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A NOTE ON THE DECOMPOSITION OF WILLE INCIDENCE GEOMETRY OF GRADE n

Dedicated to professor Shigeo Sasaki on his 60th birthday

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1. Preliminary and summary. Let p, q be two points in a lattice L with 0, then p is said to be perspective to q (in symbol $p \sim q$) if there exists an element $x \in L$ such that $q \leqq p + x$ and $qx = 0$ (F. Maeda [1]).

For a matroid lattice L, U. Sasaki and S. Fujiwara [2] have proved that

(1) $p \sim q$ is an equivalence relation,

(2) L is irreducible if and only if any two points of L are perspective to each other,

(3) L is a direct union of irreducible matroid lattices.

On the other hand, it was proved by R. Wille [3] that a lattice $\mathscr{L}$ is isomorphic to the lattice of subspaces of a Wille geometry of grade n if and only if the lattice $\mathscr{L}$ is matroid, and moreover for each element x of rank n the interval $[0\ x]$ is distributive and the interval $[x\ 1]$ is modular.

Thus, the facts (2) and (3) above can be applied for the study of the decomposition of the lattice of subspaces of a Wille geometry of order n.

Actually, in the case of projective geometry of infinite dimension (the special case of Wille geometry of grade $n = 0$), the perspectivity has more concrete geometrical interpretation, as can be shown easily by using the so-called join theorem, that *two distinct points p, q are perspective ($p \sim q$) if and only if the line pq is not degenerate*, that is the line pq contains at least three distinct points.

Under such interpretation of *perspectivity* in projective geometry, (2) and (3) reduce respectively to the following propositions:

(2′) A projective space (or an atomic, upper-continuous, complemented modular lattice) is irreducible if and only if it does not contain any degenarate line.

(3′) Any atomic, upper-continuous complemented modular lattice is a direct union of irreducible sublattices.

(3′) was proved by O. Frink [4] and (2′) was given by him as a definition.

It is intended in this note to prove the following analogous rather concrete geometrical interpretation of perspectivity and related results for the Wille geometry of grade n:

Let $\mathscr{L}$ be a matroid lattice such that for any n distinct points $\{p_1, \cdots, p_n\}$, $[0\ p_1 + \cdots + p_n]$ is distributive and $[p_1 + \cdots + p_n\ 1]$ is modular. In such a lattice $\mathscr{L}$, $p_1 + \cdots + p_{n+1}$ is called a *curve* if the $n + 1$ points $\{p_1, \cdots, p_n, p_{n+1}\}$ are distinct, and $p_1 + \cdots + p_{n+1} + p_{n+2}$ is called a *surface* if the $(n + 2)$ points $\{p_1, \cdots, p_{n+2}\}$ are distinct and $p_1 + \cdots + p_{n+2}$ is not contained in a curve. Here a point r is said to be contained in a curve $p_1 + \cdots + p_{n+1}$ if $p \leqq p_1 + \cdots + p_{n+1}$, and a curve $p_1 + \cdots + p_{n+1}$ is said to be contained in a surface $q_1 + \cdots + q_{n+2}$ if $p_1 + \cdots + p_{n+1} \leqq q_1 + \cdots + q_{n+2}$, and so forth.

THEOREM 1. *Two distinct points p, q in $\mathscr{L}$ are perspective ($p \sim q$) if and only if for any $n + 2$ distinct points $\{p, q, p_1, \cdots, p_n\}$ either they are contained in a curve or the surface determined by these points contains another point r distinct from these points and such that p, q are not contained in the curve determined by $\{p_1, \cdots, p_n, r\}$.*

COROLLARY. *If $\mathscr{L}$ (stated above) is irreducible, then every surface contains at least $(n + 3)$ distinct points. Converse does not hold generally.*

THEOREM 2. *The lattice $\mathscr{L}$ (stated above) is irreducible if and only if the sublattice $[p_1 + \cdots + p_n\ 1]$ is irreducible for any set of n distinct points $\{p_1, \cdots, p_n\}$.*

2. **Proofs of the results.** We need two lemmas for the proof of Theorem 1. Since $[0\ p_1 + \cdots + p_n]$ is distributive, it follows that $p_1 + \cdots + p_n$ contains only these n distinct points $\{p_1, \cdots, p_n\}$. Hence, if $\{p_1, \cdots, p_n, p_{n+1}\}$ are $n + 1$ distinct points, then $p_{n+1} \nleqq p_1 + \cdots + p_n$, and by the semi-modularity of $\mathscr{L}$, $p_1 + \cdots + p_n \prec p_1 + \cdots + p_n + p_{n+1}$. Then it follows:

LEMMA 1. *If $b \in \mathscr{L}$ contains at least n distinct points $\{p_1, \cdots, p_n\}$ and point $p \nleqq b$, then for any point $q \leqq p + b$, there is a point $r \leqq b$ such that $q \leqq p + r + p_1 + \cdots + p_n$.* (C. J. Hsu [5]).

LEMMA 2. *Let $q, q_1, \cdots, q_{n+1}, p_1$ be $n + 3$ distinct points such that $q + q_1 + \cdots + q_{n+1}$ is a surface. Then $q \leqq q_1 + \cdots + q_{n-1} + q_n + p_1$* and *$q \leqq q_1 + \cdots + q_{n-1} + q_{n+1} + p_1$ do not hold simultaneously.*

PROOF. Suppose the contrary that these two relations hold simultaneously, then $q_1 + \cdots + q_{n-1} + p_1 + q \leqq q_1 + \cdots + q_{n-1} + q_n + p_1$, $q_1 + \cdots + q_{n-1} + q_{n+1} + p_1$. Since $q, q_1, \cdots, q_{n-1}, p$ are distinct, $q \nleqq q_1 + \cdots + q_{n-1} + p_1$. Hence, by semi-modularity, $q_1 + \cdots + q_{n-1} + p_1 \prec q_1 + \cdots + q_{n-1} + p_1 + q$, and $q_1 + \cdots + q_{n-1} + p_1 \prec q_1 + \cdots + q_{n-1} + q_n + p_1$. Hence, $q_1 + \cdots + q_{n-1} + p_1 + q = q_1 + \cdots + q_{n-1} + p_1 + q_n$. Similarly, $q_1 + \cdots + q_{n-1} + p_1 + q = q_1 + \cdots + q_{n-1} + p_1 + q_{n+1}$. Thus, $q, q_1, \cdots, q_{n+1}, p_1$ are contained in the curve $q_1 + \cdots + q_{n+1}$, contradictory to $q \not\leqq q_1 + \cdots + q_{n+1}$.

PROOF OF THEOREM 1. Suppose that p, q are distinct and that $p \sim q$. Let $\{p, q, p_1, \cdots, p_n\}$ be any $n+2$ distinct points. Since $p \sim q$, there exists an element $x \in \mathscr{L}$ such that $q \leqq p + x$ and $q \not\leqq x$ (hence $p \not\leqq x$). If x contains at most $n-1$ distinct points, then $p + x$ contains at most n distinct points and $q \leqq p + x$, $p \neq q$ imply that q must coincide with a point contained in x contradicting $q \not\leqq x$. Thus x contains at least n distinct points.

If x contains exactly n distinct points $\{p_1, \cdots, p_n\}$, then $x = p_1 + \cdots + p_n$, and $q \leqq p + p_1 + \cdots + p_n$. Hence, $q + p_1 + \cdots + p_n \leqq p + p_1 + \cdots + p_n$. Since $p_1 + \cdots + p_n \prec q + p_1 + \cdots + p_n$, $p + p_1 + \cdots + p_n$, it follows that $p + p_1 + \cdots + p_n = q + p_1 + \cdots + p_n$ and $\{p, q, p_1, \cdots, p_n\}$ are contained in a curve.

If x containes at least $n+1$ distinct points, then by the above Lemma 1, there exist $q_1, \cdots, q_{n+1} \leqq x$ such that $q \leqq p + q_1 + \cdots + q_{n+1}$ and $q \not\leqq q_1 + \cdots + q_{n+1}$.

Now if a) $p_1 \leqq q_1 + \cdots + q_{n+1}$, then by the above Lemma 1, there exist $n+1$ distinct points $p_1, q'_1, \cdots, q'_n \leqq q_1 + \cdots + q_{n+1}$ such that $q \leqq p + p_1 + q'_1 + \cdots + q'_n$ and $q \not\leqq p_1 + q'_1 + \cdots + q'_n$.

Suppose next that b) $p_1 \not\leqq q_1 + \cdots + q_{n+1}$, then $q \leqq p + q_1 + \cdots + q_{n+1} + p_1$.

If b_1) $q \not\leqq q_1 + \cdots + q_{n+1} + p_1$, then by the Lemma 1 again, there exist $n+1$ distinct points $p_1, q'_1, \cdots, q'_n \leqq q_1 + \cdots q_{n+1} + p_1$ such that $q \leqq p + p_1 + q'_1 + \cdots + q'_n$ and $q \not\leqq p_1 + q'_1 + \cdots + q'_n$.

If b_2) $q \leqq q_1 + \cdots + q_{n+1} + p_1$, then $q_1 + \cdots + q_{n+1} \prec q + q_1 + \cdots + q_{n+1} \leqq p_1 + q_1 + \cdots + q_{n+1}$, but $q_1 + \cdots + q_{n+1} \prec p_1 + q_1 + \cdots + q_{n+1}$. Hence $q + q_1 + \cdots + q_{n+1} = p_1 + q_1 + \cdots + q_{n+1}$. Similarly $q + q_1 + \cdots + q_{n+1} = p + q_1 + \cdots + q_{n+1}$. Since $q, p, p_1, q_1, \cdots, q_{n+1}$ are distinct, $p + p_1 + q_1 + \cdots + q_{n-1}$ is a curve. By semi-modularity, we have either $\text{b}_{2\text{a}}$) $p + p_1 + q_1 + \cdots + q_{n-1} + q_n = p + p_1 + q_1 + \cdots + q_{n-1}$ or $\text{b}_{2\text{b}}$) $p + p_1 + q_1 + \cdots + q_{n-1} + q_n \succ p + p_1 + q_1 + \cdots + q_{n-1}$.

For the case $\text{b}_{2\text{a}}$), $q \leqq p + p_1 + q_1 + \cdots + q_{n-1} + q_n + q_{n+1} = p + p_1 + q_1 + \cdots + q_{n-1} + q_{n+1}$.

In the case $\text{b}_{2\text{b}}$) we have $p + p_1 + q_1 + \cdots + q_{n-1} + q_n + q_{n+1} \geqq p + p_1 + q_1 + \cdots + q_{n-1} + q_{n+1}$. If the "=" holds, then we have the same result as in $\text{b}_{2\text{a}}$). If ">" holds, then since $p + p_1 + q_1 + \cdots + q_{n-1} + q_n + q_{n+1} = p + q_1 + \cdots + q_{n+1}$ is a surface, $p + p_1 + q_1 + \cdots + q_{n-1} +$

q_{n+1} is a curve, hence $p + p_1 + q_1 + \cdots + q_{n-1} = p + p_1 + q_1 + \cdots + q_{n-1} + q_{n+1}$. From this, it follows that $p + p_1 + q_1 + \cdots + q_{n-1} + q_{n+1} + q_n = p + p_1 + q_1 + \cdots + q_{n-1} + q_n$.

Thus, in the case b$_2$) it is proved that either $q \leqq p + p_1 + q_1 + \cdots + q_n + q_{n+1} = p + p_1 + q_1 + \cdots + q_{n-1} + q_n$ or $q \leqq p + p_1 + q_1 + \cdots + q_n + q_{n+1} = p + p_1 + q_1 + \cdots + q_{n-1} + q_{n+1}$ hold. Now by the Lemma 2, if $q \leqq p_1 + q_1 + \cdots + q_{n-1} + q_n$, then $q \not\leqq p_1 + q_1 + \cdots + q_{n-1} + q_{n+1}$. Thus, it is proved that there exist $q'_1, \cdots, q'_n$ such that $q \leqq p + p_1 + q'_1 + \cdots + q'_n$ but $q \not\leqq p_1 + q'_1 + \cdots + q'_n$.

By the same process, we can replace (q'_i)'s one by one by $p_2, \cdots, p_n$ and finally we will get $q \leqq p + p_1 + \cdots + p_n + r$ with $q \not\leqq p_1 + \cdots + p_n + r$ and hence $p \not\leqq p_1 + \cdots + p_n + r$. Converse is obvious.

PROOF OF COROLLARY. If $\mathscr{L}$ is irreducible, by (2), every pair of distinct points are perspective. Let $p_1 + \cdots + p_{n+2}$ be any surface. Since p_1, p_2 are perspective, by Theorem 1, there exists a point p_{n+3} such that $p_1 \leqq p_2 + \cdots + p_{n+2} + p_{n+3}$ but $p_1 \not\leqq p_3 + \cdots + p_{n+3}$. Then $p_1 + p_3 + \cdots + p_{n+3} = p_2 + p_3 + \cdots + p_{n+3} = p_1 + p_2 + p_3 + \cdots + p_{n+2}$.

In the case $n = 0$, by (2'), if every line pq contains at least three points, then $\mathscr{L}$ is irreducible. But generally, the converse of the corollary does not hold as shown by the following counter example.

For the case $n = 1$, suppose that $\mathscr{L}$ consists of six points: p, q, q_1, q_2, q_3, q_4; eleven lines: $p + q$, $p + q_1$, $p + q_2$, $p + q_3 + q_4$, $p + q_1 + q_2$, $q + q_3$, $q + q_4$, $q_1 + q_3$, $q_1 + q_4$, $q_2 + q_3$, $q_2 + q_4$; and six planes: $p + q + q_1 + q_2$, $p + q + q_3 + q_4$, $p + q_1 + q_3 + q_4$, $p + q_2 + q_3 + q_4$, $p + q_1 + q_2 + q_3$ and $q + q_1 + q_2 + q_4$. Then $\mathscr{L}$ is a matroid lattice of the nature under consideration, each of whose planes contains four distinct points. But it is easily seen that in $\mathscr{L}$, p is not perspective to q.

PROOF OF THEOREM 2. Let $\mathscr{L}$ be the lattice stated above, then by considering a curve in $\mathscr{L}$ which contains $\{p_1, \cdots, p_n\}$ a *new point*, and a surface in $\mathscr{L}$ which contains $\{p_1, \cdots, p_n\}$ a *new line*, then $[p_1 + \cdots + p_n\ 1]$ is the lattice of subspaces of a projective geometry (of infinite dimension) with such new elements (C. J. Hsu [6]).

Suppose now that $\mathscr{L}$ is irreducible, and let $p + p_1 + \cdots + p_n$ and $q + p_1 + \cdots + p_n$ be new points in the corresponding projective geometry. Since $p \sim q$, by Theorem 1, either $p + p_1 + \cdots + p_n = q + p_1 + \cdots + p_n$ or there is a point $r \in \mathscr{L}$ such that $q \leqq p + p_1 + \cdots + p_n + r$ and $q \not\leqq p_1 + \cdots + p_n + r$, $p \not\leqq p_1 + \cdots + p_n + r$. In the former case, the two new points $p + p_1 + \cdots + p_n$ and $q + p_1 + \cdots + p_n$ coincide. In the latter case these two new points and the new point $r + p_1 + \cdots + p_n$ are distinct and they are contained in the same new line $p + p_1 + \cdots + p_n + r = q + p_1 + \cdots + p_n + r$. Thus, by (2′) the lattice $[p_1 + \cdots + p_n \ 1]$ is irreducible. Conversely, let $p, q, \in \mathscr{L}$ be any two distinct points and let $p_1, \cdots, p_n \in \mathscr{L}$ be any n points such that $p, q, p_1, \cdots, p_n$ are distinct. Then either the two new points $p + p_1 + \cdots + p_n$ and $q + p_1 + \cdots + p_n$ coincide or on the new line $p + p_1 + \cdots + p_n + q$ determined by these two new points, there is a new point $r + p_1 + \cdots + p_n$ which is distinct from these two new points. Then $p + p_1 + \cdots + p_n + q = r + p_1 + \cdots + p_n + q = r + p_1 + \cdots + p_n + p$ is a surface in $\mathscr{L}$, and hence $q \leqq p + p_1 + \cdots + p_n + r$ and $q \nleqq p_1 + \cdots + p_n + r$. Thus p is perspective to q, and $\mathscr{L}$ is irreducible.

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