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ON COMPACT MINIMAL SURFACES WITH NON-NEGATIVE GAUSSIAN CURVATURE IN A SPACE OF CONSTANT CURVATURE: 1

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. Introduction. In a series of papers entitled the above, we mainly treat compact minimal surfaces with non-negative Gaussian curvature in a space of constant curvature. If the Gaussian curvature is non-negative and not identically zero, by the Gauss-Bonnet's theorem, the genus of the surface is zero. Such minimal immersions have been studied by [7], [8], [9], especially, Calabi [4] and [5]. Their main results are as follows: Let $S^N(1)$ be an N-dimensional unit sphere in an $(N+1)$-dimensional Euclidean space and let $S^2 \to S^N(1)$ be a minimal immersion of the differentiable two-sphere in $S^N(1)$ such that the image is not contained in a great hypersphere. Then

(1) N is even;

(2) The total area is an integral multiple of 2π;

(3) If the Gaussian curvature K is constant, $K = 2/m(m+1)$, where $N = 2m$. Such a immersion is uniquely determined up to motions of $S^N(1)$, and the image is the generalized Veronese surface of Borůvka [3] and Ōtsuki [15].

(4) There are minimal immersions of $S^2 \to S^N(1)$ of which the induced metric has non-constant Gaussian curvature.

This article is a first step for the classification of minimal tori in a Euclidean sphere, i.e., minimal immersions of a torus into $S^N(1)$. Our main results are as follows: If the Gaussian curvature is identically zero and the image does not lie in any great hypersphere of $S^N(1)$, then N is an odd integer, say $N = 2m+1$, and under the additional assumption, the immersion is rigid.

If $N = 3$, the above minimal surface is the Clifford minimal torus in $S^3(1)$, and if $N = 5$, such a surface was studied by Borůvka [1]. For each odd N, we can describe explicitly examples of the flat minimal surfaces (cf. [15]).

The even dimensionality of $S^N(1)$ in the first case is an implication of the topological condition to the effect that the genus of $M = 0$. Contrary to it the fact that N is an odd integer in our case is not an topological implication: There is a minimal immersion of a torus into $S^4(1)$ which is not contained in the $S^3(1) \subset S^4(1)$ (see, Lawson [12, p. 363]).

To study the problem, we adopt Bochner's method, i.e., we use scalar fields $f_{(b)}$, $K_{(b)}$ and $N_{(b)}$, $b = 2, 3, \cdots$, on the surface, which will be defined later, and calculate their Laplacians. In § 2, we fix notations used in this paper. In § 3, we describe the concept of the n-th fundamental form and using it, we define above mentioned scalar fields. The Laplacians of $f_{(b)}$ and $K_{(b)}$, are calculated in § 4, and the Codazzi's equation of higher order is proved. We remark that these result in § 4 have known under a certain global assumption of M, but our works show that a part of the results in S. S. Chern's paper [7] is local. In § 5, we assume that M is oriented, compact and of zero Gaussian curvature.

Then an application of the results in § 4 gives a proof of one of our main theorems. In § 6, we consider Frenet-Borůvka formula of a flat minimal surface and prove a rigidity theorem. In § 7, we consider the case when the ambient space is the N-sphere. We show that the generalized Clifford torus on S^{2m+1} is algebraic. In § 8, we study compact minimal surfaces with non-negative Gaussian curvature. § 8 is closely related to the S. S. Chern's paper [7]. In the Appendix, using an inequality proved in § 3, we show an extrinsic rigidity theorem. This result generalizes the De Giorgi-Simons-Reilly's theorem partially. In the part 1, we treat § 1 ~ § 4.

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2. Preliminaries. Let $\bar{M}$ be an n-dimensional Riemannian manifold of constant curvature c and e_A, $A, B, \cdots = 1, 2, \cdots, N$, local orthonormal frame fields on $\bar{M}$. The Levi-Civita connection defines the covariant differentials

$$De_A = \sum_B w_{AB} e_B \,, \tag{2.1}$$

where $w_{AB} + w_{BA} = 0$. If w_B is a coframe field dual to e_A, the structure equations of the space are

$$dw_A = \sum_B w_B \wedge w_{BA} \,, \tag{2.2}$$

$$dw_{AB} = \sum_C w_{AC} \wedge w_{CB} - cw_A \wedge w_B \,. \tag{2.3}$$

Let M be a two dimensional oriented Riemannian manifold and

$$x\colon M \to \bar{M} \tag{2.4}$$

be an isometric minimal immersion of M into $\bar{M}$. In this paper we will agree on the following ranges of indices: $1 \leqq i, j, \cdots \leqq 2$; $3 \leqq \alpha, \beta, \cdots \leqq N$. To study the geometry of the immersed surface M we restrict ourselves to orthonormal frame fields over M such that e_i are tangent vectors of M at each point of its domain of definitions. Then we have

$$w_\alpha = 0 \,. \tag{2.5}$$

By (2.2) and the Cartan's lemma, we can put

$$w_{i\alpha} = \sum_j h_{\alpha ij} w_j \,, \qquad h_{\alpha ij} = h_{\alpha ji} \,. \tag{2.6}$$

The condition that x is minimal is expressed by

$$\sum_i h_{\alpha ii} = 0 \,. \tag{2.7}$$

We define the covariant derivatives $h_{\alpha ij,k}$'s of $h_{\alpha ij}$'s by

$$\begin{aligned} Dh_{\alpha ij} &= \sum_k h_{\alpha ij,k} w_k \\ &= dh_{\alpha ij} + \sum_s h_{\alpha sj} w_{si} + \sum_s h_{\alpha is} w_{sj} + \sum_\beta h_{\beta ij} w_{\beta\alpha} \,. \end{aligned} \tag{2.8}$$

Taking the exterior derivative of (2.6) and using (2.2) and (2.3), we get $\sum_j Dh_{\alpha ij} \wedge w_j = 0$, and so

$$h_{\alpha ij,k} = h_{\alpha ik,j} \,. \tag{2.9}$$

As the dimension of M is 2, we see by (2.7) that (2.9) is equivalent to

$$h_{\alpha 11,2} = h_{\alpha 12,1} \,, \qquad h_{\alpha 12,2} = -h_{\alpha 11,1} \,. \tag{2.10}$$

We put

$$\phi = w_1 + iw_2 \,, \tag{2.11}$$

$$H_\alpha^{(2)} = h_{\alpha 11} + ih_{\alpha 12} \text{ and } H_{\alpha,1}^{(2)} = h_{\alpha 11,1} + ih_{\alpha 12,1} \,. \tag{2.12}$$

By (2.8) and (2.10), we obtain

$$dH_\alpha^{(2)} + 2iH_\alpha^{(2)} w_{12} + \sum_\beta H_\beta^{(2)} w_{\beta\alpha} = H_{\alpha,1}^{(2)} \bar{\phi} \,. \tag{2.13}$$

From (2.13) we derive

$$d \sum (H_\alpha^{(2)})^2 + 4i \sum (H_\alpha^{(2)})^2 w_{12} = 2 \sum H_\alpha^{(2)} H_{\alpha,1}^{(2)} \bar{\phi} \,. \tag{2.14}$$

By the structure equations of M, we find

$$d\phi = -iw_{12} \wedge \phi \,, \tag{2.15}$$

$$dw_{12} = -\left(\frac{i}{2}\right)K\phi \wedge \bar{\phi}\,, \tag{2.16}$$

where K denotes the Gaussian curvature of M.

REMARK 1. The formula (2.14) gives a local differential geometric characterization of the formula (48) in [7].

3. Higher osculating spaces and n-th fundamental forms. In the study of minimal surfaces with higher codimension in a space of constant curvature, the concept of osculating spaces plays an important role. In [2] Borůvka studied such spaces extensively. At present there are good descriptions of osculating spaces in [7], [10], and [15]. For the purpose of our calculations, we shall adopt the notation developed in the paper [7] and we define the covariant differentiation of n-th fundamental tensors: Let $x(s)$ be a smooth curve C through $x \in M$ parametrized by its arc length. By the covariant differentiation along C we get the vector fields

$$\frac{Dx}{ds}, \frac{D^2x}{ds^2}, \cdots, \frac{D^nx}{ds^n}, \cdots. \tag{3.1}$$

The first n vectors in (3.1) at $s = 0$ are said to span the osculating space of order n of $x(s)$ at $x = x(0)$. The n-th osculating space $T_x^{(n)}$ of M at $x \in M$ is defined to be the space spanned by all the osculating spaces of order n at x of curves through x and lying on M. We then have $T_x^{(1)}(=T_x) \subset T_x^{(2)} \subset \cdots \subset T_x^{(n)} \subset \cdots$. Put

$$\begin{aligned} &p_{-1}(x) = 0\,, \qquad p_0(x) = 2\,, \\ &p_a(x) = \dim. T_x^{(a+1)} - \dim. T_x^{(a)},\ a = 1, 2, \cdots, n-1\,. \end{aligned} \tag{3.2}$$

Then we have

$$\dim. T_x^{(n)} = \sum_{a=0}^{n-1} p_a(x)\,, \qquad (n \geqq 1)\,. \tag{3.3}$$

A point $x \in M$ is called a regular point of order b, if $p_a(x)$ is constant for each $a = 1, 2, \cdots, b-1$, in a neighborhood of x.

Suppose now that x is a regular point of order $n - 1 \geqq 2$. We shall use the following ranges of indices:

$$\begin{aligned} &1 + \sum_{a=-1}^{b-3} p_a(x) \leqq \lambda_{b-2} \leqq \sum_{a=0}^{b-2} p_a(x)\,, \qquad b = 2, 3, \cdots, n\,, \\ &1 + \sum_{a=-1}^{n-2} p_a(x) \leqq \lambda_{n-1} \leqq N\,. \end{aligned} \tag{3.4}$$

Let e_A be local orthonormal frame fields, such that $e_{\lambda_0}, e_{\lambda_1}, \cdots, e_{\lambda_b}$ span $T_x^{(b+1)}$, $b = 0, 1, \cdots, n-2$. We then have

$$(3.5)_{b-1} \qquad w_{\lambda_{b-1}\lambda_{a+1}} = 0, \text{ for } a = b, b+1, \cdots, n-2, \quad b = 1, 2, \cdots, n-2 .$$

By the exterior differentiation of (3.5) and making use of (2.3), we get

$$\sum_{\lambda_b} w_{\lambda_{b-1}\lambda_b} \wedge w_{\lambda_b\lambda_{b+1}} = 0 , \qquad b = 1, 2, \cdots, n-2 ,$$

where the sum extends over the range of indices of λ_b not in the range of b. This allows us to introduce recurrently the quantities $h_{\lambda_{b+1}i_1i_2\cdots i_{b+2}}$ defined by the equations

$$(3.6) \qquad \sum_{\lambda_b} h_{\lambda_b i_1 i_2\cdots i_{b+1}} w_{\lambda_b\lambda_{b+1}} = \sum_{i_{b+2}} h_{\lambda_{b+1}i_1\cdots i_{b+2}} w_{i_{b+2}} .$$

$h_{\lambda_{b+1}i_1\cdots i_{b+2}}$ are symmetric in the set of indices $i_1, i_2, \cdots, i_{b+2}$. Let Ω_b be the open set of all regular points of order b. We set $\Omega_1 = M$. Then

$$(3.7) \qquad \Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots \supset \Omega_{n-1} .$$

We find

$$(3.8) \qquad (e_{\lambda_a}, D^b x) = \begin{cases} \sum_{i_1,\cdots,i_b} h_{\lambda_{b-1}i_1\cdots i_b} w_{i_1} \cdots w_{i_b}, & \text{if } a = b-1,\ b \leqq n , \\ 0, & \text{if } a \geqq b,\ n > b , \end{cases}$$

which are differential forms of degree b and are to be called the b-th fundamental forms of M into $\bar{M}$.

From (2.7) and (3.6) it follows that

$$(3.9) \qquad \sum_j h_{\lambda_b j j i_3\cdots i_{b+1}} = 0,\ b = 1, \cdots, n-1 .$$

Since $h_{\lambda_b i_1\cdots i_{b+1}}$ are symmetric in the $i_1, \cdots, i_{b+1}$, the same relation holds when contracted with respect to any two Latin indices. The integer $p_a(x)$ is equal to the number of linearly independent vectors among

$$(3.10) \qquad \sum_{\lambda_a} h_{\lambda_a i_1 i_2\cdots i_{a+1}} e_{\lambda_a},\ i_1 \leqq i_2 \leqq \cdots \leqq i_{a+1} .$$

Therefore, at each point of Ω_a, we have

$$(3.11) \qquad p_a(x) \leqq 2,\ a = 1, \cdots, n-1 .$$

Let

$$(3.12) \qquad H_\alpha^{(b)} = h_{\alpha\underbrace{1\cdots 1}_{b}} + i h_{\alpha\underbrace{1\cdots 1}_{b-1}2},\ \alpha \geqq \mu_{b-1} ,$$

where $\mu_{b-1} = \sum_{a=0}^{b-2} p_a(x) + 1$. We define the covariant derivatives $h_{\alpha i_1\cdots i_b,k}$'s of $h_{\alpha i_1\cdots i_b}$'s, $\alpha \geqq \mu_{b-1}$, by

$$Dh_{\alpha i_1\cdots i_b} = \sum_k h_{\alpha i_1\cdots i_b,k} w_k = dh_{\alpha i_1\cdots i_b} + \sum_s h_{\alpha s i_2\cdots i_b} w_{s i_1} + \cdots + \sum_s h_{\alpha i_1\cdots i_{b-1}s} w_{s i_b} + \sum_{\beta \geqq \mu_{b-1}} h_{\beta i_1\cdots i_b} w_{\beta\alpha} . \tag{3.13}$$

Then we have

$$dH_\alpha^{(b)} + biH_\alpha^{(b)} w_{12} + \sum_{\beta \geqq \mu_{b-1}} H_\beta^{(b)} w_{\beta\alpha} = H_{\alpha,1}^{(b)} w_1 + H_{\alpha,2}^{(b)} w_2 , \tag{3.13$'$}$$

where $H_{\alpha,k}^{(b)} = h_{\alpha 1\cdots 1,k} + ih_{\alpha 1\cdots 12,k}$.

LEMMA 1.

$$H_\alpha^{(b)} = H_{\alpha,1}^{(b-1)} , \quad H_\alpha^{(b)} = iH_{\alpha,2}^{(b-1)} , \quad \alpha \geqq \mu_{b-1} , \quad n \geqq b \geqq 3 . \tag{3.14}$$

PROOF. From (3.6) we have

$$\sum_{\lambda_{b-2}} H_{\lambda_{b-2}}^{(b-1)} w_{\lambda_{b-2}\lambda_{b-1}} = H_{\lambda_{b-1}}^{(b)} \bar{\phi} , \qquad b = 2, 3, \cdots, n , \tag{3.15$_b$}$$

where we have put $H_1^{(1)} = 1$ and $H_2^{(1)} = i$. Since $H_{\lambda_{b-1}}^{(b-1)} = 0$, by (3.8), we get

$$\sum_k H_{\lambda_{b-1},k}^{(b-1)} w_k = H_{\lambda_{b-2}}^{(b-1)} w_{\lambda_{b-2}\lambda_{b-1}} \text{ and } H_{\lambda_a,k}^{(b-1)} = 0 \text{ for } a \geqq b . \tag{3.16}$$

From (3.15) and (3.16) we get (3.14). q.e.d.

Now we shall construct scalar invariants of the isometric minimal immersion x. The vector $E_1 = e_1 + ie_2$ is defined up to the transformation $E_1 \to E_1^* = e^{i\tau} E_1$, where τ is real. Under such a change,

$$H_\alpha^{(b)} \to H_\alpha^{(b)*} = e^{bi\tau} H_\alpha^{(b)} . \tag{3.17$_b$}$$

In fact, by a direct calculation, we have $(3.17)_2$. When $b \geqq 3$, from (3.15), we get $(3.17)_b$ by an induction for b.

The system of normal vectors $\{e_\alpha\}$, $\alpha \in \mu_{b-1}$, is defined up to the transformation

$$\tilde{e}_\alpha = \sum_{\beta \in \mu_{b-1}} A_{\alpha\beta} e_\beta, \ \alpha \in \mu_{b-1} , \tag{3.18}$$

where $(A_{\alpha\beta})$ is an orthogonal matrix. Under such a change we have

$$\tilde{H}_\alpha^{(b)} = \sum_{\beta \in \mu_{b-1}} A_{\alpha\beta} H_\beta^{(b)}, \ \alpha \in \mu_{b-1} ,$$

and so

$$\sum_{\alpha \in \mu_{b-1}} (\tilde{H}_\alpha^{(b)})^2 = \sum_{\beta \in \mu_{b-1}} (H_\beta^{(b)})^2 . \tag{3.19}$$

It follows from (3.17) and (3.19) that the real valued scalar field,

$$f_{(b)} = \Big(\sum_{\alpha \in \mu_{b-1}} (H_\alpha^{(b)})^2\Big)\Big(\overline{\sum_{\alpha \in \mu_{b-1}} (H_\alpha^{(b)})^2}\Big) \tag{3.20}$$

is globally defined on the connected components of Ω_{b-1}, being independent of the choice of the frame field. $f_{(2)}$ is a globally defined smooth function on M.

We have the following decomposition of $f_{(b)}$ on Ω_{b-1}: Let

$$K_{(b)} = \sum_{\alpha\in\mu_{b-1}} \Big(h^2_{\alpha\underbrace{1\cdots1}_{b}} + h^2_{\alpha\underbrace{1\cdots12}_{b-1}}\Big) \quad \text{and}$$
$$N_{(b)} = \sum_{\alpha\in\mu_{b-1}} h^2_{\alpha\underbrace{1\cdots1}_{b}} \sum_{\alpha} h^2_{\alpha\underbrace{1\cdots12}_{b-1}} - \Big(\sum_{\alpha} h_{\alpha\underbrace{1\cdots1}_{b}} h_{\alpha\underbrace{1\cdots12}_{b-1}}\Big)^2 . \tag{3.21}$$

Then we have, by a direct calculation,

$$f_{(b)} = K^2_{(b)} - 4N_{(b)} \geqq 0 , \quad b = 2, 3, \cdots, n , \tag{3.22$_b$}$$

where $K_{(b)}$ and $N_{(b)}$ are also invariants of the isometric minimal immersion x of M defined on Ω_{b-1}. Especially we have

$$K_{(2)} = \frac{1}{2} \sum_{\alpha,i,j} h^2_{\alpha ij} , \qquad N_{(2)} = \frac{1}{16} \sum_{\alpha,\beta,i,j} R^2_{\alpha\beta ij} , \tag{3.23}$$

where $R_{\alpha\beta ij} = \sum_s (h_{\alpha is}h_{\beta js} - h_{\beta is}h_{\alpha js})$ are components of the curvature tensor of the normal connection $w_{\alpha\beta}$.

When the differentiable 2-sphere is minimally immersed into a space of constant curvature, we have $f_{(b)} = 0$, $b = 2, 3, \cdots$. This fact is essential in the papers of S. S. Chern [7], [8]. Geometrically $f_{(b)} = 0$ means that the vectors $\sum_{\alpha=2b-1}^{2b} h_{\alpha1\cdots1}e_\alpha$ and $\sum_{\alpha=2b-1}^{2b} h_{\alpha1\cdots12}e_\alpha$ in the osculating space of b-th order are perpendicular to each other and are of the same length. On the other hand, $N_{(b)} = 0$ means that the above two vectors are linearly dependent, by the Cauchy-Schwartz equality. Moreover, for the geometric meaning of $K_{(b)}$ and $N_{(b)}$, we have the following lemma by T. Ōtsuki ([15, p. 96]).

LEMMA 2 (T. Ōtsuki). *If $M = \Omega_b$ and $N_{(b)} > 0$ and $K_{(b+1)} = 0$ on M, then there is a $2b$-dimensional totally geodesic submanifold of $\bar{M}$ such that M is contained in the submanifold.*

If $M = \Omega_{b-1}$ and $N_{(b)} = 0$ and $K_{(b)} > 0$ on M, then there is a $(2b-1)$-dimensional totally geodesic submanifold of $\bar{M}$ such that M is contained in the submanifold.

4. **Laplacians of $f_{(b)}$ and $K_{(b)}$.** We use the operators ∂, $\bar\partial$ relative to a complex structure induced by an isothermal coordinate on M and

$$d^c = i(\bar\partial - \partial) . \tag{4.1}$$

For any real valued smooth function f its Laplacian Δf is defined by

$$dd^c f = \left(\frac{i}{2}\right)\Delta f \phi \wedge \bar{\phi} . \tag{4.2}$$

The following lemma is usefull to the study of $f_{(b)}$.

LEMMA 3. *Let M be a 2-dimensional oriented Riemannian manifold. Let H be a complex valued smooth function on M and $f = H\bar{H}$. Suppose that*

$$dH + niHw_{12} = \overline{A\phi} \tag{4.3}$$

holds, where n is a real constant and A is a smooth function M. Then we have

$$\Delta f = 2\{nfK + 2A\bar{A}\} , \tag{4.4}$$

$$\Delta \log f = 2nK, \text{ if } f \neq 0 . \tag{4.5}$$

PROOF. By (4.3), we have

$$df = Hd\bar{H} + \bar{H}dH = AH\phi + \overline{AH\phi} , \tag{4.6}$$

and so

$$d^c f = i(\overline{AH\phi} - AH\phi) . \tag{4.7}$$

Taking the exterior derivative of (4.3) and making use of (2.15) and (2.16), we get

$$d\bar{A} \wedge \bar{\phi} = nidH \wedge w_{12} - i\bar{A}w_{12} \wedge \bar{\phi} + \frac{n}{2}HK\phi \wedge \bar{\phi} . \tag{4.8}$$

From (4.3), (4.6), and (4.8), we derive

$$d(\overline{AH}) \wedge \bar{\phi} - d(AH) \wedge \phi = idf \wedge w_{12} + (nfK + 2A\bar{A})\phi \wedge \bar{\phi} . \tag{4.9}$$

Thus (4.4) follows from (2.15), (4.7) and (4.9). (4.5) follows from $d^c \log f = i(\overline{A\phi}/H - A\phi/\bar{H})$ and

$$d\left(\frac{\bar{A}}{H}\right) \wedge \bar{\phi} - d\left(\frac{A}{\bar{H}}\right) \wedge \phi = id \log f \wedge w_{12} + nK\phi \wedge \bar{\phi} . \tag{4.10}$$

q.e.d.

The Codazzi equation (2.10) implies (2.13). In general we have

$$\left\{dH_\alpha^{(b)} + ibH_\alpha^{(b)}w_{12} + \sum_{\beta \geqq \mu_{b-1}} H_\beta^{(b)}w_{\beta\alpha}\right\} \wedge \bar{\phi} = 0 , \quad \text{for } \alpha \geqq \mu_{b-1} . \tag{4.11$_b$}$$

To prove $(4.11)_b$ by an induction we assume, noting that (3.13)′,

$$H_{\alpha,1}^{(b-1)} = iH_{\alpha,2}^{(b-1)} , \quad \text{for } \alpha \geqq \mu_{b-2} . \tag{4.12}$$

Then we see

$$dH_\alpha^{(b-1)} + i(b-1)H_\alpha^{(b-1)}w_{12} + \sum_{\beta \geqq \mu_{b-2}} H_\beta^{(b-1)} w_{\beta\alpha} = H_{\alpha,1}^{(b-1)}\bar{\phi} \,, \tag{4.13}$$

where $\alpha \geqq \mu_{b-2}$ and $b \geqq 3$. We put

$$dH_{\alpha,1}^{(b-1)} + ibH_{\alpha,1}^{(b-1)}w_{12} + \sum_{\beta \geqq \mu_{b-2}} H_{\beta,1}^{(b-1)} w_{\beta\alpha} = H_{\alpha,1,1}^{(b-1)}w_1 + H_{\alpha,1,2}^{(b-1)}w_2 \,. \tag{4.14}$$

Taking the exterior derivative of (4.13), we have

$$\begin{aligned} &\Delta H_\alpha^{(b-1)} w_1 \wedge w_2 \\ &= (b-1)KH_\alpha^{(b-1)} w_1 \wedge w_2 + i \sum_{\beta \geqq \mu_{b-2}} H_\beta^{(b-1)} \Big(\sum_{B \in \tilde{\lambda}_{b-3}} w_{\beta B} \wedge w_{B\alpha} \Big) \,, \end{aligned} \tag{4.15}$$

where $\tilde{\lambda}_{b-3} = \{A \mid 1 \leqq A \leqq \sum_{a=0}^{b-3} p_a(x)\}$ and

$$\Delta H_\alpha^{(b-1)} = (h_{\alpha\underbrace{1\cdots1}_{b-1},1,1} + h_{\alpha\underbrace{1\cdots1}_{b-1},2,2}) + i(h_{\alpha\underbrace{1\cdots1}_{b-2}2,1,1} + h_{\alpha\underbrace{1\cdots1}_{b-2}2,2,2}) \,. \tag{4.16}$$

(4.15) implies $\Delta H_\alpha^{(b-1)} = 0$ for $\alpha \geqq \mu_{b-1}$, by $(3.4)_{b-3}$ and (3.7). By Lemma 1 and (4.16), we get (4.11). The formula (4.11) is the Codazzi equation of higher order. (4.11) implies

$$dH_\alpha^{(b)} + biH_\alpha^{(b)}w_{12} + \sum_{\beta \geqq \mu_{b-1}} H_\beta^{(b)} w_{\beta\alpha} = H_{\alpha,1}^{(b)}\bar{\phi} \,,$$

and thus we have

$$d\sum (H_\alpha^{(b)})^2 + 2bi \sum (H_\alpha^{(b)})^2 w_{12} = \overline{A_{(b)}\phi} \,, \tag{4.17}$$

where

$$\bar{A}_{(b)} = 2 \sum_{\alpha \geqq \mu_{b-1}} H_\alpha^{(b)} H_{\alpha,1}^{(b)} \,. \tag{4.18}$$

From (4.17) and Lemma 3, we have

$$\Delta f_{(b)} = 4\{bf_{(b)}K + A_{(b)}\overline{A_{(b)}}\} \,, \qquad b = 2, 3, \cdots, n \,. \tag{4.19}$$

By the Lemma 2, we shall study the case of $N_{(b)} \neq 0$, for $b = 2, 3, \cdots, n-1$. Then we have $p_b(x) = 2$, for $b = 1, \cdots, n-2$, $2b+1 \leqq \lambda_b \leqq 2b+2$, $2n-1 \leqq \lambda_{n-1} \leqq N$ and $\alpha \geqq \mu_{b-1}$ is equivalent to $\alpha \geqq 2b-1$.

Next, we calculate the term $\sum_{A \leqq 2b-4} w_{\beta A} \wedge w_{A\alpha}$ in (4.15): From $(3.15)_{b-1}$ with $\lambda_{b-2} = 2b-3$ and its conjugate, we have

$$\begin{cases} B_{(b-2)} w_{2b-5,2b-3} = \bar{H}_{(2b-4)}^{(b-2)} H_{(2b-3)}^{(b-1)} \bar{\phi} - H_{(2b-4)}^{(b-2)} \bar{H}_{(2b-3)}^{(b-1)} \phi \,, \\ \bar{B}_{(b-2)} w_{2b-4,2b-3} = \bar{H}_{(2b-5)}^{(b-2)} H_{(2b-3)}^{(b-1)} \bar{\phi} - H_{(2b-5)}^{(b-2)} \bar{H}_{(2b-3)}^{(b-1)} \phi \end{cases} \tag{4.20}$$

where

$$\begin{cases} B_{(b-2)} = \bar{H}_{(2b-4)}^{(b-2)} H_{(2b-5)}^{(b-2)} - H_{(2b-4)}^{(b-2)} \bar{H}_{(2b-5)}^{(b-2)} = -2i\sqrt{N_{(b-2)}} \,, \\ \sqrt{N_{(b-2)}} = h_{(2b-5)1\cdots1} h_{(2b-4)1\cdots12} - h_{(2b-5)1\cdots12} h_{(2b-4)1\cdots1} \,. \end{cases} \tag{4.21}$$

From $(3.15)_{b-1}$ with $\lambda_{b-2} = 2b - 2$ we find

$$(4.22)\qquad \begin{aligned} B_{(b-2)} w_{(2b-5)(2b-2)} &= \bar{H}^{(b-2)}_{(2b-4)} H^{(b-1)}_{(2b-2)} \bar{\phi} - H^{(b-2)}_{(2b-4)} \bar{H}^{(b-1)}_{(2b-2)} \phi \\ \bar{B}_{(b-2)} w_{(2b-4)(2b-2)} &= \bar{H}^{(b-2)}_{(2b-5)} H^{(b-1)}_{(2b-2)} \bar{\phi} - H^{(b-2)}_{(2b-5)} \bar{H}^{(b-1)}_{(2b-2)} \phi . \end{aligned}$$

By (4.20) and (4.22), we have

$$\sum_{A \leqq 2b-4} w_{(2b-3)A} \wedge w_{(2b-2)A} = \left\{ \frac{\| H^{(b-2)}_{(2b-5)} \|^2}{\bar{B}^2_{(b-2)}} + \frac{\| H^{(b-2)}_{(2b-4)} \|^2}{B^2_{(b-2)}} \right\} \bar{B}_{(b-1)} \bar{\phi} \wedge \phi .$$

By (4.21) it follows that

$$\sum_{A \leqq 2b-4} w_{(2b-3)A} \wedge w_{A(2b-2)} = -\frac{K_{(b-2)}}{N_{(b-2)}} \sqrt{N_{(b-1)}}\, w_1 \wedge w_2, \text{ for } b \geqq 3 ,$$

where we have put $H^{(1)}_1 = 1$, $H^{(1)}_2 = i$ and thus $N_{(1)} = 1$, $K_{(1)} = 2$. Thus we find

$$(4.23)\qquad \sum_{\alpha \geqq 2b-3} \bar{H}^{(b-1)}_\alpha \Delta H^{(b-1)}_\alpha = (b-1) K K_{(b-1)} - 2 \frac{K_{(b-2)}}{N_{(b-2)}} N_{(b-1)} .$$

On the other hand, by virtue of (4.13) and (4.14), we have

$$(4.24)\qquad \begin{aligned} dK_{(b-1)} &= \sum_{\alpha \geqq 2b-3} \{ H^{(b-1)}_\alpha \bar{H}^{(b-1)}_{\alpha,1} \phi + \bar{H}^{(b-1)}_\alpha H^{(b-1)}_{\alpha,1} \bar{\phi} \} , \\ dd^c K_{(b-1)} &= i \sum_{\alpha \geqq 2b-3} \{ \bar{H}^{(b-1)}_\alpha \Delta H^{(b-1)}_\alpha + 2 H^{(b-1)}_{\alpha,1} \bar{H}^{(b-1)}_{\alpha,1} \} \phi \wedge \bar{\phi} . \end{aligned}$$

From these formulas, (4.23) and Lemma 1, we get

$$\frac{1}{2} \Delta K_{(b-1)} = (b-1) K K_{(b-1)} - 2 \frac{K_{(b-2)}}{N_{(b-2)}} N_{(b-1)} + 2 K_{(b)} + 2 \sum_{\lambda_{b-2}} (H^{(b-1)}_{\lambda_{b-2},1} \overline{H^{(b-1)}_{\lambda_{b-2},1}}) .$$

Summarizing these results, we get

Theorem 1. *Let M be a minimal surface in a space $\bar{M}$ of constant curvature. Then on a neighborhood of a regular point of order $n - 1 \geqq 2$, we have*

$$(4.25)\qquad H^{(b)}_{\alpha,1} = i H^{(b)}_{\alpha,2}, \text{ for } \alpha \geqq \mu_{b-1} \text{ and } b = 2, 3, \cdots, n ;$$

$$(4.26)_b\qquad \Delta f_{(b)} = 4 \{ b f_{(b)} K + A_{(b)} \overline{A_{(b)}} \} ;$$

$$(4.27)_b\qquad \begin{aligned} \frac{1}{2} \Delta K_{(b)} = &-2 \frac{N_{(b)}}{N_{(b-1)}} K_{(b-1)} + b K K_{(b)} + 2 K_{(b+1)} \\ &+ 2 \sum_{\lambda_{b-1}} (H^{(b)}_{\lambda_{b-1},1} \overline{H^{(b)}_{\lambda_{b-1},1}}), \text{ if } N_{(b-1)} \neq 0,\ 2 \leqq b \leqq n-1 . \end{aligned}$$

Remark 2. If M is a 2-sphere, then (4.27) is proved by S. S. Chern [7, p. 38].

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Added in proof (May, 1973). Since the completion of this paper, the author obtained n-dimensional generalizations of (4.25) and (4.27); the details are to appear in a sequel.