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ON ALMOST COMPLEX HYPERSURFACES OF A K-SPACE

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Introduction. Let M be a $2n$-dimensional almost Hermitian manifold with almost Hermitian metric tensor g_{ji} and almost complex structure tensor $F_j{}^i$ [1]. By ∇_j, $R_{kji}{}^h$ and R_{ji} we denote the operator of covariant differentiation with respect to Christoffel symbols $\left\{ {h \atop ji} \right\}$ formed with g_{ji}, the curvature tensor and the Ricci tensor respectively, and put $R^*{}_{ji} = (1/2)F^{ab}R_{abci}F_j{}^c$ where $R_{abci} = g_{il}R_{abc}{}^l$ etc.. Then M is said to be a K-space (or almost Tachibane space, nearly Kähler manifold) provided

$$\nabla_j F_i{}^h + \nabla_i F_j{}^h = 0 .$$

It is well known that a Kähler manifold is a K-space but a K-space is not necessarily a Kähler manifold [3]. The main purpose of the present paper is to prove the following

THEOREM A. *Let $\tilde{M}$ be a K-space of constant holomorphic sectional curvature $\tilde{c}$ and M a connected non-Kähler almost complex hypersurface of $\tilde{M}$. If M is of complex dimension $n > 2$, then the following statements are equivalent*:

(i) *M is totally geodesic in $\tilde{M}$,*

(ii) *M is of constant holomorphic sectional curvature,*

(iii) *M is an Einstein space satisfying $R^*{}_{ji} = (\rho^*/2n)g_{ji}$,*

(iv) $\tilde{c} = 0$

where $c = 1/(4n(n+1))(\rho + 3\rho^*)$, $\rho = g^{ji}R_{ji}$ and $\rho^* = g^{ji}R^*{}_{ji}$.

THEOREM B. *Let $\tilde{M}$ be an 8-dimensional K-space of constant holomorphic sectional curvature and M a non-Kähler almost complex hypersurface of $\tilde{M}$. Then M is a space of constant curvature.*

For Theorem A, we shall prove that if an almost complex hypersurface M of a K-space $\tilde{M}$ is a Kähler manifold, then $\tilde{M}$ is also a Kähler manifold (Proposition 2). But we know Smyth's result [7; p. 257] for the case where M is a Kähler manifold. The analogous problems to those in

[1] The Latin indices run over the range $1, 2, \cdots, 2n$.

Theorem A in the case of Kähler manifold were studied by A. Gray [1], B. Smyth [7], T. Takahashi [9] and others. One of the present authors (Sawaki and Sekigawa [5]) proved the equivalence of the first two statements of Theorem A in a more general case than a K-space. In § 1 we shall state some well known properties of a K-space and some recent results on a K-space of constant holomorphic sectional curvature. In § 2, differential geometric properties of almost complex hypersurfaces of an almost Hermitian manifold and as a special case, a K-space will be stated. In § 3 we shall prove some lemmas on almost complex hypersurfaces of a K-space with constant holomorphic sectional curvature. In particular, we shall obtain Codazzi equation which will play an important role in this papar. § 4 will be devoted to the proof of Theorems A and B.

1. K-space. In a $2n$-dimensional almost Hermitian manifold M, we define the following linear operators

$$O_{ih}^{ml} = \frac{1}{2}(\delta_i^m \delta_h^l - F_i{}^m F_h{}^l) , \qquad {}^*O_{ih}^{ml} = \frac{1}{2}(\delta_i^m \delta_h^l + F_i{}^m F_h{}^l) .$$

Then a tensor T_{ji}(resp. $T_j{}^i$) is said to be pure in j, i if it satisfies

$${}^*O_{ji}^{ab} T_{ab} = 0 \qquad (\text{resp. } {}^*O_{jb}^{ai} T_a{}^b = 0) ,$$

and hybrid in j, i if it satisfies

$$O_{ji}^{ab} T_{ab} = 0 \qquad (\text{resp. } O_{jb}^{ai} T_a{}^b = 0) .$$

This definition is the same as a general tensor, for example, $T_{ji}{}^h$ and we can easily verify the following

PROPOSITION 1. (i) *If $T_j{}^i$ is pure (resp. hybrid) in j, i then*

$$F_t{}^i T_j{}^t = F_j{}^t T_t{}^i \qquad (\textit{resp. } F_t{}^i T_j{}^t = -F_j{}^t T_t{}^i) .$$

(ii) *Let T_{ji} be pure in j, i. If $S_j{}^i$ is pure (resp. hybrid) in j, i then $T_{jr} S_i{}^r$ is pure (resp. hybrid) in j, i.*

(iii) *If T_{ji} is pure and at the same time hybrid in j, i then T_{ji} vanishes.*

In particular, let M be a K-space. Then it is easily verified that

$$(1.1) \qquad {}^*O_{ji}^{ab} \nabla_a F_b{}^h = 0 , \qquad O_{ib}^{ah} \nabla_j F_a{}^b = 0 ,$$

and we know

$$(1.2) \qquad (\nabla_j F_{ih}) \nabla^j F^{ih} = \rho - \rho^* = \text{constant} \geqq 0 \qquad [8] .$$

Moreover, for a K-space, we know the following

LEMMA 1.1. (Gray [2]) *There does not exist a 4-dimensional non-Kähler K-space.*

LEMMA 1.2. (Takamatsu [10]) *In a 6-dimensional K-space, we have*

$$R_{ji} - R^*{}_{ji} = \frac{\rho - \rho^*}{6} g_{ji} .$$

LEMMA 1.3. (Yamaguchi, Chuman, and Matsumoto [13]) *A 6-dimensional non-Kähler K-space is an Einstein space.*

LEMMA 1.4. (Watanabe and Takamatsu [12]) *In a K-space of constant holomorphic sectional curvature, we have*

$$R_{ji} = \frac{\rho}{2n} g_{ji} , \qquad R^*{}_{ji} = \frac{\rho^*}{2n} g_{ji} .$$

LEMMA 1.5. (Tanno [11]) *A 6-dimensional non-Kähler K-space of constant holomorphic sectional curvature is a space of constant curvature.*

2. Almost complex hypersurfaces of an almost Hermitian manifold. Let $\tilde{M}$ be an almost Hermitian manifold of dimension $2n + 2$ with almost Hermitian metric tensor g and almost complex structure tensor F. Moreover, let M be an almost complex hypersurface of $\tilde{M}$, i.e., suppose that there exists an almost complex analytic mapping $f: M \rightarrow \tilde{M}$. Then we identify, for each $x \in M$, the tangent space $T_x(M)$ with $f_*(T_x(M)) \subset T_{f(x)}(\tilde{M})$ by means of f_*. Since $f^* \circ g = g'$ and $F \circ f_* = f_* \circ F'$ where g' and F' are the almost Hermitian metric tensor and the almost complex structure tensor of M respectively, g' and F' are respectively identified with the restrictions of the structures g and F to the subspace $f_*(T_x(M))$. Henceforth, under this consideration, we use g and F instead of g' and F' respectively.

As is well known, there exists a local coordinate system $(\tilde{x}^1, \cdots, \tilde{x}^{2n}, \tilde{x}^{2n+1}, \tilde{x}^{2n+2})$ on a neighborhood $\tilde{U}$ of $f(x)$ in $\tilde{M}$ such that $(x^1, \cdots, x^{2n})$ is a local coordinate system on the neighborhood U of x in M given by $U = \{y \in M \mid x^{2n+1}(y) = x^{2n+2}(y) = 0\}$, where $x^\lambda = \tilde{x}^\lambda \circ f$ $(\lambda = 1, 2, \cdots, 2n + 2)$.

By $\tilde{\nabla}$ we always mean the Riemannian covariant differentiation on $\tilde{M}$ and ξ a differentiable unit vector field normal to M at each point of $U(x)$. If X and Y are vector fields on the neighborhood $U(x)$, we may write

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi + k(X, Y)F\xi \tag{2.1}$$

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to M. Then the following lemma is easily verified (for example see [5]).

LEMMA 2.1. (i) *∇ is the covariant differentiation of the almost Hermitian manifold M.*

(ii) *h and k are symmetric covariant tensor fields of degree 2 on $U(x)$.*

The identities $g(\xi, \xi) = 1$ and $g(F\xi, F\xi) = 1$ imply $g(\tilde{\nabla}_X \xi, \xi) = 0$ and $g(\tilde{\nabla}_X(F\xi), F\xi) = 0$ respectively. Therefore, we may put

$$\tilde{\nabla}_X \xi = -A(X) + s(X)F\xi , \tag{2.2}$$

$$\tilde{\nabla}_X(F\xi) = -B(X) + t(X)\xi \tag{2.3}$$

where $A(X)$ and $B(X)$ are tangent to M.

In this place, we know the following

LEMMA 2.2. (i) *A, B and s, t are tensor fields on $U(x)$ of type* (1.1) *and* (0.1) *respectively.*

(ii) *A and B are symmetric with respect to g and satisfy*

$$h(X, Y) = g(AX, Y) , \tag{2.4}$$

$$k(X, Y) = g(BX, Y) . \tag{2.5}$$

LEMMA 2.3. *Let $\tilde{R}$ and R be Riemannian curvature tensors of $\tilde{M}$ and an almost complex hypersurface M of $\tilde{M}$ respectively. Then for any vector fields X, Y, Z, and W on $U(x) \subset M$, we have*

$$\begin{aligned} \tilde{R}(X, Y)W = R(X, Y)W &- \{h(Y, W)AX - h(X, W)AY\} - \{k(Y, W)BX \\ &- k(X, W)BY\} + \{(\nabla_X h)(Y, W) - (\nabla_Y h)(X, W) + k(Y, W)t(X) \\ &- k(X, W)t(Y)\}\xi + \{(\nabla_X k)(Y, W) - (\nabla_Y k)(X, W) \\ &+ h(Y, W)s(X) - h(X, W)s(Y)\}F\xi , \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\tilde{R}(X, Y, Z, W) \\ &= R(X, Y, Z, W) - \{g(AX, Z)g(AY, W) - g(AY, Z)g(AX, W)\} \\ &\quad - \{g(BX, Z)g(BY, W) - g(BY, Z)g(BX, W)\} . \end{aligned} \tag{2.7}$$

(Gauss)

In particular, let $\tilde{M}$ be a K-space, i.e., suppose that

$$(\tilde{\nabla}_X F)Y + (\tilde{\nabla}_Y F)X = 0 \qquad \text{(or equivalently } (\tilde{\nabla}_X F)X = 0)$$

for any vector fields X and Y on $\tilde{M}$.

It is well known that an almost complex hypersurface M of a K-space $\tilde{M}$ is also a K-space [4].

LEMMA 2.4. (Sawaki and Sekigawa [5]) *In an almost complex hypersurface M of a K-space $\tilde{M}$, we have*

(i) $FA = -AF, \quad FB = -BF,$

(ii) *FA and FB are symmetric with respect to g,*

(iii) *$B = FA$ (or equivalently $h(X, Y) = k(X, FY)$ for any vector fields X and Y on M).*

LEMMA 2.5. (Sawaki and Sekigawa [5]) *In an almost complex hyper-*

surface M of a K-space $\tilde{M}$, for a unit vector X tangent to M, we have

$$\tilde{H}(X) = H(X) + 2\{g(AX, X)^2 + g(FAX, X)^2\}$$

where $\tilde{H}(X)$ (resp. $H(X)$) is holomorphic sectional curvature in $\tilde{M}$ (resp. M).

LEMMA 2.6. *Let M be an almost complex hypersurface of a K-space $\tilde{M}$. If X and Y belong to $T_y(M)$ ($y \in U(x)$), then $(\tilde{\nabla}_X F)Y$ also belongs to $T_y(M)$.*

PROOF. By (2.1), from $\tilde{\nabla}_X(FY) = (\tilde{\nabla}_X F)Y + F\tilde{\nabla}_X Y$, we have

$$\begin{aligned} \nabla_X(FY) &+ h(X, FY)\xi + k(X, FY)F\xi \\ &= (\tilde{\nabla}_X F)Y + F\nabla_X Y + h(X, Y)F\xi - k(X, Y)\xi \,. \end{aligned} \tag{2.8}$$

On the other hand, from Lemma 2.4 (iii), we have $h(X, FY) = -k(X, Y)$ or $h(X, Y) = k(X, FY)$ and therefore (2.8) turns out to be

$$\nabla_X(FY) = (\tilde{\nabla}_X F)Y + F\nabla_X Y$$

from which we have

$$(\tilde{\nabla}_X F)Y = (\nabla_X F)Y \in T_y(M) \,. \tag{2.9}$$

Moreover, we prove the following lemma and proposition which we owe much to Prof. Tanno.

LEMMA 2.7. $s(X) + t(X) = 0$.

PROOF. By (2.3) $-$ F(2.2) we have

$$(\tilde{\nabla}_X F)\xi = (s(X) + t(X))\xi \,,$$

where we have used Lemma 2.4 (iii). Since $\tilde{\nabla}_X F$ is skew-symmetric with respect to g, we obtain $s(X) + t(X) = 0$.

PROPOSITION 2. *Let M be an almost complex hypersurface of a K-space $\tilde{M}$. If M is a Kähler manifold, then $\tilde{M}$ is also a Kähler manifold.*

PROOF. In the proof of Lemma 2.7, we have $(\tilde{\nabla}_X F)\xi = 0$. Similarly, $(\tilde{\nabla}_X F)F\xi = 0$. Since $\tilde{M}$ is a K-space we have $(\tilde{\nabla}_\xi F)X = (\tilde{\nabla}_{F\xi} F)X = 0$. By (2.9) we have $(\tilde{\nabla}_X F)Y = (\nabla_X F)Y = 0$, since M is Kählerian. Next we extend ξ to a vector field on a neighborhood of $y \in M$ in $\tilde{M}$. Then we have

$$(\tilde{\nabla}_\xi F)F\xi = -\tilde{\nabla}_\xi \xi - F(\tilde{\nabla}_\xi(F\xi)) = -F(\tilde{\nabla}_\xi F)\xi$$

from which it follows that $(\tilde{\nabla}_\xi F)F\xi = 0$ because $\tilde{M}$ is a K-space. Similarly $(\tilde{\nabla}_{F\xi} F)\xi = 0$. Therefore, $\tilde{\nabla} F = 0$ at y (or on its neighborhood in M). Thus, by (1.2) we have $\tilde{\rho} - \tilde{\rho}* = 0$ at y, and hence $\tilde{\nabla} F = 0$ on $\tilde{M}$.

The following corollary follows immediately from Proposition 2.

COROLLARY. *Let $\tilde{M}$ be a non-Kähler K-space and M an almost complex submanifold of $\tilde{M}$ which is a Kähler manifold. Then* $\dim M \leqq \dim \tilde{M} - 4$.

3. Almost complex hypersurfaces of a K-space with constant holomorphic sectional curvature. Recently Sawaki, Watanabe, and Sato [6] proved the following

LEMMA 3.1. *Let $\tilde{M}$ be a K-space of constant holomorphic sectional curvature $\tilde{c}$ and $\tilde{R}$ the Riemannian curvature tensor. Then we have*

$$\begin{aligned}\tilde{R}(X, Y, Z, W) & \\ &= \frac{1}{4}\tilde{c}\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + g(X, FZ)g(Y, FW) \\ &\quad - g(X, FW)g(Y, FZ) + 2g(X, FY)g(Z, FW)\} + \frac{1}{4}\{g((\tilde{\nabla}_X F)Z, (\tilde{\nabla}_Y F)W) \\ &\quad - g((\tilde{\nabla}_X F)W, (\tilde{\nabla}_Y F)Z) + 2g((\tilde{\nabla}_X F)Y, (\tilde{\nabla}_Z F)W)\}\end{aligned} \tag{3.1}$$

for any tangent vectors X, Y, Z, and W at every point of $\tilde{M}$.

From (3.1) it follows that

$$\begin{aligned}\tilde{R}(X, Y)W & \\ &= \frac{1}{4}\tilde{c}\{g(Y, W)X - g(X, W)Y + g(FY, W)FX - g(FX, W)FY \\ &\quad + 2g(X, FY)FW\} + \frac{1}{4}\{-(\tilde{\nabla}_X F)(\tilde{\nabla}_Y F)W + (\tilde{\nabla}_Y F)(\tilde{\nabla}_X F)W \\ &\quad + 2(\tilde{\nabla}_W F)(\tilde{\nabla}_X F)Y\}\,.\end{aligned} \tag{3.2}$$

In particular, let M be an almost complex hypersurface of a K-space $\tilde{M}$ and X, Y, W belong to $T_y(M)$ ($y \in U(x)$). Then, by Lemma 2.6, $(\tilde{\nabla}_X F)Y$ belongs to $T_y(M)$ and therefore $(\tilde{\nabla}_X F)(\tilde{\nabla}_Y F)W$ also belongs to $T_y(M)$. Hence, from (3.2) it follows that if X, Y, and W belong to $T_y(M)$, then $\tilde{R}(X, Y)W$ belongs to $T_y(M)$.

Thus, for an almost complex hypersurface M of a K-space $\tilde{M}$ with constant holomorphic sectional curvature $\tilde{c}$, from (2.6) we have

$$\begin{aligned}\tilde{R}(X, Y)W = R(X, Y)W &- \{h(Y, W)AX - h(X, W)AY\} \\ &- \{k(Y, W)BX - k(X, W)BY\} \\ &\text{for} \quad X, Y, W \in T_y(M),\ y \in U(x)\end{aligned}$$

or by (2.4), (2.5), and Lemma 2.4 (iii), this equation reduces to

$$\tilde{R}(X, Y)W = R(X, Y)W - \{g(AY, W)AX - g(AX, W)AY\} - \{g(FAY, W)FAX - g(FAX, W)FAY\} . \tag{3.3}$$

Now, from (3.2) it follows that the linear endomorphism $T_y(M)$ $(y \in M)$ determined by $X \rightarrow \tilde{R}(X, Y)W$ has the trace (for example see [7] p. 254)

$$\frac{\tilde{c}}{4}\{2ng(Y, W) - g(Y, W) + g(Y, W) + 2g(Y, W)\} + \frac{3}{4}\{\sigma(Y, W) - \sigma^*(Y, W)\}$$

where

$$\sigma(Y, W) = R_{ji}Y^jW^i , \ \sigma^*(Y, W) = R^*{}_{ji}Y^jW^i , \ Y = Y^j\frac{\partial}{\partial x^j} , \ W = W^j\frac{\partial}{\partial x^j} .$$

On the other hand, from (3.3), we see that the same linear endomorphism has the trace

$$\sigma(Y, W) + 2g(A^2Y, W)$$

where we have used the fact that M is minimal in $\tilde{M}$.

Thus, we have

$$\frac{\tilde{c}}{2}(n + 1)g(Y, W) + \frac{3}{4}(\sigma(Y, W) - \sigma^*(Y, W)\} = \sigma(Y, W) + 2g(A^2Y, W)$$

or with local components this equation can be written as

$$\frac{c}{2}(n + 1)g_{ji} + \frac{3}{4}(R_{ji} - R^*{}_{ji}) = R_{ji} + 2H_{jl}H_i^{\ l}$$

where

$$A(X) = X^lA_l^{\ j}\frac{\partial}{\partial x^j} , \quad H_{ji} = g_{il}A_j^{\ l} \quad \text{and} \quad H_j^{\ i} = g^{il}H_{jl} .$$

At the same time, from the coefficient of ξ in (2.6), we have

$$(\nabla_X h)(Y, W) - (\nabla_Y h)(X, W) + k(Y, W)t(X) - k(X, W)t(Y) = 0$$

or

$$(\nabla_X h)(Y, W) - (\nabla_Y h)(X, W) - h(Y, FW)t(X) + h(X, FW)t(Y) = 0 . \qquad \text{(Codazzi)}$$

Since $h(X, Y) = H_{ji}X^jY^i$, with local components Codazzi equation can be written as

$$\nabla_i H_{jk} - \nabla_j H_{ik} + t_jF_k^{\ l}H_{il} - t_iF_k^{\ l}H_{jl} = 0$$

where $t(X) = t_iX^i$.

Similarly, from the coefficient of $F\xi$, we have

$$(\nabla_X k)(Y, W) - (\nabla_Y k)(X, W) + h(Y, W)s(X) - h(X, W)s(Y) = 0 . \qquad \text{(Codazzi)}$$

Later, we shall see that these two Codazzi equations are equivalent (see Remark).

With local components, the second Codazzi equation can be written as

$$\nabla_i(F_k{}^l H_{jl}) - \nabla_j(F_k{}^l H_{il}) - s_i H_{jk} + s_j H_{ik} = 0$$

where $s(X) = s_i X^i$ and we have used that $k(Y, W) = -h(Y, FW) = -F_k{}^l H_{jl} Y^j W^k$.

Gathering these results, we have the following

LEMMA 3.2. *Let M be an almost complex hypersurface of a K-space $\tilde{M}$ with constant holomorphic sectional curvature $\tilde{c}$. Then we have*

$$\frac{\tilde{c}}{2}(n+1)g_{ji} - \frac{1}{4}(R_{ji} + 3R^*{}_{ji}) = 2H_{jl}H_i{}^l , \tag{3.5}$$

$$\nabla_i H_{jl} - \nabla_j H_{il} + t_j F_l{}^h H_{ih} - t_i F_l{}^h H_{jh} = 0 , \tag{3.6}$$

$$\nabla_i(F_k{}^l H_{jl}) - \nabla_j(F_k{}^l H_{il}) - s_i H_{jk} + s_j H_{ik} = 0 . \tag{3.7}$$

LEMMA 3.3. *Let M be an almost complex hypersurface of a K-space $\tilde{M}$ with constant holomorphic sectional curvature. Then we have*

$$H_{jl}\nabla_i F_k{}^l = 0 . \tag{3.8}$$

PROOF. By Lemma 2.7, we have $s_j + t_j = 0$. From (3.7) we have

$$F_k{}^l(\nabla_i H_{jl} - \nabla_j H_{il}) + H_{jl}\nabla_i F_k{}^l - H_{il}\nabla_j F_k{}^l - s_i H_{jk} + s_j H_{ik} = 0 . \tag{3.9}$$

Multiplying (3.6) by $F_k{}^l$ and making use of $t_j = -s_j$, we have

$$F_k{}^l(\nabla_i H_{jl} - \nabla_j H_{il}) + s_j H_{ik} - s_i H_{jk} = 0 .$$

Therefore, (3.9) reduces to

$$H_{jl}\nabla_i F_k{}^l = H_{il}\nabla_j F_k{}^l . \tag{3.10}$$

Now, in (3.10) $\nabla_i F_k{}^l$ is pure in i, k and $\nabla_j F_k{}^l$ is hybrid in k, l by virtue of (1.1). On the other hand, since Lemma 2.4 (i) means that H_{il} is pure in i, l, $H_{il}\nabla_j F_k{}^l$ is hybrid in i, k by virtue of Proposition 1 (ii). Thus, the left hand side of (3.10) is pure in i, k and at the same time the right hand side is hybrid in i, k. Hence, by Proposition 1 (iii), we have $H_{jl}\nabla_i F_k{}^l = 0$.

REMARK. By Lemma 2.7, from (3.9) and (3.10) it follows that (3.6) and (3.7) are equivalent.

4. Proof of Theorems.

PROOF OF THEOREM A. (i) implies (ii): by the assumption $A = 0$, from Lemma 2.5, we have

$$\tilde{c} = H(X) \quad \text{for any unit vector} \quad X \quad \text{tangent to} \quad M .$$

(ii) implies (iii): this follows immediately from Lemma 1.4.

(iii) implies (iv): making use of $R_{ji} = (\rho/2n)\, g_{ji}$ and $R^*{}_{ji} = (\rho^*/2n) g_{ji}$, from (3.5), we have

$$H_{jl}H_i^l = \left\{\frac{n+1}{4}\tilde{c} - \frac{1}{16n}\Big(\rho + 3\rho^*\Big)\right\} g_{ji}$$

or

$$H_{jl}H_i^l = \frac{n+1}{4}(\tilde{c} - c) g_{ji} \tag{4.1}$$

where

$$c = \frac{1}{4n(n+1)}(\rho + 3\rho^*) .$$

Multiplying (3.8) by $H_h{}^j$ and making use of (4.1), we have

$$\frac{n+1}{4}(\tilde{c} - c)\nabla_i F_{kh} = 0 .$$

In this place, we notice that $\tilde{c} - c =$ constant by virtue of (1.2). Hence if $\tilde{c} - c \neq 0$, then we have $\nabla_i F_{kh} = 0$. This contradicts the assumption that M is non-Kähler. Thus, we have $\tilde{c} = c$.

(iv) implies (i): transacting (3.5) with g^{ji}, we have

$$\tilde{c}n(n+1) - \frac{1}{4}(\rho + 3\rho^*) = 2H_{jl}H^{jl} \quad \text{i.e.,} \quad n(n+1)(\tilde{c} - c) = 2H_{jl}H^{jl}$$

from which it follows that $H_{jl} = 0$, that is, M is totally geodesic. q.e.d.

PROOF OF THEOREM B. By Lemmas 1.2 and 1.3, we have

$$R_{ji} = \frac{\rho}{6} g_{ji} , \qquad R^*{}_{ji} = \frac{\rho^*}{6} g_{ji} ,$$

because M must be 6-dimensional by virtue of Lemma 1.1. Thus, by Theorem A, we see that M is a K-space of constant holomorphic sectional curvature. Hence, Theorem B follows from Lemma 1.5.

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