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SUBGROUP OF SOME LIE GROUP AS A RIEMANNIAN SUBMANIFOLD

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Let G be a connected Lie group such that $\mathrm{Ad}(G)$ is compact. Then G admits a (positive definite) Riemannian metric g which is bi-invariant (left and right invariant). A submanifold H of G is endowed with the induced Riemannian metric g' by means of g. We consider G and H in such a situation. The purpose of this paper is to prove the following theorem.

THEOREM. *Let G be a connected Lie group such that* $\mathrm{Ad}(G)$ *is compact. Then an abstract subgroup H of G is a Lie subgroup (of dimension > 0) if and only if H is a totally geodesic submanifold of G.*

This is applicable, of course, in the case where G is connected and compact.

We keep in mind on the following facts. Let ∇ (resp. ∇_X) denote the covariant differential (resp. derivative) with respect to the Riemannian connection on G induced from g, then $\nabla_X Y = (1/2)[X, Y]$. Any 1-parameter subgroup $a(t)$, $-\infty < t < +\infty$ is a geodesic in G and the canonical parameter t is an affine parameter on the geodesic. Conversely a geodesic through e (unit element of G) is contained in a 1-parameter subgroup of G.

PROOF OF THE THEOREM. The necessity is easily verified. Conversely assume that H is an abstract subgroup of G which is a totally geodesic submanifold of G. The identity injection $H \to G$ is denoted by f. G and H are metric spaces by means of the Riemannian metric g and the induced Riemannian metric g' respectively, whose distance functions are denoted by $d_G(x, y)$, $x, y \in G$ and $d_H(x, y)$, $x, y \in H$ respectively. The topology of G (resp. H) coincides with that given by the distance d_G (resp. d_H), which is denoted by $\tilde{\Sigma}$ (resp. Σ). In general, Σ is stronger than the induced topology from $\tilde{\Sigma}$.

Let p be an arbitrary point of H and let V be an arbitrary open set (with respect to Σ) containing $q = L_a p (\in H)$. There exists an open ball

$B_\varepsilon(q) = \{x' \mid d_H(q, x') < \varepsilon\}$ around q with radius $\varepsilon > 0$ contained in V. On the other hand, there exists a neighborhood $U \subset H$ of p such that

$$(*) \qquad d_H(y, z) = d_G(y, z) \qquad \text{for } y, z \in U ,$$

([1], p. 79), since H is totally geodesic. Let $B_{\varepsilon'}(p) = \{x \mid d_H(p, x) < \varepsilon'\}$ be an open ball around p with radius $\varepsilon' > 0$ contained in U, then (*) holds in $B_{\varepsilon'}(p)$. We choose $\varepsilon' < \varepsilon$. If furthermore we choose ε' sufficiently small, then any point $x \in B_{\varepsilon'}(p)$ can be joined to p by the minimizing geodesic γ_{px} of H lying in $B_{\varepsilon'}(p)$: length of $\gamma_{px} = d_H(p, x)$. Since H is totally geodesic, $f(\gamma_{px})$ is a geodesic in G which is minimizing in G by virtue of (*). $L_{f(a)}$, $a \in H$ being an isometry on G, the left translation of $f(\gamma_{px})$ by $L_{f(a)} : L_{f(a)}(f(\gamma_{px})) = f(L_a(\gamma_{px}))$ is a minimizing geodesic in G joining q to $x' = L_a x$. We denote this geodesic by $\tilde{\gamma}_{qx'}$ whose length is equal to $d_G(q, x') = d_G(p, x) < \varepsilon' < \varepsilon$. Since H is an abstract subgroup of G, $\tilde{\gamma}_{qx'}$ lies in H so that the tangent vector of $\tilde{\gamma}_{qx'}$ at q must be tangent to H. We denote $\tilde{\gamma}_{qx'}$ as a subset of H by $\gamma_{qx'} : \tilde{\gamma}_{qx'} = f(\gamma_{qx'})$. Since H is totally geodesic, $\gamma_{qx'}$ is a geodesic in H, which is also minimizing because the metric on H is the induced one. Then we have

$$\begin{aligned} d_H(q, x') &= \text{length of } \gamma_{qx'} \text{ in } H \\ &= \text{length of } \tilde{\gamma}_{qx'} \text{ in } G = d_G(q, x') < \varepsilon . \end{aligned}$$

Hence $x' = L_a x \in B_\varepsilon(q)$. The point $x \in B_{\varepsilon'}(p)$ being arbitrary, we have

$$L_a(B_{\varepsilon'}(p)) \subset B_\varepsilon(q) \subset V .$$

This shows that L_a, $a \in H$ is continuous on H. Since $L_{f(a)}$ is differentiable on G, L_a, $a \in H$ is differentiable on H.

The right translation R_a, $a \in G$ and the diffeomorphism

$$\psi : G \to G \quad \text{given by} \quad x \to x^{-1} \quad (x \in G) ,$$

give isometries of G onto itself. Hence we can prove that R_a, $a \in H$ and $\psi \mid H$ (restriction of ψ to H) are both differentiable on H, quite similarly as in the case of L_a, $a \in H$. Thus L_a, R_a $(a \in H)$ and $\psi \mid H$ give isometries on H. Making use of these facts, we can prove that H is a Lie subgroup. One method is as follows.

Suppose that $xy = z$, $x, y \in H$ and we shall prove that the mapping $H \times H \to H$ given by $(x, y) \to z$ is differentiable on H. Let U be an arbitrary open set containing z in H and $B_\varepsilon(z)$ be an open ball around z with radius $\varepsilon > 0$ contained in U. Let $B_{\varepsilon/2}(y)$ be an open ball around y with radius $\varepsilon/2$. Since the right translation R_y is continuous, we can choose a ball $B_\rho(x)$ around x with radius $\rho > 0$ such that $R_y(B_\rho(x)) \subset B_{\varepsilon/2}(z)$, namely for any $x' \in B_\rho(x)$, $R_y x' = x'y \in B_{\varepsilon/2}(z)$. Let $x' \in B_\rho(x)$ and $y' \in B_{\varepsilon/2}(y)$

be arbitrary, then

$$d_H(z, x'y') \leqq d_H(z, x'y) + d_H(x'y, x'y') .$$

Since $x'y \in B_{\varepsilon/2}(z)$, we have $d_H(z, x'y) < \varepsilon/2$. And since $L_{x'}$ is an isometry on H, we have $d_H(x'y, x'y') = d_H(y, y') < \varepsilon/2$, so that

$$d_H(z, x'y') < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

This means that $x'y' \in B_\varepsilon(z)$, namely $B_\rho(x)\cdot B_{\varepsilon/2}(y) \subset B_\varepsilon(z) \subset U$. Hence the mapping $H \times H \rightarrow H$ given by $(x, y) \rightarrow z$ is continuous and so differentiable because it is differentiable on $G \times G$ onto G. q.e.d.

REFERENCE

[1] S. HELGASON, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

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