

CURVATURE AND SPECTRUM OF RIEMANNIAN MANIFOLD

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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(Received April 15, 1973)

1. Introduction. Let M be an n -dimensional compact Riemannian manifold without boundary, and Δ the Laplacian acting on smooth p -forms on M , $0 \leq p \leq n$. Being elliptic, the operator Δ has an infinite sequence

$$0 \leq \lambda_{p,1} \leq \lambda_{p,2} \leq \dots \leq \lambda_{p,k} \leq \dots \uparrow \infty$$

of eigenvalues, each of which is repeated as many times as its multiplicity indicates, and the corresponding sequence $\{\phi_{p,k}\}_{k=1}^{\infty}$ of eigenforms forms a complete orthonormal set in the space of p -forms with Riemannian inner product. The series

$$\sum_{k=1}^{\infty} \exp(-\lambda_{p,k}t) \phi_{p,k}(x) \otimes \phi_{p,k}(y)$$

converges uniformly on compact subsets of $(0, \infty) \times M \times M$ to the fundamental solution $e^p(t, x, y)$ of the heat operator $\partial/\partial t + \Delta$ acting on p -forms, and we have,

$$(1) \quad \sum_{k=1}^{\infty} \exp(-\lambda_{p,k}t) = \int_M \text{Tr } e^p(t, x, x), \quad t > 0,$$

Tr denoting the trace. The geometric and probabilistic interpretation of $e^p(t, x, y)$ may be seen in Itô [2]. Moreover, we have the following Minakshisundaram's asymptotic expansion for $\text{Tr } e^p(t, x, x)$:

$$(2) \quad \text{Tr } e^p(t, x, x) \underset{t \rightarrow 0^+}{\sim} (4\pi t)^{-n/2} \{u_{p,0}(x) + tu_{p,1}(x) + \dots + t^k u_{p,k}(x) + \dots\},$$

where the coefficients $u_{p,k}(x)$ are local Riemannian invariants. Let $a_{p,k} = \int_M u_{p,k}$. Then by (1) and (2) we have,

$$(3) \quad \sum_{k=1}^{\infty} \exp(-\lambda_{p,k}t) \underset{t \rightarrow 0^+}{\sim} (4\pi t)^{-n/2} (a_{p,0} + ta_{p,1} + \dots + t^k a_{p,k} + \dots).$$

We call the sequence $\text{Spec}^p(M) = \{\lambda_{p,1}, \lambda_{p,2}, \dots, \lambda_{p,k}, \dots\}$ of eigenvalues the spectrum of M for p -forms. Many of the relations between spectra and geometric quantities of Riemannian manifolds have been obtained by cal-

culating the coefficients of the Minakshisundaram's expansion in terms of curvatures of M . In the present paper we shall calculate the coefficient $u_{1,3}$ in (2) for the Einsteinian case to obtain the following:

THEOREM. *Let M be a compact locally symmetric Einstein space and M' a compact Riemannian manifold. Suppose that M and M' have the same spectra for functions and for 1-forms respectively, i.e., $\text{Spec}^0(M) = \text{Spec}^0(M')$ and $\text{Spec}^1(M) = \text{Spec}^1(M')$. Then M' is also locally symmetric.*

I should like to express my hearty thanks to Professor T. Sakai who has kindly read through this manuscript to point out several errors.

2. Preliminaries. Let M be a Riemannian manifold with Riemannian metric \langle , \rangle and Levi-Civita connection ∇ . Let $R(X, Y)Z = \nabla_{[X, Y]}Z - [\nabla_X, \nabla_Y]Z$ and $\rho(X, Y) = \text{Tr}(Z \rightarrow R(X, Z)Y)$ be curvature tensor and Ricci tensor respectively. We denote; the covariant differentiation. Throughout this paper we use the Einstein's summation convention.

Let us fix a point m of M and choose a normal coordinate system (V, x^i) originated at the point m . We shall omit (m) for the quantities at the point m , for example,

$$R_{ijkl} = R_{ijkl}(m) = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle (m),$$

$$\rho_{ij} = \rho_{ij}(m) = \rho\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)(m).$$

Let $x \in V$ be a point with the coordinates (x^i) . Then $g_{ij}(x) = \langle \partial/\partial x^i, \partial/\partial x^j \rangle(x)$ is represented in terms of curvature tensor and its successive covariant derivatives at the point m , see Sakai [5].

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{kijl} x^k x^l + \frac{1}{3!} R_{kijl;p} x^k x^l x^p + \frac{1}{5!} \left(6R_{kijl;pq} \right. \\ \left. + \frac{16}{3} R_{kilm} R_{pjqu} \right) x^k x^l x^p x^q + \frac{1}{6!} (8R_{kijl;pqr} + 16R_{kilm} R_{pjqu;r} \\ (4) \quad + 16R_{kijlu} R_{piqu;r}) x^k x^l x^p x^q x^r + \frac{1}{7!} (10R_{kijl;pqrs} \\ + 34R_{kilm;pq} R_{rjsu} + 34R_{kijlu;pq} R_{risu} + 55R_{kilm;p} R_{qjru;s} \\ - 16R_{kilm} R_{pjqu} R_{ruvs}) x^k x^l x^p x^q x^r x^s + o(|x|^6).$$

From $g_{ij}(x)g^{jk}(x) = \delta_i^k$ and (4) we have,

$$(5) \quad g^{ij}(x) = \delta^{ij} + A_{ki}^{ij} x^k x^l + A_{klp}^{ij} x^k x^l x^p + A_{klpq}^{ij} x^k x^l x^p x^q \\ + A_{klpq;r}^{ij} x^k x^l x^p x^q x^r + o(|x|^5),$$

where

$$\begin{aligned}
 A_{kl}^{ij} &= -\frac{1}{3} R_{kijl} , \\
 A_{klp}^{ij} &= -\frac{1}{3!} R_{kijl;p} , \\
 A_{klpq}^{ij} &= \frac{1}{5!} (-6R_{kijl;pq} + 8R_{kilu}R_{pjqu}) , \\
 A_{klpqr}^{ij} &= \frac{1}{6!} (-8R_{kijl;pqr} + 24R_{kilu}R_{pjqu;r} + 24R_{kiju}R_{piqu;r}) .
 \end{aligned}$$

Then from

$$\Gamma_{ab}^c(x) = \frac{1}{2} g^{ci}(x) \left(\frac{\partial g_{ia}}{\partial x^b} + \frac{\partial g_{ib}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^i} \right) (x) ,$$

(4), (5), and Bianchi's identity we have,

$$\begin{aligned}
 (6) \quad \Gamma_{ab}^c(x) &= B_k^{cab} x^k + B_{kl}^{cab} x^k x^l + B_{klp}^{cab} x^k x^l x^p + B_{klpq}^{cab} x^k x^l x^p x^q \\
 &\quad + B_{klpqr}^{cab} x^k x^l x^p x^q x^r + o(|x|^5) ,
 \end{aligned}$$

where

$$\begin{aligned}
 B_k^{cab} &= -\frac{1}{3} (R_{kabc} + R_{kbac}) , \\
 B_{kl}^{cab} &= -\frac{1}{3!} \left\{ (R_{kabc;l} + R_{kbac;l}) + \frac{1}{2} (R_{kalc;b} + R_{kblic;a} - R_{kalb;c}) \right\} , \\
 B_{klp}^{cab} &= -\frac{1}{5!} \left\{ 6(R_{kabc;lp} + R_{kbac;lp}) + 3(R_{kalc;bp} + R_{kalc;pb}) \right. \\
 &\quad \left. + 3(R_{kblic;ap} + R_{kblic;pa}) - 3(R_{kalb;cp} + R_{kalb;pc}) \right. \\
 &\quad \left. + \frac{32}{3} R_{kclu} (R_{pabu} + R_{pbau}) - 8(R_{kalu}R_{bcpu} + R_{kbliu}R_{acpu}) \right\} , \\
 B_{klpq}^{cab} &= -\frac{1}{6!} \{ 8(R_{kabc;lpq} + R_{kbac;lpq}) + 4(R_{kalc;bpq} + R_{kalc;pbq} + R_{kalc;pqb}) \\
 &\quad + 4(R_{kblic;apq} + R_{kblic;pqa} + R_{kblic;pqa}) - 4(R_{kalb;cpq} + R_{kalb;peq} + R_{kalb;pqc}) \\
 &\quad + 44R_{kclu} (R_{pabu;q} + R_{pbau;q}) + 4R_{kclu} R_{paqb;u} \\
 &\quad - 32(R_{kalu}R_{bcpu;q} + R_{kbliu}R_{acpu;q}) \\
 &\quad - 24(R_{bcku}R_{lapu;q} + R_{acku}R_{lbpu;q}) + 32(R_{kabu} + R_{kba u})R_{lcpu;q} \} , \\
 B_{klpqr}^{cab} &= -\frac{1}{7!} \left[10(R_{kabc;lpqr} + R_{kbac;lpqr}) \right. \\
 &\quad \left. + 5(R_{kalc;bpqr} + R_{kalc;pbqr} + R_{kalc;pqbr} + R_{kalc;pqrb}) \right]
 \end{aligned}$$

$$\begin{aligned}
& + 5(R_{kblc;apqr} + R_{kblc;pqar} + R_{kblc;pqar} + R_{kblc;pqra}) \\
& - 5(R_{kalb;cpqr} + R_{kalb;pqcr} + R_{kalb;pqcr} + R_{kalb;pqrc}) \\
& - 51(R_{acku;lp}R_{qbrv} + R_{bcku;lp}R_{qaru} + R_{kalu;pq}R_{bcrv} + R_{kblu;pq}R_{acr u}) \\
& - 17R_{kalu}(R_{bcpu;qr} + R_{pcqu;rb} - R_{pbqu;rc}) \\
& - 17R_{kblu}(R_{acpu;qr} + R_{pcqu;ra} - R_{paqu;rc}) + 67(R_{kabu} + R_{kbbu})R_{lcpv;qr} \\
& + R_{kclu}\{67(R_{pabu;qr} + R_{pbau;qr}) + 25(R_{paqu;br} + R_{paqu;rb}) \\
& + 25(R_{pbqu;ar} + R_{pbqu;ra}) - 42(R_{paqb;ur} + R_{paqb;ru})\} \\
& - 110(R_{kalu;p}R_{bcqu;r} + R_{kblu;p}R_{acqu;r}) + 155R_{kalu;p}(R_{qabu;r} + R_{qbau;r}) \\
& - 15R_{kclu;p}R_{qabr;u} - 8R_{kalu}R_{pbqv}(R_{curv} + R_{crru}) \\
& + 24R_{kulv}(R_{acpu}R_{qbrv} + R_{bcpu}R_{qarv}) \\
& + R_{kclu}\left\{-104(R_{paqv}R_{burv} + R_{pbqv}R_{aurv}) + 8(R_{paqv}R_{brru} + R_{pbqv}R_{avru})\right. \\
& \left. + \frac{248}{3}R_{puqv}(R_{rabv} + R_{rbav})\right\}.
\end{aligned}$$

Let us define $g(x) = \det(g_{ij}(x))$ on V . Then we have,

$$(7) \quad g^{-1/4}(x) = 1 + G_{kl}x^kx^l + G_{klp}x^kx^lx^p + G_{klpq}x^kx^lx^px^q \\
+ G_{klpqr}x^kx^lx^px^qx^r + G_{klpqrs}x^kx^lx^px^qx^rs + o(|x|^6),$$

where

$$\begin{aligned}
G_{kl} &= \frac{1}{12}\rho_{kl}, \\
G_{klp} &= \frac{1}{24}\rho_{kl;p}, \\
G_{klpq} &= \frac{1}{4!}\left(\frac{3}{10}\rho_{kl;pq} + \frac{1}{12}\rho_{kl}\rho_{pq} + \frac{1}{15}R_{kulv}R_{puqv}\right), \\
G_{klpqr} &= \frac{1}{5!}\left(\frac{1}{3}\rho_{kl;pqr} + \frac{5}{12}\rho_{kl}\rho_{pq;r} + \frac{1}{3}R_{kulv}R_{puqv;r}\right), \\
G_{klpqrs} &= \frac{1}{6!}\left(\frac{5}{14}\rho_{kl;pqrs} + \frac{3}{4}\rho_{kl}\rho_{pq;rs} + \frac{4}{7}R_{kulv}R_{puqv;rs} + \frac{5}{8}\rho_{kl;p}\rho_{qr;s}\right. \\
& + \frac{15}{28}R_{kulv;p}R_{quvr;s} + \frac{5}{72}\rho_{kl}\rho_{pq}\rho_{rs} \\
& \left. + \frac{1}{6}\rho_{kl}R_{puqv}R_{rsv} + \frac{8}{63}R_{kulv}R_{pvqw}R_{rvsu}\right).
\end{aligned}$$

For the proof, see Sakai [5].

3. Coefficients $u_{1,k}$. Let $\sigma_x: [0, 1] \rightarrow M$ be the minimal geodesic joining

the points m and x . By the parallel translation the curve σ_x defines an isomorphism $P_{x,t}$ of the cotangent space $T_{\sigma_x(t)}^*(M)$ at $\sigma_x(t)$ onto the cotangent space $T_x^*(M)$ at x . For a cotangent vector $\alpha = \alpha(m) \in T_m^*(M)$, let us define

$$(8) \quad U_0(\alpha, x) = g^{-1/4}(x)P_{x,0}(\alpha) .$$

Then $U_0(\alpha, x)$ is a smooth 1-form in the neighborhood V , and we define $U_k(\alpha, x)$, $k \geq 1$, inductively as follows:

$$(9) \quad U_k(\alpha, x) = -g^{-1/4}(x) \int_0^1 g^{1/4}(\sigma_x(t))P_{x,t}(\Delta U_{k-1}(\alpha, \sigma_x(t)))t^{k-1}dt .$$

See Berger [1] and Patodi [4]. The map $\alpha \rightarrow U_k(\alpha, m)$ is a linear endomorphism on $T_m^*(M)$ and the coefficients $u_{1,k}(m)$ is given by $\text{Tr}(\alpha \rightarrow U_k(\alpha, m))$.

LEMMA 1. For a cotangent vector $\alpha = \alpha(m) = \alpha_i dx^i(m) \in T_m^*(M)$, $P_{x,0}(\alpha) = \alpha_i(x) dx^i$ is represented as follows:

$$(10) \quad \begin{aligned} \alpha_i(x) = & \alpha_i + \frac{1}{3!} \alpha_j R_{kijl} x^k x^l + \frac{2}{4!} \alpha_j R_{kijl;p} x^k x^l x^p \\ & + \frac{1}{5!} \alpha_j (3R_{kijl;pq} + R_{kilu} R_{pjqu}) x^k x^l x^p x^q \\ & + \frac{2}{6!} \alpha_j (2R_{kijl;pqr} + R_{kilu;p} R_{qjru} + 2R_{kilu} R_{pjqu;r}) x^k x^l x^p x^q x^r \\ & + \frac{1}{7!} \alpha_j (5R_{kijl;pqrs} + 10R_{kilu} R_{pjqu;rs} + 3R_{kilu} R_{pjqu;rs} \\ & + 10R_{kilu;p} R_{qjru;s} - R_{kilu} R_{pjqu} R_{ruvs}) x^k x^l x^p x^q x^r x^s + o(|x|^6) . \end{aligned}$$

PROOF. Since $P_{x,0}(\alpha)$ is parallel along σ_x , we have

$$(11) \quad \frac{d\alpha_i(\sigma_x(t))}{dt} - \alpha_k(\sigma_x(t)) \Gamma_{ij}^k(\sigma_x(t)) x^j = 0 .$$

On the other hand, by the Taylor's expansion,

$$(12) \quad \alpha_i(x) = \alpha_i + \frac{d\alpha_i(\sigma_x(t))}{dt} \Big|_{t=0} + \frac{1}{2!} \frac{d^2\alpha_i(\sigma_x(t))}{dt^2} \Big|_{t=0} + \dots .$$

The lemma follows from (11), (12), and (6).

Then from (7), (8), and (10), $U_0(\alpha, x) = (U_0(\alpha, x))_i dx^i$ is represented as follows:

$$(13) \quad \begin{aligned} (U_0(\alpha, x))_i = & E^i + E_{ki}^i x^k x^l + E_{klp}^i x^k x^l x^p + E_{klpq}^i x^k x^l x^p x^q \\ & + E_{klpqr}^i x^k x^l x^p x^q x^r + E_{klpqr}^i x^k x^l x^p x^q x^r x^s + o(|x|^6) , \end{aligned}$$

where

$$\begin{aligned}
E^i &= \alpha_i, \\
E_{kl}^i &= \frac{1}{3!} \left(\frac{1}{2} \alpha_i \rho_{kl} + \alpha_j R_{kijl} \right), \\
E_{klp}^i &= \frac{1}{4!} (\alpha_i \rho_{kl;p} + 2\alpha_j R_{kijl;p}), \\
E_{klpq}^i &= \frac{1}{5!} \left\{ \alpha_i \left(\frac{3}{2} \rho_{kl;pq} + \frac{5}{12} \rho_{kl} \rho_{pq} + \frac{1}{3} R_{kulu} R_{puqv} \right) \right. \\
&\quad \left. + \alpha_j \left(3R_{kijl;pq} + \frac{5}{3} \rho_{kl} R_{pijq} + R_{kilu} R_{pjqu} \right) \right\}, \\
E_{klpqr}^i &= \frac{1}{6!} \left\{ \alpha_i \left(2\rho_{kl;pqr} + \frac{5}{2} \rho_{kl} \rho_{pqr} + 2R_{kulu} R_{puqvr} \right) + \alpha_j \left(4R_{kijl;pqr} \right. \right. \\
&\quad \left. \left. + 4R_{kilu} R_{pjqu;r} + 5\rho_{kl} R_{pijq;r} + 5\rho_{kl;p} R_{qijr} + 2R_{kulu;p} R_{qjru} \right) \right\}, \\
E_{klpqrs}^i &= \frac{1}{7!} \left\{ \alpha_i \left(\frac{5}{2} \rho_{kl;pqrs} + \frac{21}{4} \rho_{kl} \rho_{pqrs} + 4R_{kulu} R_{puqvr;s} + \frac{35}{8} \rho_{kl;p} \rho_{qrs} \right) \right. \\
&\quad \left. + \frac{15}{4} R_{kulu;p} R_{quvr;s} + \frac{35}{72} \rho_{kl} \rho_{pq} \rho_{rs} \right. \\
&\quad \left. + \frac{7}{6} \rho_{kl} R_{puqv} R_{rusv} + \frac{8}{9} R_{kulu} R_{pvqu} R_{rwsu} \right) \\
&\quad \left. + \alpha_j \left(5R_{kijl;pqrs} + \frac{21}{2} \rho_{kl;pq} R_{rijs} + \frac{21}{2} \rho_{kl} R_{pijqrs} \right. \right. \\
&\quad \left. \left. + 10R_{kilu} R_{pjqu;rs} + 3R_{kiju} R_{piqu;rs} \right. \right. \\
&\quad \left. \left. + \frac{35}{2} \rho_{kl;p} R_{qijr;s} + 10R_{kulu;p} R_{qjru;s} + \frac{35}{12} \rho_{kl} \rho_{pq} R_{rijs} \right. \right. \\
&\quad \left. \left. + \frac{7}{2} \rho_{kl} R_{piqu} R_{rjsu} + \frac{7}{3} R_{kulu} R_{puqv} R_{rijs} - R_{kilu} R_{pjqu} R_{rusv} \right) \right\}.
\end{aligned}$$

4. Calculation. From now on we assume that M is a compact Einstein space, i.e., $\rho(X, Y) = c\langle X, Y \rangle$, $c \in R$. Then the Laplacian Δ acting on 1-forms is locally represented as follows:

$$\begin{aligned}
(14) \quad (\Delta\omega)_h(x) &= -g^{ij}(x)\omega_{h;ij}(x) + c\omega_h(x) \\
&= -g^{ij}(x) \left\{ \frac{\partial^2 \omega_h(x)}{\partial x^i \partial x^j} - \omega_u(x) \frac{\Gamma_{hj}^u(x)}{\partial x^i} \right. \\
&\quad \left. - \frac{\partial \omega_u(x)}{\partial x^i} \Gamma_{hj}^u(x) - \frac{\partial \omega_u(x)}{\partial x^j} \Gamma_{hi}^u(x) - \frac{\partial \omega_h(x)}{\partial x^u} \Gamma_{ij}^u(x) \right. \\
&\quad \left. + \omega_u(x) (\Gamma_{vj}^u(x) \Gamma_{hi}^v(x) + \Gamma_{vh}^u(x) \Gamma_{ij}^v(x)) \right\} + c\omega_h(x),
\end{aligned}$$

for a 1-form $\omega(x) = \omega_h(x)dx^h$. Therefore, from (5), (6), (13), and (14), $\Delta U_0(\alpha, x) = (\Delta U_0(\alpha, x))_h dx^h$ is given as follows:

$$(15) \quad (\Delta U_0(\alpha, x))_h = S^h + S^h_k x^k x^l + S^h_{kl} x^k x^l x^p + S^h_{klpq} x^k x^l x^p x^q + o(|x|^4),$$

where

$$\begin{aligned} S^h &= H^{hii} + cE^h, \\ S^h_{kl} &= H^{hii}_{kl} + A^{ij}_{kl} H^{hij} + cE^h_{kl}, \\ S^h_{klpq} &= H^{hii}_{klpq} + A^{ij}_{kl} H^{hij}_{pq} + A^{ij}_{klp} H^{hij}_q + A^{ij}_{klpq} H^{hij} + cE^h_{klpq}, \\ H^{hij} &= E^u B^{uhj}_i - (E^h_{ij} + E^h_{ji}), \\ H^h_{kl} &= E^u (B^{uhj}_{ik} + B^{uhj}_{ki}) - Q^h_{ijk}, \\ H^{hij}_{kl} &= E^u (B^{uhj}_{kli} + B^{uhj}_{kil} + B^{uhj}_{kli} - B^{uvj}_k B_l^{vhi} - B_k^{uvh} B_l^{vij}) + (E^u_{ik} + E^u_{ki}) B_l^{uhj} \\ &\quad + (E^u_{jk} + E^u_{kj}) B_l^{uhi} + (E^h_{uk} + E^h_{ku}) B_l^{uij} + E^u_{kl} B_i^{uhj} - Q^h_{ijkl}, \\ H^{hij}_{klpq} &= E^u (B^{uhj}_{iklpq} + B^{uhj}_{kilpq} + B^{uhj}_{klij} + B^{uhj}_{klpiq} + B^{uhj}_{klpqi} - B^{uvj}_k B_{lpq}^{vhi} \\ &\quad - B^{uvj}_k B_{pq}^{vhi} - B^{uvj}_k B_q^{vhi} - B_k^{uvh} B_{lpq}^{vij} - B_{kl}^{uvh} B_{pq}^{vij} - B_{klp}^{uvh} B_q^{vij}) \\ &\quad + E^u (B^{uhj}_{ipq} + B^{uhj}_{piq} + B^{uhj}_{pqi} - B_p^{uhj} B_q^{vhi} - B_p^{uvh} B_q^{vij}) \\ &\quad + (E^u_{ik} + E^u_{ki}) B_{lpq}^{uhj} + (E^u_{jk} + E^u_{kj}) B_{lpq}^{uhi} + (E^h_{uk} + E^h_{ku}) B_{lpq}^{uij} \\ &\quad + (E^u_{ikl} + E^u_{kil} + E^u_{kli}) B_{pq}^{uhj} + (E^u_{jkl} + E^u_{kjl} + E^u_{klj}) B_{pq}^{uhi} \\ &\quad + (E^h_{ukl} + E^h_{kul} + E^h_{klu}) B_{pq}^{uij} + E^u_{klp} (B_{iq}^{uhj} + B_{qi}^{uhj}) \\ &\quad + (E^u_{iklp} + E^u_{kilp} + E^u_{kli p} + E^u_{klpi}) B_q^{uhj} \\ &\quad + (E^u_{jklp} + E^u_{kjl p} + E^u_{kljp} + E^u_{klpj}) B_q^{uhi} \\ &\quad + (E^h_{uklp} + E^h_{kulp} + E^h_{klup} + E^h_{klpu}) B_q^{uij} + E^u_{klpq} B_i^{uhj} - Q^h_{ijklpq}, \\ Q^h_{ijk} &= E^h_{ijk} + E^h_{jik} + E^h_{jki} + E^h_{kij} + E^h_{kji} + E^h_{kji}, \\ Q^h_{ijkl} &= E^h_{ijkl} + E^h_{jikl} + E^h_{jkil} + E^h_{jkl i} + E^h_{ikjl} + E^h_{kijl} \\ &\quad + E^h_{kji l} + E^h_{kjl i} + E^h_{iklj} + E^h_{kijl} + E^h_{kl ij} + E^h_{klji}, \\ Q^h_{ijklpq} &= E^h_{ijklpq} + E^h_{jiklpq} + E^h_{jkilpq} + E^h_{jklipq} + E^h_{jklpiq} + E^h_{jklpqi} \\ &\quad + E^h_{ikjlpq} + E^h_{kijlpq} + E^h_{kijlpq} + E^h_{kjlipq} + E^h_{kjlpiq} + E^h_{kjl pqi} \\ &\quad + E^h_{iklj pq} + E^h_{klijpq} + E^h_{klijpq} + E^h_{kljipq} + E^h_{kljpiq} + E^h_{klj pqi} \\ &\quad + E^h_{iklpjq} + E^h_{ktilpq} + E^h_{ktilpq} + E^h_{klpijq} + E^h_{klpijq} + E^h_{klpijq} \\ &\quad + E^h_{iklpqj} + E^h_{ktilpqj} + E^h_{ktilpqj} + E^h_{klpiqj} + E^h_{klpiqj} + E^h_{klpiqj}. \end{aligned}$$

By the parallel translation along σ_x we have,

$$(P_{x,t}(\Delta U_0(\alpha, \sigma_x(t))))_h = S^h + \left(\frac{1-t^2}{2} S^u B_l^{uhk} + t^2 S^h_k \right) x^k x^l$$

$$\begin{aligned}
 (16) \quad & + \left(\frac{1-t^3}{3} S^u B_{lp}^{u,h,k} + t^3 S_{klp}^h \right) x^k x^l x^p \\
 & + \left\{ \frac{(1-t^2)^2}{8} S^u B_i^{u,v,k} B_q^{v,h,p} + \frac{1-t^4}{4} S^u B_{lpq}^{u,h,k} \right. \\
 & \left. + \frac{t^2(1-t^2)}{2} S_{kl}^u B_q^{u,h,p} + t^4 S_{klpq}^h \right\} x^k x^l x^p x^q + o(|x|^4).
 \end{aligned}$$

Therefore, by (9), (7), and (16), $U_1(\alpha, x) = (U_1(\alpha, x))_h dx^h$ is represented as follows:

$$(17) \quad (U_1(\alpha, x))_h = T^h + T_{kl}^h x^k x^l + T_{klp}^h x^k x^l x^p + T_{klpq}^h x^k x^l x^p x^q + o(|x|^4),$$

where

$$\begin{aligned}
 T^h &= -S^h, \\
 T_{kl}^h &= -\frac{1}{3} (S^u B_l^{u,h,k} + S_{kl}^h + 2S^h G_{kl}), \\
 T_{klpq}^h &= -\frac{1}{15} (S^u B_l^{u,v,k} B_q^{v,h,p} + 3S^u B_{lpq}^{u,h,k} + S_{kl}^u B_q^{u,h,p} + 3S_{klpq}^h \\
 &\quad + 4S^u B_l^{u,h,k} G_{pq} + 2S_{kl}^h G_{pq} - 2S^h G_{kl} G_{pq} + 12S^h G_{klpq}).
 \end{aligned}$$

Similarly $U_2(\alpha, x) = (U_2(\alpha, x))_h dx^h$ is represented as follows:

$$(18) \quad (U_2(\alpha, x))_h = F^h + F_k^h x^k + F_{kl}^h x^k x^l + o(|x|^2),$$

where

$$\begin{aligned}
 F^h &= -\frac{1}{2} (T^u B_i^{u,h,i} - 2T_{ii}^h + cT^h), \\
 F_{kl}^u &= -\frac{1}{4} \left\{ T^u (B_{ikl}^{u,h,i} + B_{kii}^{u,h,i} + B_{kli}^{u,h,i} - B_k^{u,vi} B_l^{vhi} - B_k^{u,vh} B_i^{vii}) \right. \\
 &\quad + 2(T_{ik}^u + T_{ki}^u) B_i^{u,h,i} + (T_{uk}^h + T_{ku}^h) B_i^{u,ii} + T_{ki}^u B_i^{u,h,i} \\
 &\quad - 2(T_{iikl}^h + T_{ikil}^h + T_{ikli}^h + T_{kiii}^h + T_{kili}^h + T_{klli}^h) \\
 &\quad + T^u A_{kl}^{ij} B_i^{u,h,j} - (T_{ij}^h + T_{ji}^h) A_{kl}^{ij} + cT_{kl}^h \\
 &\quad \left. + (T^u B_i^{u,h,i} - 2T_{ii}^h + cT^h) G_{kl} + \frac{1}{2} (T^u B_i^{u,vi} - 2T_{ii}^v + cT^v) B_i^{v,h,k} \right\}.
 \end{aligned}$$

Finally $U_3(\alpha, m) = (U_3(\alpha, m))_h dx^h(m)$ is represented as follows:

$$\begin{aligned}
 (19) \quad (U_3(\alpha, m))_h &= -\frac{1}{6} \left\{ T^u (B_{ijj}^{u,h,i} + B_{jij}^{u,h,i} + B_{jji}^{u,h,i} - B_j^{u,vi} B_j^{vhi} - B_j^{u,vh} B_j^{vii}) \right. \\
 &\quad + 2(T_{ij}^u + T_{ji}^u) B_j^{u,h,i} + (T_{uj}^h + T_{ju}^h) B_j^{u,ii} + T_{jj}^u B_j^{u,h,i} \\
 &\quad \left. - 4(T_{iijj}^h + T_{ijij}^h + T_{ijji}^h) + T^u A_{kk}^{ij} B_j^{u,h,i} \right\}.
 \end{aligned}$$

$$\begin{aligned}
 & - 2T_{ij}^h A_{kk}^{ij} + cT_{ii}^h - \frac{1}{2} (T^u B_i^{uv} - 2T_{ii}^v + cT^v) B_j^{vh} \\
 & + (T^u B_i^{uhi} - 2T_{ii}^h + cT^h) G_{jj} - c(T^u B_i^{uhi} - 2T_{ii}^h + cT^h) \}.
 \end{aligned}$$

For the calculation of $u_{1,3}(m) = \text{Tr}(\alpha \rightarrow U_3(\alpha, m))$ we use the formulas (2.1) through (2.11) in Sakai [5] and the following:

$$(20) \quad R_{abcd} R_{aucv} R_{budv} = \frac{1}{2} c |R|^2 - \frac{1}{4} R_{abcd} R_{abuv} R_{cduv} + \frac{1}{4} |\nabla R|^2 - \frac{1}{8} \Delta |R|^2,$$

$$(21) \quad R_{abcd} R_{aucv} R_{bvdu} = \frac{1}{2} c |R|^2 - \frac{1}{2} R_{abcd} R_{abuv} R_{cduv} + \frac{1}{4} |\nabla R|^2 - \frac{1}{8} \Delta |R|^2,$$

which are easily derived from the Ricci's identity in the Einsteinian case. Thus we have,

$$\begin{aligned}
 (22) \quad u_{1,3}(m) &= \frac{1}{6!} \left\{ \left(\frac{5}{18} n^4 - \frac{76}{9} n^3 + \frac{572}{9} n^2 - \frac{1172}{9} n \right) c^3 \right. \\
 &+ \left(\frac{1}{9} n^2 - \frac{29}{3} n + \frac{632}{9} \right) c |R|^2 - 5R_{abcd} R_{abuv} R_{cduv} \\
 &\left. + \frac{5}{3} |\nabla R|^2 - \frac{5}{6} \Delta |R|^2 \right\} + \frac{2}{3} \text{Tr}(\alpha_h \rightarrow T_{ijj}^h + T_{ijij}^h + T_{ijji}^h),
 \end{aligned}$$

$$\begin{aligned}
 (23) \quad & \text{Tr}(\alpha_h \rightarrow T_{ijj}^h + T_{ijij}^h + T_{ijji}^h) \\
 &= \frac{1}{6!} \left\{ \left(\frac{1}{3} n^4 - \frac{38}{15} n^3 + \frac{8}{15} n^2 + \frac{176}{15} n \right) c^3 \right. \\
 &+ \left(\frac{8}{15} n^2 - \frac{16}{5} n - \frac{146}{15} \right) c |R|^2 + 3R_{abcd} R_{abuv} R_{cduv} - |\nabla R|^2 \\
 &\left. + \frac{1}{2} \Delta |R|^2 \right\} - \frac{1}{5} \text{Tr}(\alpha_h \rightarrow S_{ijj}^h + S_{ijij}^h + S_{ijji}^h),
 \end{aligned}$$

$$\begin{aligned}
 (24) \quad & \text{Tr}(\alpha_h \rightarrow S_{ijj}^h + S_{ijij}^h + S_{ijji}^h) \\
 &= \frac{1}{6!} \left\{ \left(-\frac{5}{6} n^3 + 13n^2 - \frac{2}{3} n \right) c^3 + (27n + 69) c |R|^2 \right. \\
 &\left. + 90R_{abcd} R_{abuv} R_{cduv} \right\} + \text{Tr}(\alpha_h \rightarrow H_{ijj}^{huv} + H_{ijij}^{huv} + H_{ijji}^{huv}),
 \end{aligned}$$

$$\begin{aligned}
 (25) \quad & \text{Tr}(\alpha_h \rightarrow H_{ijj}^{huv}) = \frac{1}{6!} \left\{ \left(-\frac{5}{12} n^4 + \frac{26}{9} n^3 + \frac{442}{63} n^2 - \frac{76}{3} n \right) c^3 \right. \\
 &+ \left(-n^2 + \frac{10}{21} n + \frac{121}{7} \right) c |R|^2 + \left(-\frac{4}{7} n - \frac{405}{14} \right) R_{abcd} R_{abuv} R_{cduv}
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{30}{7}n + \frac{20}{7} \right) |\nabla R|^2 + \left(-\frac{15}{7}n - \frac{10}{7} \right) \Delta |R|^2 \Big\} , \\
 \text{Tr} (\alpha_h \rightarrow H_{ijij}^{huv}) & = \frac{1}{6!} \left\{ \left(-\frac{5}{12}n^3 - \frac{46}{3}n^2 - \frac{26}{3}n \right) c^3 \right. \\
 (26) \quad & + \left(-\frac{1}{3}n^2 - 8n + \frac{437}{21} \right) c |R|^2 + \left(-\frac{8}{7}n + 20 \right) R_{abcd}R_{abuv}R_{cduv} \\
 & \left. + \left(-\frac{95}{84}n - \frac{529}{84} \right) |\nabla R|^2 + \left(-\frac{50}{21}n + \frac{5149}{168} \right) \Delta |R|^2 \right\} ,
 \end{aligned}$$

$$\begin{aligned}
 \text{Tr} (\alpha_h \rightarrow H_{ijji}^{huv}) & = \frac{1}{6!} \left\{ \left(\frac{85}{36}n^3 - \frac{31}{9}n^2 + \frac{50}{3}n \right) c^3 \right. \\
 (27) \quad & + \left(-\frac{1}{6}n^2 - \frac{85}{42}n - \frac{1031}{21} \right) c |R|^2 + \left(-\frac{8}{7}n - \frac{410}{7} \right) R_{abcd}R_{abuv}R_{cduv} \\
 & \left. + \left(-\frac{65}{28}n - \frac{341}{84} \right) |\nabla R|^2 + \left(-\frac{5}{7}n + \frac{5801}{168} \right) \Delta |R|^2 \right\} .
 \end{aligned}$$

From (22) through (27) we have,

$$\begin{aligned}
 u_{1,3}(n) & = \frac{1}{6!} \left\{ \left(\frac{5}{9}n^4 - \frac{32}{3}n^3 + \frac{4016}{63}n^2 - 120n \right) c^3 \right. \\
 (28) \quad & + \left(\frac{2}{3}n^2 - \frac{890}{63}n + 56 \right) c |R|^2 + \left(\frac{8}{21}n - 6 \right) R_{abcd}R_{abuv}R_{cduv} \\
 & \left. + \left(-\frac{1}{9}n + 2 \right) |\nabla R|^2 + \left(\frac{44}{63}n - 9 \right) \Delta |R|^2 \right\} .
 \end{aligned}$$

Integrating $u_{1,3}$ over M , we have our main lemma.

LEMMA 2. *Let M be an n -dimensional compact Einstein space of constant scalar curvature nc . Then the coefficient $a_{1,3}$ in (3) for $p = 1$ is given as follows:*

$$\begin{aligned}
 a_{1,3} & = \frac{1}{6!} \left\{ \left(\frac{5}{9}n^4 - \frac{32}{3}n^3 + \frac{4016}{63}n^2 - 120n \right) c^3 \text{vol} (M) \right. \\
 (29) \quad & + \left(\frac{2}{3}n^2 - \frac{890}{63}n + 56 \right) c \int_M |R|^2 \\
 & \left. + \left(\frac{8}{21}n - 6 \right) \int_M R_{abcd}R_{abuv}R_{cduv} + \left(-\frac{1}{9}n + 2 \right) \int_M |\nabla R|^2 \right\} .
 \end{aligned}$$

For the proof of our theorem, we employ the following two lemmas.

LEMMA 3. (V. K. Patodi) *Let M be an n -dimensional compact Einstein space of constant scalar curvature nc , and M' a compact Riemannian manifold. Suppose that*

$$\text{Spec}^0(M) = \text{Spec}^0(M') \quad \text{and} \quad \text{Spec}^1(M) = \text{Spec}^1(M'),$$

then M' is also an n -dimensional Einstein space of constant scalar curvature nc .

For the proof, see Patodi [3]. Under the same condition as Lemma 3, the following are well known:

$$(30) \quad \text{vol}(M) = \text{vol}(M'),$$

$$(31) \quad \int_M |R|^2 = \int_{M'} |R'|^2.$$

LEMMA 4. (T. Sakai) *For an n -dimensional compact Einstein space M of constant scalar curvature nc , the coefficient $a_{0,3}$ in (3) for $p = 0$ is given as follows:*

$$(32) \quad a_{0,3} = \frac{1}{6!} \left\{ \left(\frac{5}{9} n^3 - \frac{2}{3} n^2 - \frac{16}{63} n \right) c^3 \text{vol}(M) + \left(\frac{2}{3} n - \frac{8}{63} \right) c \int_M |R|^2 \right. \\ \left. + \frac{8}{21} \int_M R_{abcd} R_{abuv} R_{cduv} - \frac{1}{9} \int_M |\nabla R|^2 \right\}.$$

For the proof, see Sakai [5]. From Lemma 3, (30), (31), and Lemma 4, we have,

$$\int_M |\nabla R|^2 = \int_{M'} |\nabla R'|^2,$$

which completes the proof of our theorem.

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