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SOME REMARKS AND QUESTIONS CONCERNING THE INTRINSIC DISTANCE

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. Introduction. Let X be a connected complex space and d_X be the intrinsic pseudo-distance, ([4], see also the definition given in § 3). We shall show that if d_X is a distance, i.e., if X is hyperbolic, then d_X is inner in the sense of Rinow [6], see also § 2. This is not surprising since d_X is defined in such a way that it is *essentially* the integrated form of an infinitesimal pseudo-metric. In fact, Royden [7] has shown that if X is a complex manifold, then d_X is *precisely* the integrated form of an intrinsic differential metric F_X. But it is perhaps of some interest to give a direct proof of the fact that d_X is inner without assuming that X is non-singular. The proof works also for infinite dimensional complex spaces X. The fact that d_X is inner allows us to talk about geodesics and curvature when X is hyperbolic. Although I cannot do very much with the geodesics and the curvature thus introduced, some of the results in [4] proved directly can be derived from the general theory of metric spaces with inner distance.

2. Inner distances. Let X be a metric space with distance function d. Given a curve $\gamma(t)$, $a \leqq t \leqq b$, in X, the *length* $L(\gamma)$ of γ is defined by

$$L(\gamma) = \sup \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)) ,$$

where the supremum is taken with respect to all partitions $a = t_0 < t_1 < \cdots < t_k = b$ of the interval $[a, b]$. A curve γ is said to be *rectifiable* if its length $L(\gamma)$ is finite. A metric space X is said to be *finitely arc-wise connected* if every pair of points x, y of X can be joined by a rectifiable curve. It is said to be *without detour* ("ohne Umwege" in Rinow [6]) if for every point $x \in X$ and for every positive number ε, there exists a positive number δ such that every point $y \in X$ with $d(x, y) < \delta$ can be joined to x by a rectifiable curve γ of length $L(\gamma) < \varepsilon$.

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Let (X, d) be a finitely arc-wise connected metric space. The induced inner distance d^i is defined by

$$d^i(x, y) = \inf L(\gamma) ,$$

where the infimum is taken with respect to all rectifiable curves γ joining x and y. From the definition of d^i, it follows immediately that

$$d(x, y) \leqq d^i(x, y) \quad \text{for} \quad x, y \in X .$$

For the proofs of the following facts, the reader is refered to Rinow [6; pp. 119–120]:

(1) *Let (X, d) be a finitely arc-wise connected metric space. Then d and d^i define the same topology on X if and only if X is without detour.*

(2) *Let (X, d) be a finitely arc-wise connected metric space. Then $L(\gamma) = L^i(\gamma)$ for all curves γ, where L^i is the length defined by d^i.*

A metric space X is said to be *complete* (or Cauchy-complete) if every Cauchy sequence converges. A stronger concept is that of finitely compact space. A metric space X is said to be *finitely compact* if every bounded infinite set has at least one accumulation point.

(3) *Let (X, d) be a finitely arc-wise connected metric space without detour. Then (X, d) is complete (resp. finitely compact) if and only if (X, d^i) is complete (resp. finitely compact).*

A distance d is said to be *inner* if $d = d^i$. If (X, d) is finitely arc-wise connected, then d^i is always inner (see Rinow [6; p. 121]). Hence, the term "the inner distance d^i induced by d" is justified. Since the definition of d^i assumes that X is finitely arc-wise connected, we shall agree that a metric space X with inner distance d is finitely arc-wise connected. By (1), such a space is also without detour. A curve γ from x to y is called a *minimizing geodesic* from x to y if $L(\gamma) = d(x, y)$. A curve γ is a *geodesic* if for every $t \in [a, b]$, there exists a small number $\delta > 0$ such that $\gamma \mid [t - \delta, t + \delta]$ is a minimizing geodesic from $\gamma(t - \delta)$ to $\gamma(t + \delta)$. The following result is essentially due to Hilbert, (see Rinow [6; p. 141]):

(4) *If X is a finitely compact metric space with inner distance d, then any two points x, y of X can be joined by a minimizing geodesic.*

The following result goes back to Hopf-Rinow, (see Rinow [6; p. 172]):

(5) *For a locally compact metric space X with inner distance d, the following conditions are mutually equivalent:*

(a) *X is finitely compact;*

(b) *X is (Cauchy) complete;*

(c) *Every geodesic can be infinitely extended.*

3. Hyperbolic complex spaces. Let X be a connected complex space. We recall the definition of the intrinsic pseudo-distance d_X of X. Given two points x, y of X, choose points $x = x_0, x_1, \cdots, x_{k-1}, x_k = y$ in X, points $a_1, \cdots, a_k, b_1, \cdots, b_k$ in the unit disk $D = \{z \in C; |z| < 1\}$ and holomorphic mappings $f_1, \cdots, f_k$ of D into X such that

$$f_i(a_i) = x_{i-1} \quad \text{and} \quad f_i(b_i) = x_i \quad \text{for} \quad i = 1, 2, \cdots, k .$$

Using the Poincaré distance (i.e., non-Euclidean distance) ρ of D, we define

$$d_X(x, y) = \inf \sum_{i=1}^{k} \rho(a_i, b_i) ,$$

where the infimum is taken with respect to all possible choices of above points and mappings. Then d_X is a pseudo-distance on X.

We say that X is *hyperbolic* if d_X is a distance. If X is hyperbolic, then the topology defined by d_X coincides with the given topology of X, (see Barth [1]).

THEOREM. *If X is a hyperbolic complex space, then its intrinsic distance d_X is inner, i.e., $d_X = d_X^i$.*

PROOF. Since we have $d(x, y) \leqq d^i(x, y)$ for any distance d, it suffices to prove $d_X(x, y) \geqq d_X^i(x, y)$. Let $x = x_0, x_1, \cdots, x_k = y$, $a_1, \cdots, a_k$, $b_1, \cdots, b_k$, $f_1, \cdots, f_k$ be as above. It suffices to construct a rectifiable curve γ from x to y such that

$$L(\gamma) \leqq \sum_{i=1}^{k} \rho(a_i, b_i) .$$

Let C_i be the geodesic from a_i to b_i in the disk D. Joining $f_1(C_1), \cdots, f_k(C_k)$ consecutively, we obtain a curve γ from x to y. Since $L(\gamma) = \sum_{i=1}^{k} L(f_i(C_i))$, it suffices to prove

$$L(f_i(C_i)) \leqq \rho(a_i, b_i) .$$

In proving the inequality above, we omit the subscript i; let $C(t)$, $t_0 \leqq t \leqq t_1$, be the geodesic in D from a to b. Consider a partition $t_0 = s_0 < s_1 < \cdots < s_m = t_1$ of the interval $[t_0, t_1]$. Since

$$L(f(C)) = \sup \sum d_X(f(C(s_{j-1})), f(C(s_j)))$$

and

$$\rho(a, b) = \sum \rho(C(s_{j-1}), C(s_j)) ,$$

it suffices to prove

$$d_X(f(C(s_{j-1})), f(C(s_j))) \leqq \rho(C(s_{j-1}), C(s_j)) .$$

But this is clear from the definition of d_X. q.e.d.

This allows us to apply a number of results in the book of Rinow [6], some of which are listed in § 2, to hyperbolic complex spaces.

It is not clear if a hyperbolic complex space X with d_X is a G-space in the sense of Busemann [2].

As in [4] we define an intrinsic infinitesimal pseudo-metric F_X. For every tangent vector ξ of X at a point x, let v be a tangent vector of the unit disk D at the origin 0 and f be a holomorphic mapping of D into X such that $f_*(v) = \xi$. We set

$$F_X(\xi) = \inf \| v \| ,$$

where $\| v \|$ denotes the length of v with respect to the Poincaré metric of D and the infimum is taken with respect to all possible choices of v and f. Then F_X is a non-negative upper semicontinuous function on the tangent bundle $T(X)$ such that $F_X(c\xi) = |c| \cdot F_X(\xi)$ for $c \in \boldsymbol{C}$ and $\xi \in T(X)$. Royden [7] has shown that d_X is the integrated form of F_X, i.e.,

$$d_X(x, y) = \inf \int_\gamma F_X ,$$

where the infimum is taken over all piecewise differentiable curves γ from x to y. This result of Royden shows at once that d_X is an inner distance and X is also a G-space of Busemann if X is a hyperbolic complex *manifold*. The question remains unanswered for a complex space X with singularities.

It is not known if the metric F_X satisfies the convexity condition:

$$F_X(\xi + \eta) \leqq F_X(\xi) + F_X(\eta) .$$

This condition is usually assumed in the theory of Finsler metrics. Relatedly, *it is not known if a sufficiently small spherical neighborhood $U(x; \varepsilon) = \{y \in X; d_X(x, y) < \varepsilon\}$ is geodesically convex.*

Another (pseudo-) distance of interest is the *Carathéodory (pseudo-distance c_X defined by*

$$c_X(x, y) = \sup \rho(f(x), f(y)) ,$$

where the supremum is taken over all holomorphic mappings $f: X \to D$. In general, c_X is not an inner distance. The induced inner distance c'_X was studied by Reiffen [5] as well as by Carathéodory himself, [3]. Since $c_X \leqq d_X$, (see [4]) and d_X is inner, we have

$$c_X \leqq c'_X \leqq d_X .$$

Reiffen has shown that c'_X is the integrated form of a (pseudo-) differential metric.

4. Curvature. Let X be a metric space with inner distance d. A geometric configuration consisting of three distinct points and three minimizing geodesics is called simply a *triangle*. The three points and the three minimizing geodesics are called *vertices* and *edges* of the triangle. Let Δ be a triangle with vertices x, y, z and edges e_x, e_y, e_z. (The edge facing x, i.e., joining y, z, is denoted by e_x. It may not be determined uniquely by y, z). We denote the midpoints of e_x, e_y, e_z by $\bar{x}, \bar{y}, \bar{z}$, respectively. Let S_K be a 2-dimensional simply connected, complete riemannian space of constant curvature $K \leqq 0$. Let Δ' be a triangle with vertices x', y', z' and edges $e_{x'}, e_{y'}, e_{z'}$ in S_K which is congruent to the triangle Δ. Such a triangle Δ' is unique up to a motion in S_K. (For $K > 0$, Δ' may not exist or may not be unique if it exists). The midpoints of $e_{x'}, e_{y'}, e_{z'}$ will be denoted by $\bar{x}', \bar{y}', \bar{z}'$, respectively. We say that the *curvature* of the metric space X is $\leqq K$ at a point x if there exists a neighborhood U of x such that for every triangle Δ with vertices x, y, z in U the distance $d(\bar{y}, \bar{z})$ is less than or equal to the distance between $\bar{y}'$ and $\bar{z}'$. This definition is essentially the same as the one in Rinow [6; p. 310].

If X is a complex manifold with inner distance d, then we define the holomorphic sectional curvature as follows. Let V be a complex submanifold of X. The restriction of d to V is not usually an inner distance. So we consider the inner distance induced from the restriction of d to V. We say that the *holomorphic sectional curvature* of (X, d) is $\leqq K$ at a point x if the curvature of every 1-dimensional complex submanifold V through x is $\leqq K$ at x.

We know [4] that a hermitian manifold whose holomorphic sectional curvature is bounded above by a negative constant is hyperbolic. *It is not known if, conversely, every hyperbolic complex manifold admits such a hermitian metric.* In connection with this question, *one can ask if a hyperbolic complex manifold X with d_X has holomorphic sectional curvature $\leqq K < 0$ in the sense defined above.*

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