the present note is to show the existence of hyperinvariant subspaces for an operator T with a spectrum condition and with a growth condition on the resolvent of T. The result will be considered as a generalization of those results on invariant subspaces in [2], [5]. In the case of a Hilbert space, such an operator, $T = A + B$, where A is a normal operator with some spectrum condition and B is a compact operator with some condition, has been studied in [2], [5].

Throughout this note, we denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$ and $\rho(T)$ the spectrum, the point spectrum, the continuous spectrum, the residual spectrum and the resolvent set of T, respectively (see Chap. VII of Part I of [3]). For any complex number λ and a subset E of the complex plane, we put $d(\lambda, E) = \inf_{z\in E}|\lambda - z|$. We understand a smooth arc to be such a one that has a continuous second derivative when parametrized with respect to arc length. We assume a Jordan curve J is positively oriented and for a fixed λ_0 on J, where J has a parametrization $\lambda = g(s)$ $(0 \leqq s \leqq l(J))$, in terms of arc length s from λ_0, $g(0) = \lambda_0$, $g(s) = g(s + l(J))$, and $g(s)$ is continuous on J and $g'(s)$, $g''(s)$ are continuous except points $\lambda_k = g(s_k)$, $s_k < s_{k+1}$, $k = 1, 2, \cdots, n$ on J, where $l(J)$ denotes the whole length of J.

2. The existence of hyperinvariant subspaces for some operators. We denote by J, a Jordan curve, which consists of a finite number of rectifiable smooth arcs in the complex plane. The set of all operators $T \in \mathfrak{B}(\mathfrak{X})$ such that $\sigma(T) \subset J$ will be denoted by $\mathfrak{B}(\mathfrak{X}; J)$, (it may well be the case that the spectrum separates the complex plane).

Let $T \in \mathfrak{B}(\mathfrak{X}; J)$.

DEFINITION 2.1. We say that the resolvent of T has the growth condition (C) near J if

$$\int_0^\varepsilon \log \log M(\delta) d\delta < \infty$$

for some sufficiently small $\varepsilon > 0$, where $M(\delta) = \sup_{d(\lambda, J) \geqq \delta} \|(\lambda I - T)^{-1}\|$.

In [2], the resolvent is said to have polar growth of order n (respectively, exponential growth of order p) near J if $d(\lambda, J)^n \|(\lambda I - T)^{-1}\|$ (respectively, $d(\lambda, J)^p \log \|(\lambda I - T)^{-1}\|$), $\lambda \notin J$ is bounded near J. If the resolvent has exponential growth of order p (for any $p > 0$) near J, then it has the growth condition (C) near J. (Obviously, the resolvent has polar growth of order n near J, then it has exponential growth of order p for any $p > 0$). This growth condition in Definition 2.1 plays an important role in the future. In this section we have the following result.

THEOREM 2.2. *Let $T \in \mathfrak{B}(\mathfrak{X}; J)$ such that the resolvent of T has the growth condition (C) near J. If $\sigma(T)$ is not reduced to a single point, then T has a proper closed hyperinvariant subspace.*

We shall divide the proof into portions. The assertion follows immediately in the case that $\sigma(T)$ is disconnected. In fact, if $\sigma(T)$ is disconnected, let σ_0 be a proper open and closed subset of its spectrum $\sigma(T)$ and let Γ_0 be a contour enclosing σ_0 but no other point of $\sigma(T) - \sigma_0$ and lying in $\rho(T)$, then the spectral projection $E(\sigma_0)$ corresponding to σ_0,

$$E(\sigma_0) = \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda I - T)^{-1} d\lambda$$

is an idempotent distinct from 0 and I which commutes with every operator commuting with T and $E(\sigma_0)$ would define a proper closed hyperinvariant subspace $E(\sigma_0)\mathfrak{X}$ of $\mathfrak{X}$ with respect to T (see Chap. XI of [11]). Therefore we have only to show that there can exist no operator $T \in \mathfrak{B}(\mathfrak{X}; J)$ such that the resolvent of T has the growth condition (C) near J and $\sigma(T)$ is connected subarc of J and $\mathfrak{X}$ admits no proper closed hyperinvariant subspace under T.

Now we need the following generalization of Levinson's theorem [7, Lemma 27.1, p. 135].

LEMMA 2.3. (Ljubič and Macaev [8]). *Suppose that $M(\delta)$ is non-increasing function $(\delta > 0)$ satisfying the following conditions, $M(\delta) \to \infty$ as $\delta \to 0$ and*

$$\int_0^\varepsilon \log \log M(\delta) d\delta < \infty$$

for some sufficiently small $\varepsilon > 0$, *and suppose that* J_0 *is a smooth arc in the complex plane, beginning at some point* z_0. *Then there exists a function analytic except for* $z = z_0$ *satisfying the following conditions*:

(i) $|f(z)| \leqq 1/M(\delta)$ *along some beak* Γ (Fig. 1) *formed by two smooth*

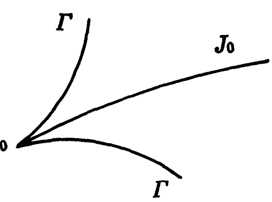

Fig. 1

arcs tangent to J_0 *on opposite sides at the point* z_0, *where* $\delta = d(z, J_0)$;

(ii) $f(z)$ *is continuous on the inside and on the beak* Γ;

(iii) $f(z) \neq 0$ (*for* $z \neq \infty$), $f(\infty) = 0$.

Using Lemma 2.3 we have the followings. For any point $\mu \in J$, which is not a singular point of J, we denote $f_\mu^+(\lambda)$ and $f_\mu^-(\lambda)$ analytic functions whose existence is assured by Lemma 2.3, constructed with respect to the majorant $M(\delta)$ and curves J_μ^+ and J_μ^- respectively (J_μ^+ and J_μ^- are the Jordan curve J issuing from the point $\mu \in J$ respectively in positive and negative directions). We denote corresponding beaks by Γ_μ^+ and Γ_μ^- respectively. Let $\Delta = [\alpha, \beta]$ be any segment of the curve J, where α, β are not singular points of J (α precedes β in a positive direction along J). Put

$$F_\Delta(\lambda) = f_\alpha^+(\lambda) f_\beta^-(\lambda)$$

and denote by Γ_Δ the contour obtained by joining Γ_α^+ and Γ_β^- by means of any two simple curves, which consists of a finite number of rectifiable smooth arcs (Fig. 2). The function $F_\Delta(\lambda)$ is evidently analytic on the

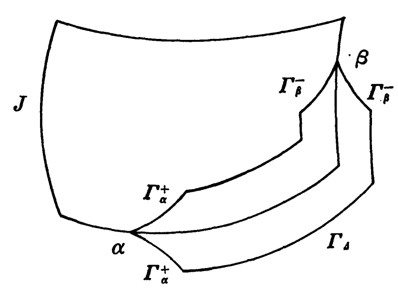

Fig. 2

inside of Γ_Δ, continuous on the inside and on Γ_Δ and does not vanish on the inside of Γ_Δ. From the construction following estimates hold

$$F_\Delta(\lambda) = O(1/M(d(\lambda, J))) \quad \text{for} \quad \lambda \longrightarrow \alpha, \lambda \in \Gamma_\Delta ,$$
$$F_\Delta(\lambda) = O(1/M(d(\lambda, J))) \quad \text{for} \quad \lambda \longrightarrow \beta, \lambda \in \Gamma_\Delta ,$$

where $O(\cdot)$ is Landau's symbol.

LEMMA 2.4. *Let $T \in \mathfrak{B}(\mathfrak{X}; J)$ such that $\sigma(T)$ is a connected set (not a single point) and the resolvent of T has the growth condition (C) near J. For each pair of points α, β where α precedes β in a positive direction along J and they are not singular points of J, we put*

$$\mathfrak{S}(\alpha, \beta) = \{x; x \in \mathfrak{X}, (\lambda I - T)^{-1}x \textit{ is continuable to a function which is analytic on } \rho(T) \cup \{\textit{the arc } (\alpha, \beta)\}\} ,$$
$$\tilde{\mathfrak{S}}(\alpha, \beta) = \{x; x \in \mathfrak{X}, (\lambda I - T)^{-1}x \textit{ is continuable to a function which is analytic on } \rho(T) \cup \{\textit{the arc } (\beta, \alpha)\}\} .$$

Then both $\mathfrak{S}(\alpha, \beta)$ and $\tilde{\mathfrak{S}}(\alpha, \beta)$ are closed subspaces of $\mathfrak{X}$, invariant under T.

PROOF. Because both of the invariance and the linearity of $\mathfrak{S}(\alpha, \beta)$ are plain, we have only to prove that $\mathfrak{S}(\alpha, \beta)$ is closed. Let $x_n \in \mathfrak{S}(\alpha, \beta)$, $(n = 1, 2, 3, \cdots)$ and $x_n \to x$ as $n \to \infty$. Let $R(\lambda, T: x_n)$ denote the analytic continuation of $(\lambda I - T)^{-1}x_n$, then for every positive number ε, let $\Delta = [\alpha + \varepsilon, \beta - \varepsilon]$ in above argument. Then $R(\lambda, T: x_n)$ are analytic on the inside of Γ_Δ. Here we define the function such that

$$g_n(\lambda) = \begin{cases} F_\Delta(\lambda)(\lambda - \alpha - \varepsilon)(\lambda - \beta + \varepsilon)R(\lambda, T: x_n) \\ \qquad \text{if} \quad \lambda \neq \alpha + \varepsilon \quad \text{and} \quad \lambda \neq \beta - \varepsilon , \\ 0 \qquad \text{if} \quad \lambda = \alpha + \varepsilon \quad \text{or} \quad \lambda = \beta - \varepsilon , \end{cases}$$

then $g_n(\lambda)$ are analytic on the inside of Γ_Δ and strongly continuous on Γ_Δ, which follows from estimates of $F_\Delta(\lambda)$. By the maximum modulus principle, $\{g_n(\lambda)\}$ is a uniform Cauchy sequence with respect to λ, hence the limit function $g_0(\lambda)$ is analytic on the inside of Γ_Δ and so

$$x_\infty(\lambda) = g_0(\lambda)\{F_\Delta(\lambda)(\lambda - \alpha - \varepsilon)(\lambda - \beta + \varepsilon)\}^{-1}$$

is also analytic on the inside of Γ_Δ. It follows that $(\lambda I - T)^{-1}x$ has an analytic continuation to a neighbourhood of the arc $(\alpha + \varepsilon, \beta - \varepsilon)$ for any sufficiently small $\varepsilon > 0$, and hence to a neighbourhood of the arc (α, β). Thus $x \in \mathfrak{S}(\alpha, \beta)$. The assertion for $\tilde{\mathfrak{S}}(\alpha, \beta)$ will be proved in just the same way. The proof of Lemma 2.4 is now completed.

LEMMA 2.5. *Let $T, \mathfrak{S}(\alpha, \beta)$ and $\tilde{\mathfrak{S}}(\alpha, \beta)$ be the same as in Lemma*

2.4 *and* $\tilde{\Delta} = [\beta, \alpha]$. *For any vector* $x \in \mathfrak{X}$, *the function* $g(\lambda)$ *is defined such that*

$$g(\lambda) = \begin{cases} F_{\tilde{\Delta}}(\lambda)(\lambda - \alpha)(\lambda - \beta)(\lambda I - T)^{-1}x & \text{if } \lambda \in \Gamma_{\Delta} - \{\alpha, \beta\}\,, \\ 0 & \text{if } \lambda = \alpha \text{ or } \lambda = \beta\,. \end{cases}$$

If $b(z)$ *is any numerical-valued function analytic in and continuous on* $|z| \leqq 1$ *and if* τ *is the conformal mapping from the inside of* $\Gamma_{\tilde{\Delta}}$ *to the unit disc, then the contour integral*

$$(*) \qquad y = \int_{C_0} b(z) g(\tau^{-1}(z)) dz$$

belongs to the space $\mathfrak{S}(\alpha, \beta)$, *where* C_0 *denotes the boundary of the unit disc (suppose that* C_0 *is positively oriented). Moreover, unless* x *belongs to the space* $\tilde{\mathfrak{S}}(\alpha, \beta)$ *there exists a function* $b(z)$ *analytic in and continuous on* $|z| \leqq 1$ *such that the vector* y *defined by* (*) *is non-zero.*

PROOF. Clearly, the function $g(\tau^{-1}(z))$ is continuous on C_0. Using the resolvent equation, we have

$$(\mu I - T)^{-1} g(\lambda) = (\mu - \lambda)^{-1} g(\lambda) - (\mu - \lambda)^{-1} F_{\tilde{\Delta}}(\lambda)(\lambda - \alpha)(\lambda - \beta)(\mu I - T)^{-1}x$$

for $\mu \in \rho(T) \cap \operatorname{Ext} \Gamma_{\tilde{\Delta}}$, where $\operatorname{Ext} \Gamma_{\tilde{\Delta}}$ denotes the outside of $\Gamma_{\tilde{\Delta}}$, thus

$$\begin{aligned}(\mu I - T)^{-1} y &= \int_{C_0} \frac{b(z) g(\tau^{-1}(z))}{\mu - \tau^{-1}(z)} dz \\ &\quad - \int_{C_0} \frac{b(z) F_{\tilde{\Delta}}(\tau^{-1}(z))(\tau^{-1}(z) - \alpha)(\tau^{-1}(z) - \beta)(\mu I - T)^{-1}x}{\mu - \tau^{-1}(z)} dz\end{aligned}$$

for the vector y of (*). By Cauchy's theorem, the second term is zero. Therefore we have

$$(**) \qquad (\mu I - T)^{-1} y = \int_{C_0} \frac{b(z) g(\tau^{-1}(z))}{\mu - \tau^{-1}(z)} dz$$

for the vector y of (*). Since the final expression of (**) is plainly analytic on the outside of $\Gamma_{\tilde{\Delta}}$, it follows that $y \in \mathfrak{S}(\alpha, \beta)$. Next, we suppose that the vector y defined by (*) is zero for each $b(z)$ which is analytic in and continuous on $|z| \leqq 1$, i.e., for all such $b(z)$

$$\int_{C_0} b(z) g(\tau^{-1}(z)) dz = 0\,.$$

Therefore the vector-valued function $g(\tau^{-1}(z))$ defined on C_0 must be the boundary value of a vector-valued function analytic in and continuous on $|z| \leqq 1$. Therefore $g(\lambda)$ must be analytically continuable to the inside of $\Gamma_{\tilde{\Delta}}$. Thus $(\lambda I - T)^{-1}x$ must be continuable across the arc (β, α), i.e.,

$x \in \tilde{\mathfrak{S}}(\alpha, \beta)$. The proof of Lemma 2.4 is now completed.

Let $T, \mathfrak{S}(\alpha, \beta), \tilde{\mathfrak{S}}(\alpha, \beta), \Delta, \tilde{\Delta}$ and C_0 be the same as above. Let τ (respectively, ν) be a conformal mapping from the inside of $\Gamma_{\tilde{\Delta}}$ (respectively, Γ_{Δ}) to the unit disc. Now we define the function $g_{\tilde{\Delta}}(\lambda)$ such that

$$g_{\tilde{\Delta}}(\lambda) = \begin{cases} F_{\tilde{\Delta}}(\lambda)(\lambda - \alpha)(\lambda - \beta)(\lambda I - T)^{-1} & \text{if } \lambda \in \Gamma_{\tilde{\Delta}} - \{\alpha, \beta\} , \\ 0 & \text{if } \lambda = \alpha \text{ or } \lambda = \beta . \end{cases}$$

Let $b_{\tilde{\Delta}}(z)$ be a function analytic in and continuous on $|z| \leqq 1$ such that the contour integral

$$\int_{C_0} b_{\tilde{\Delta}}(z) g_{\tilde{\Delta}}(\tau^{-1}(z)) x dz$$

is non-zero for some vector $x \notin \tilde{\mathfrak{S}}(\alpha, \beta)$. In just the same way we can define $g_{\Delta}(\lambda)$ and $b_{\Delta}(z)$ with respect to Δ instead of $\tilde{\Delta}$.

LEMMA 2.6. *If we put*

$$S = \int_{C_0} b_{\tilde{\Delta}}(z) g_{\tilde{\Delta}}(\tau^{-1}(z)) dz \quad \text{and} \quad W = \int_{C_0} b_{\Delta}(z) g_{\Delta}(\nu^{-1}(z)) dz ,$$

then $\overline{S\mathfrak{X}}$, $\overline{W\mathfrak{X}}$, where bar denotes the closure, are hyperinvariant subspaces under T and

$$\{0\} \subset \overline{S\mathfrak{X}} \subset \mathfrak{S}(\alpha, \beta) , \qquad \{0\} \subset \overline{W\mathfrak{X}} \subset \tilde{\mathfrak{S}}(\alpha, \beta) .$$

PROOF. From the proof of preceding Lemmas, the last assertions follows. Let A be any operator in $\mathfrak{B}(\mathfrak{X})$ such that $AT = TA$. Since both operators S, W are elements of the bi-commutant of T, it follows

$$A(\overline{S\mathfrak{X}}) \subset \overline{AS\mathfrak{X}} = \overline{SA\mathfrak{X}} \subset \overline{S\mathfrak{X}} ,$$
$$A(\overline{W\mathfrak{X}}) \subset \overline{AW\mathfrak{X}} = \overline{WA\mathfrak{X}} \subset \overline{W\mathfrak{X}} ,$$

The proof of Lemma 2.6 is now completed.

PROOF OF THEOREM. Suppose that T is an operator satisfying the hypotheses of Lemma 2.4. By Lemma 2.4 and Lemma 2.6, we have only to show that $\mathfrak{S}(\alpha, \beta)$ and $\tilde{\mathfrak{S}}(\alpha, \beta)$ are non-trivial. We may assume $\sigma(T)$ lies on both arcs (α, β) and (β, α), because we can choose the pair of points α, β arbitrary on J. This implies $\mathfrak{S}(\alpha, \beta) \neq \mathfrak{X}$ and $\tilde{\mathfrak{S}}(\alpha, \beta) \neq \mathfrak{X}$. Thus we have only to show that $\mathfrak{S}(\alpha, \beta) \neq \{0\}$ and $\tilde{\mathfrak{S}}(\alpha, \beta) \neq \{0\}$. By Lemma 2.5, $\mathfrak{S}(\alpha, \beta) \neq \mathfrak{X}, \tilde{\mathfrak{S}}(\alpha, \beta) \neq \mathfrak{X}$ imply $\tilde{\mathfrak{S}}(\alpha, \beta) \neq \{0\}, \mathfrak{S}(\alpha, \beta) \neq \{0\}$ respectively. Since both $\mathfrak{S}(\alpha, \beta), \tilde{\mathfrak{S}}(\alpha, \beta)$ are non-trivial and by definitions in Lemma 2.6 $S \neq 0$, $W \neq 0$. Thus from Lemma 2.6 the assertion follows.

3. Perturbation and hyperinvariant subspaces. In the sequel J will stand a finite union of simple rectifiable smooth arcs in the complex plane (not separating the complex plane). Instead of $\mathfrak{X}$ we consider a Hilbert space $\mathfrak{H}$. Let C_∞ be the set of all compact operators acting in $\mathfrak{H}$. Let $T \in C_\infty$. We denote by C_ω the class of compact operators for which

$$\| T \|_\omega = \sum_{n=1}^{\infty} \mu_n(T)/n < \infty ,$$

where $\mu_1(T), \mu_2(T), \cdots, \mu_n(T), \cdots$ are the eigenvalues of $(T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity (see [4, §15 of Chap. III]) and the class C_p is the set of all compact operators such that $\| T \|_p = \{\sum_{n=1}^{\infty} \mu_n(T)^p\}^{1/p}$, $(1 \leqq p < \infty)$ is finite (see [3, §9 of Chap. XI]). It is easy to see that $C_p \subset C_\omega$ for arbitrary p $(1 \leqq p < \infty)$. In fact, from Hölder's inequality it follows that

$$\| T \|_\omega \leqq \left\{\sum_n 1/n^q\right\}^{1/q} \left\{\sum_n \mu_n(T)^p\right\}^{1/p} ,$$

where $1/p + 1/q = 1$. The class C_ω was introduced in the paper of V. I. Macaev [9]. To an operator $T \in C_\infty$ we associate the sequences $\lambda_1(T)$, $\lambda_2(T), \cdots, \lambda_n(T), \cdots$ of eigenvalues of T numbered according to decreasing values of their moduli and repeated according to multiplicity and the function $n(t, T)$ (respectively, $\gamma(t, T)$), for $t > 0$, giving the number of terms of the sequence $\lambda_n(T)$ (respectively, $\mu_n(T)$), $n = 1, 2, \cdots, n, \cdots$, exceeding $1/t$ in modulus.

The purpose of this section is to show that the following theorem holds.

THEOREM 3.1. *Let T be an operator in $\mathfrak{B}(\mathfrak{H})$. Write $T = A + B$, where T is the sum of a normal operator A, whose spectrum lies on J and a compact operator B which belongs to the class C_ω. If $\sigma(T)$ is not reduced to a single point, then T has a proper closed hyperinvariant subspace in $\mathfrak{H}$.*

On the existence of invariant subspace, in [5] the author has proved in the case that the perturbing term belongs to one of the classes C_p $(1 \leqq p < \infty)$. In §3 of [2], Apostol studies operators of the form $A + B$, which the resolvent of A has polar growth of order n near J and $B \in C_p$ $(1 \leqq p < \infty)$.

REMARK. In the above theorem if $\sigma(T)$ is a single point, then T has a proper closed invariant subspace. In fact, since the operator T may be translated, we may assume without loss of generality that $\sigma(T) = \{0\}$, from Lemma 2.3 of [5] it follows that the Hilbert space admits a proper closed invariant subspace under T, using Aronszajn-Smith theorem [1].

We shall use the following known result.

LEMMA 3.2. (see [4, §5 of Chap. I]) *Let T be as above (in the theorem). Then any point λ with $\lambda \notin \sigma(A)$ $(\subset J)$ is either $\lambda \in \rho(T)$ or an eigenvalue with finite multiplicity of T.*

LEMMA 3.3. *If T has an eigenvalue with finite multiplicity, then $\mathfrak{H}$ admits a proper closed subspace which is hyperinvariant under T.*

PROOF. Let λ_0 be an eigenvalue with n multiplicity and $ST = TS$ $(S \in \mathfrak{B}(\mathfrak{H}))$. Then the eigensubspace of T corresponding to λ_0 is an n-dimensional subspace invariant under S. In fact, since

$$(T - \lambda_0 I)S\{(T - \lambda_0 I)^{-1}(0)\} = S(T - \lambda_0 I)\{(T - \lambda_0 I)^{-1}(0)\} = S(0) = \{0\} .$$

Therefore we have

$$S\{(T - \lambda_0 I)^{-1}(0)\} \subset (T - \lambda_0 I)^{-1}(0) .$$

The proof of Lemma 3.3 is now completed.

The assertion in Theorem follows from Lemma 3.3 and an argument in §2 in each case that one of the following condition is satisfied:

$$\text{(i)} \quad \sigma(T) - \sigma(A) \neq \varnothing \ ; \qquad \text{(ii)} \quad \sigma(T) \text{ is disconnected} .$$

Therefore we have only to show that there can exist no operator $T = A + B$, where A is normal such that $\sigma(A) \subset J$, $B \in C_\omega$ and $\sigma(T) (\subset \sigma(A))$ is a connected subarc of J and $\mathfrak{H}$ admits no proper closed hyperinvariant subspace under T.

In the sequel, we shall show a growth condition of the resolvent of T near its spectrum $\sigma(T)$. If S has finite dimensional range, $S = PS$, where P is the orthogonal projection on the range of S. Thus $S^* = S^*P$, so that S^* also has finite dimensional range. Let $\mathfrak{H}_0$ be a finite dimensional space including both the range of S and the range of S^*. Then plainly, $\mathfrak{H}_0$ is invariant under S and S^*, and since $(S\mathfrak{H}_0^\perp, x) = (\mathfrak{H}_0^\perp, S^*x) = 0$ for all $x \in \mathfrak{H}$, we have $S\mathfrak{H}_0^\perp = \{0\}$ and similarly $S^*\mathfrak{H}_0^\perp = \{0\}$. We denote by $\Phi = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ an orthonormal basis of $\mathfrak{H}_0$ and define an $n \times n$ matrix $S_{(n)} = (a_{ij})_{1 \leqq i, j \leqq n}$ by the following relation

$$S\varphi_j = \sum_{i=1}^{n} a_{ij}\varphi_i \qquad (j = 1, 2, \cdots, n) .$$

Thus $S_{(n)}$ is the matrix of the operator $S | \mathfrak{H}_0$ (= the restriction of S to $\mathfrak{H}_0$) with respect to the basis Φ.

We define

$$D_1(S) = \det (I_{(n)} - S_{(n)})$$

the determinant of $I_{(n)} - S_{(n)}$, where $I_{(n)}$ is the $n \times n$ identity matrix.

Now, we denote by $K = (k_{ij})_{1\leq i,j\leq n}$ the cofactor matrix of $I_{(n)} - S_{(n)}$ i.e., k_{ij} is the (j, i) cofactor of $I_{(n)} - S_{(n)}$. We have

$$(I_{(n)} - S_{(n)})K = K(I_{(n)} - S_{(n)}) = D_1(S)I_{(n)} .$$

We define an $n \times n$ matrix $(h_{ij})_{1\leq i,j\leq n} = S_{(n)}K$ and an operator H on $\mathfrak{H}$ such that

$$Hx = \sum_{j=1}^{n}\sum_{i=1}^{n} h_{ij}(x, \varphi_j)\varphi_i \quad \text{for} \quad x \in \mathfrak{H} .$$

Then we have $H\mathfrak{H}_0 \subset \mathfrak{H}_0$ and $H\mathfrak{H}_0^{\perp} = \{0\}$. Therefore the restriction $H|\mathfrak{H}_0$ of H to $\mathfrak{H}_0$ is the operator whose matrix representation with respect to the basis Φ is the matrix (h_{ij}). For operators $(D_1(S)I + H)(I - S)$ and $(I - S) \times (D_1(S)I + H)$, the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_0^{\perp}$ are invariant and it follows that by the above relations

$$(D_1(S)I + H)(I - S)x = (I - S)(D_1(S)I + H)x = D_1(S)x$$

for any $x \in \mathfrak{H}_0^{\perp}$. Moreover, since

$$D_1(S)I_{(n)} + S_{(n)}K = (I_{(n)} - S_{(n)})K + S_{(n)}K = K ,$$

we have the matrix relations

$$\begin{aligned}&(D_1(S)I_{(n)} + S_{(n)}K)(I_{(n)} - S_{(n)})\\ &\qquad = (I_{(n)} - S_{(n)})(D_1(S)I_{(n)} + S_{(n)}K) = D_1(S)I_{(n)} ,\end{aligned}$$

i.e., by restriction each operator to $\mathfrak{H}_0$ we have the last above relations with respect to the basis Φ. Thus we have the following equations of operators on the subspace $\mathfrak{H}_0$,

$$\begin{aligned}&(D_1(S)I|\mathfrak{H}_0 + H|\mathfrak{H}_0)(I|\mathfrak{H}_0 - S|\mathfrak{H}_0)\\ &\qquad = (I|\mathfrak{H}_0 - S|\mathfrak{H}_0)(D_1(S)I|\mathfrak{H}_0 + H|\mathfrak{H}_0) = D_1(S)I|\mathfrak{H}_0 ,\end{aligned}$$

therefore we obtain the operator relations on $\mathfrak{H}$,

$$(D_1(S)I + H)(I - S) = (I - S)(D_1(S)I + H) = D_1(S)I .$$

So that $(I - S)^{-1}$ exists, when $D_1(S) \neq 0$, i.e., $1 \notin \sigma(S)$,

$$(I - S)^{-1} = I + D_1(S)^{-1}H .$$

Now, we define

$$D_2(S) = D_1(S)((I - S)^{-1}x, y)$$

for unit vectors x, y in $\mathfrak{H}$.

LEMMA 3.4. *Let S be a finite dimensional operator, $1 \notin \sigma(S)$, and let $D_1(S)$ and $D_2(S)$ be as above. Then we have the followings,*

$$\log|D_1(S)| \leqq \sum_{j=1}^{n} \log(1 + \mu_j(S)) ,$$

$$\log|D_2(S)| \leqq \sum_{j=1}^{n} \log(1+\mu_j(S)) + 3\log(3+\mu_1(S)) .$$

PROOF. To prove the second assertion, it is sufficient to establish the corresponding inequality for the operator obtained by restricting $I + D_1(S)^{-1}H$ to the subspace $\mathfrak{H}_0$. In fact, since $H\mathfrak{H}_0^\perp = \{0\}$, the restriction $(I-S)^{-1}$ to $\mathfrak{H}_0^\perp$ is $I|\mathfrak{H}_0^\perp$ and the second assertion follows from the first assertion. Now, the matrix representation of $(I + D_1(S)^{-1}H)|\mathfrak{H}_0$ with respect to the basis Φ of $\mathfrak{H}_0$ is $I_{(n)} + D_1(S)^{-1}S_{(n)}K$ and we have

$$(I + D_1(S)^{-1}H)\varphi_j = D_1(S)^{-1}(D_1(S)I + H)\varphi_j = D_1(S)^{-1}\sum_{i=1}^{n} k_{ij}\varphi_i$$

$(j = 1, 2, \cdots, n)$. Thus we have

$$((D_1(S)I + H)\varphi_j, \varphi_i) = k_{ij} .$$

Given any unit vectors x, y in $\mathfrak{H}_0$, we may put $x = x_1\varphi_1 + \cdots + x_n\varphi_n$ and $y = y_1\varphi_1 + \cdots + y_n\varphi_n$, where $\sum_{i=1}^{n}|x_i|^2 = \sum_{i=1}^{n}|y_i|^2 = 1$. Then

$$D_1(S)((I-S)^{-1}x, y) = ((D_1(S)I + H)x, y)$$

$$= \sum_{i,j=1}^{n} k_{ij}x_j\bar{y}_i = -\det\begin{bmatrix} 0 & \bar{y}_1 \ \bar{y}_2 \cdots \bar{y}_n \\ x_1 & \\ x_2 & I_{(n)} - S_{(n)} \\ \vdots & \\ x_n & \end{bmatrix} .$$

On the other hand, since we have the relation (by Lemma 16 of Chap. XI. 9 of [3]),

$$\log\left|\det\begin{bmatrix} 0 & \bar{y}_1 \ \bar{y}_2 \cdots \bar{y}_n \\ x_1 & \\ x_2 & I_{(n)} - S_{(n)} \\ \vdots & \\ x_n & \end{bmatrix}\right| = \sum_{j=1}^{n}\log|1-\lambda_j| \leqq \sum_{j=1}^{n}\log(1+\mu_j)$$

with the eigenvalues $\{\lambda_j\}$ (respectively, $\{\mu_j\}$) of

$$S_0 = \begin{bmatrix} 1 & \bar{y}_1 \ \bar{y}_2 \cdots \bar{y}_n \\ x_1 & \\ x_2 & S_{(n)} \\ \vdots & \\ x_n & \end{bmatrix}$$

(respectively, $(S_0{}^*S_0)^{1/2}$). If we put

$$S_1 = \begin{bmatrix} 1 & 0 & 0\cdots 0 \\ 0 & & \\ 0 & & 0 \\ \vdots & & \\ 0 & & \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 0\cdots 0 \\ x_1 & & \\ x_2 & & 0 \\ \vdots & & \\ x_n & & \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 0 & \bar{y}_1 & \bar{y}_2\cdots \bar{y}_n \\ 0 & & \\ 0 & & 0 \\ \vdots & & \\ 0 & & \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0 & 0 & 0\cdots 0 \\ 0 & & \\ 0 & & S_{(n)} \\ \vdots & & \\ 0 & & \end{bmatrix},$$

then $S_0 = S_1 + S_2 + S_3 + S_4$. By virtue of corollary 3 of Chap. XI. 9 of [3], for $\mu_j(S_0)$ we have the followings

$$\begin{aligned} \mu_1(S_0) &\leqq \mu_1(S_1) + \mu_1(S_2) + \mu_1(S_3) + \mu_1(S_4) = 3 + \mu_1(S)\ , \\ \mu_2(S_0) &\leqq \mu_1(S_1) + \mu_1(S_2) + \mu_2(S_3) + \mu_1(S_4) = 2 + \mu_1(S)\ , \\ \mu_3(S_0) &\leqq \mu_1(S_1) + \mu_2(S_2) + \mu_2(S_3) + \mu_1(S_4) = 1 + \mu_1(S)\ , \\ \mu_4(S_0) &\leqq \mu_2(S_1) + \mu_2(S_2) + \mu_2(S_3) + \mu_1(S_4) = \mu_1(S)\ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_j(S_0) &\leqq \mu_{j-3}(S)\ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_{n+1}(S_0) &\leqq \mu_{n-2}(S)\ , \end{aligned}$$

therefore we have

$$\begin{aligned} \sum_{j=1}^{n+1} \log(1 + \mu_j(S_0)) &\leqq \sum_{j=1}^{n-2} \log(1 + \mu_j(S)) + \log(4 + \mu_1(S))(3 + \mu_1(S))(2 + \mu_1(S)) \\ &\leqq \sum_{j=1}^{n} \log(1 + \mu_j(S)) + 3\log(3 + \mu_1(S))\ . \end{aligned}$$

Thus the second assertion follows. The first assertion follows immediately, in fact

$$\log|D_1(S)| = \sum_{j=1}^{n} \log|1 - \lambda_j(S)| \leqq \sum_{j=1}^{n} \log(1 + \mu_j(S))\ .$$

This completes the proof of Lemma 3.4.

Now without loss of generality, we may assume that $||B|| \leqq 1/2$. For any sufficiently small number $\delta > 0$, let P be the projection to the spectral subspace of $(B^*B)^{1/2}$ relative to $(\delta, 1/2]$ and $B_1 = BP$, $B_2 = B(I - P)$. Since A is normal, we have $||(\lambda I - A)^{-1}|| \leqq \{d(\lambda, \sigma(A))\}^{-1}$. Thus by $||B_2(\lambda I - A)^{-1}|| \leqq \delta/d(\lambda, \sigma(A))$,

$$\begin{aligned}\|(\lambda I - A - B_2)^{-1}\| &= \|(\lambda I - A)^{-1}\{I - B_2(\lambda I - A)^{-1}\}^{-1}\| \\ &\leqq \|(\lambda I - A)^{-1}\| \, \|\{I - B_2(\lambda I - A)^{-1}\}^{-1}\| \\ &\leqq \{d(\lambda, \sigma(A)) - \delta\}^{-1} \\ &\leqq 1/\delta ,\end{aligned}$$

for λ with $d(\lambda, \sigma(A)) \geqq 2\delta$. We have

$$\begin{aligned}&\log \|(\lambda I - T)^{-1}\| \\ &\quad \leqq \log \|(\lambda I - A - B_2)^{-1}\| + \log \|\{I - B_1(\lambda I - A - B_2)^{-1}\}^{-1}\| .\end{aligned}$$

Now, to obtain an estimate of the second term in the sequel, extending the method for estimating the resolvent of a compact operator in [10], we shall show a growth condition of the resolvent of T near its spectrum $\sigma(T)$ ($\subset \sigma(A)$). Let $D_k(\lambda) = D_k(B_1(\lambda I - A - B_2)^{-1})$, for $k = 1, 2$. Let $z = \tau(\lambda)$ be the conformal mapping of the domain $\{\lambda; d(\lambda, \sigma(A)) \geqq 2\delta\}$ onto the disc of radius r with center at the origin and $\tau(\infty) = 0$. We put $n(t) = n(t, B_1(\tau^{-1}(z)I - A - B_2)^{-1})$, by Jensen's formula [6, Theorem 5, p. 14]

$$\begin{aligned}\int_0^r \frac{n(t)}{t}\, dt &= \frac{1}{2\pi}\int_0^{2\pi} \log |D_1(\tau^{-1}(re^{i\theta}))|\, d\theta - \log |D_1(\tau^{-1}(0))| \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \log^- |D_1(\tau^{-1}(re^{i\theta}))|\, d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |D_1(\tau^{-1}(re^{i\theta}))|\, d\theta .\end{aligned}$$

Thus we have for any unit vectors x, y in $\mathfrak{H}$,

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} \log^+ |(\{I - B_1(\tau^{-1}(re^{i\theta})I - A - B_2)^{-1}\}^{-1}x, y)|\, d\theta + \int_0^r \frac{n(t)}{t}\, dt \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{D_2(\tau^{-1}(re^{i\theta}))}{D_1(\tau^{-1}(re^{i\theta}))}\right| d\theta + \int_0^r \frac{n(t)}{t}\, dt \\ &\quad \leqq \frac{1}{2\pi}\int_0^{2\pi} [\log^+ |D_2(\tau^{-1}(re^{i\theta}))| + \log^- |D_1(\tau^{-1}(re^{i\theta}))|]\, d\theta \\ &\qquad + \int_0^r \frac{n(t)}{t}\, dt \\ &\quad \leqq \frac{1}{2\pi}\left[\int_0^{2\pi} \log^+ |D_2(\tau^{-1}(re^{i\theta}))|\, d\theta + \int_0^{2\pi} \log^+ |D_1(\tau^{-1}(re^{i\theta}))|\, d\theta\right] \\ &\quad \leqq \sum_j \log \{1 + \mu_j(B_1(\lambda_1 I - A - B_2)^{-1})\} \\ &\qquad + \sum_j \log \{1 + \mu_j(B_1(\lambda_2 I - A - B_2)^{-1})\} \\ &\qquad + 3 \log (3 + 1/2\delta) ,\end{aligned}$$

where $\lambda_1, \lambda_2 \in \{\lambda : d(\lambda, \sigma(A)) = 2\delta\}$, since the inequality $1 + t \leqq e^t$ for a real number t, it follows

$$\leqq \sum_{i=1}^{2} \|B_1(\lambda_i I - A - B_2)^{-1}\|_1 + 3 \log (3 + 1/2\delta)$$
$$\leqq (2/\delta)\|B_1\|_1 + 3 \log (3 + 1/2\delta) .$$

Thus we have, for any unit vectors x, y in $\mathfrak{H}$ and for λ with $d(\lambda, \sigma(A)) \geqq 2\delta$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |(\{I - B_1(\tau^{-1}(re^{i\theta})I - A - B_2)^{-1}\}^{-1}x, y)|\, d\theta$$
$$\leqq (2/\delta)\|B_1\|_1 + 3 \log (3 + 1/2\delta) .$$

On the other hand by the Poisson-Jensen formula [12, Theorem 1.1, p. 1] we have

$$\log |(\{I - B_1(\tau^{-1}(\rho e^{i\varphi})I - A - B_2)^{-1}\}^{-1}x, y)| - \sum_k \log \left|\frac{r(z - z_k)}{r^2 - \bar{z}_k z}\right|$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \log |(\{I - B_1(\tau^{-1}(re^{i\theta})I - A - B_2)^{-1}x, y)|$$
$$\times \frac{r^2 - \rho^2}{r^2 - 2r\rho \cos (\theta - \varphi) + \rho^2}\, d\theta$$

where $z = \rho e^{i\varphi}$, $0 \leqq \rho < r$ and $\{z_k\}$ are zeros of the function

$$(\{I - B_1(\tau^{-1}(z)I - A - B_2)^{-1}\}^{-1}x, y)$$

in $|z| < r$. Since $0 < |r(z - z_k)/(r^2 - \bar{z}_k z)| < 1$ and

$$\frac{r - \rho}{r + \rho} \leqq \frac{r^2 - \rho^2}{r^2 - 2r\rho \cos (\theta - \varphi) + \rho^2} \leqq \frac{r + \rho}{r - \rho} ,$$

we have

$$\log |(\{I - B_1(\tau^{-1}(\rho e^{i\varphi})I - A - B_2)^{-1}\}^{-1}x, y)|$$
$$\leqq \frac{r + \rho}{r - \rho}\,\frac{1}{2\pi}\int_0^{2\pi} \log^+ |(\{I - B_1(\tau^{-1}(re^{i\theta})I - A - B_2)^{-1}\}^{-1}x, y)|\, d\theta$$
$$\leqq \frac{r + \rho}{r - \rho}\{(2/\delta)\|B_1\|_1 + 3 \log (3 + 1/2\delta)\} .$$

Since τ is a conformal mapping, therefore we have

$$\log \|\{I - B_1(\lambda I - A - B_2)^{-1}\}^{-1}\|$$
$$\leqq O(1/\delta)\{(2/\delta)\|B_1\|_1 + 3 \log (3 + 1/2\delta)\} ,$$

for λ with $d(\lambda, \sigma(A)) \geqq 3\delta$. Therefore from the already proved inequalities in above, we have

$$\log \|(\lambda I - T)^{-1}\| \leqq \log(1/\delta) + O(1/\delta) \cdot \{(2/\delta)\|B_1\|_1 + 3\log(3 + 1/2\delta)\} ,$$

for λ with $d(\lambda, \sigma(A)) \geqq 3\delta$.

We put $\gamma(t) = \gamma(t, B)$. Using this function we can write

$$\log \|(\lambda I - T)^{-1}\| \leqq \log(1/\delta) + O(1/\delta) \cdot \{(2/\delta)\gamma(1/\delta) + 3\log(3 + 1/2\delta)\} ,$$

for λ with $d(\lambda, \sigma(A)) \geqq 3\delta$. Then we have for sufficiently small $\delta > 0$,

$$\begin{aligned} &\sup_{d(\lambda,\sigma(A))\geqq 3\delta} \log\log \|(\lambda I - T)^{-1}\| \\ &\qquad \leqq \log\log(1/\delta) + O\log(1/\delta) + \log(2/\delta) + \log\gamma(1/\delta) \\ &\qquad\quad + \log\{3\log(3 + 1/2\delta)\} . \end{aligned}$$

Here, $\log\log(1/\delta)$ and $\log(1/\delta)$ are integrable on some interval $(0, \varepsilon]$, let us prove that $\log\gamma(1/\delta)$ is integrable on $(0, \varepsilon]$. This follows from the assumption, $B \in C_\omega$, in fact

$$\begin{aligned} \int_0^\varepsilon \log\gamma(1/\delta)d\delta &= \int_{1/\varepsilon}^\infty \frac{\log\gamma(t)}{t^2}\,dt \\ &= \int_{1/\varepsilon}^\infty \frac{d\gamma(t)}{\gamma(t)t} - \varepsilon\log\gamma(1/\varepsilon) + \lim_{t\to\infty}\frac{\log\gamma(t)}{t} , \end{aligned}$$

while

$$\int_{1/\varepsilon}^\infty \frac{d\gamma(t)}{\gamma(t)t} \leqq \sum_n \mu_n(B)/n$$

with $\varepsilon\mu_1(B) = 1$, since $\log n < 1 + 1/2 + 1/3 + \cdots + 1/n$, we have

$$\frac{\log\gamma(t)}{t} \leqq \sum{}' \mu_k(B)/k \leqq \sum_n \mu_n(B)/n ,$$

where $\sum'$ denotes the summation over k such that $\mu_k(B) \geqq 1/t$. Therefore we can obtain the following growth condition of the resolvent of T near its spectrum

$$\int_0^\varepsilon \log\log M(\delta)d\delta < \infty ,$$

where $M(\delta) = \sup_{d(\lambda,\sigma(A))\geqq 3\delta} \|(\lambda I - T)^{-1}\|$.

From the above growth condition of the resolvent of T near its spectrum, by virtue of Theorem 2.2 (in section 2) Theorem in this section was obtained.

COROLLARY 3.5. *Let $T \in \mathfrak{B}(\mathfrak{H})$. If $T - T^* \in C_\omega$(i.e., the imaginary part of T belongs to C_ω) and $\sigma(T)$ is not reduced to a single point, then T has a proper closed hyperinvariant subspace in $\mathfrak{H}$.*

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