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ON CERTAIN HYPERSURFACES IN A REAL SPACE FORM

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Using the formula of Simons' type, many and interested studies have recently been done for hypersurfaces in a real space from. As one of special situations, they have common pattern that the second fundamental form has distinct constant eigenvalues. On the other hand, T. Otsuki [1] has investigated the problem to determine all minimal hypersurfaces immersed in a sphere on which the number of distinct principal curvatures is equal to two.

The purpose of this paper is to study hypersurfaces in a real space form such that the second fundamental form has at most two distinct eigenvalues. In § 2, we study hypersurfaces in a real space form such that the product of two distinct eigenvalues is equal to minus of the curvature in the ambient space. In § 3, we treat connected and complete hypersurfaces in a hyperbolic space which have the same curvature as that of the ambient space and we shall show that there exist many examples of such hypersurfaces which are not totally geodesic.

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1. Preliminaries. Let M be an m-dimensional Riemannian manifold isometrically immersed in an $(m+1)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$ with the immersion $f: M \to \bar{M}$. Let $F(M)$ and $F(\bar{M})$ be the bundles of all orthonormal frames over M and $\bar{M}$ respectively. Let B be the set of all elements $b = (x, e_1, e_2, \cdots, e_m, e_{m+1}) \in F(\bar{M})$ such that $(x, e_1, e_2, \cdots, e_m) \in F(M)$, identifying $x \in M$ with $f(x)$ in $\bar{M}$ and e_i with $df(e_i)$ for $i = 1, 2, \cdots, m$. Then, B is considered as a smooth submanifold of $F(\bar{M})$. We have, as is well known, a system of differential 1-forms $\omega_i, \omega_{ij}, \omega_{im+1}$ $(i, j = 1, 2, \cdots, m)$ and ω_{m+1} on B associated with the immersion f such that

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$$
(1.1)\qquad \begin{cases} \omega_{ij} = -\omega_{ji},\ \omega_{im+1} = -\omega_{m+1i},\ \omega_{m+1} = 0\ , \\ d\omega_i = \sum_{j=1}^m \omega_{ij} \wedge \omega_j\ , \\ d\omega_{ij} = \sum_{k=1}^m \omega_{ik} \wedge \omega_{kj} + \omega_{im+1} \wedge \omega_{m+1j} - \bar{c}\omega_i \wedge \omega_j\ , \\ d\omega_{im+1} = \sum_{j=1}^m \omega_{ij} \wedge \omega_{jm+1}\ , \end{cases}
$$

and

$$
(1.2)\qquad \omega_{im+1} = \sum_{j=1}^m A_{ij}\omega_j\ , \qquad A_{ij} = A_{ji}\ .
$$

Throughout this paper, we assume that M *has at most two distinct principal curvatures*, say λ and μ. Then λ and μ are continuous on M and are differentiable on the set N of all points at which one principal curvature is different from the other. Furthermore N is clearly open in M. We can choose a neighborhood U of a point $p \in N$ where there exists $b \in B$ such that

$$
(1.3)\qquad \begin{cases} \omega_{am+1} = \lambda\omega_a\ , \qquad \omega_{\alpha m+1} = \mu\omega_\alpha\ , \\ a = 1, 2, \cdots, r, \qquad \alpha = r+1, r+2, \cdots, m\ , \end{cases}
$$

where r is the multiplicity of λ. It follows from (1.3) that we have

$$
(1.4)\qquad \begin{cases} d\lambda \wedge \omega_a + (\lambda - \mu)\sum_\alpha \omega_{a\alpha} \wedge \omega_\alpha = 0\ , \\ d\mu \wedge \omega_\alpha + (\lambda - \mu)\sum_a \omega_{a\alpha} \wedge \omega_a = 0\ . \end{cases}
$$

2. Hypersurfaces with two distinct principal curvatures. In this section, we shall study principal curvatures of hypersurfaces in a Riemannian manifold $\bar{M}$ of non-zero constant curvature $\bar{c}$ which have at most two distinct principal curvatures, say λ and μ, such that $\lambda\mu + \bar{c} = 0$ on N. We shall prove the following

THEOREM. *Let M be an $m(\geqq 3)$-dimensional connected and complete Riemannian manifold which is isometrically immersed in an $(m+1)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}(\neq 0)$. If M has at most two distinct principal curvatures λ and μ at each point of M and they satisfy $\lambda\mu + \bar{c} = 0$ where $\lambda \neq \mu$, then λ and μ are constant on M.*

PROOF. We shall use the same notation as that in § 1. To simplify the statement we may assume without any loss of generality that $\lambda > \mu$ on N and we first consider the case that there exists a point of N at which the multiplicity of λ is equal to r, where r satisfies the restriction $2 \leqq r \leqq m-2$. Let E be the set of all points of N at which the multiplicity of λ is equal to r. Then E is open in N, so it is also open in M. Since the multiplicities of the principal curvatures are both constant on E, it is well known in [1] and [2] that λ and μ are constant on E. It follows from the continuity of λ that the multiplicity of λ is equal to r on the boundary of E, so it is closed in M. Since M is connected, the multiplicity of λ is constant r on M, so that λ and μ are constant on M.

Next, we shall consider the case that one of the principal curvatures is simple, that is, the multiplicity of one of the principal curvatures is 1 on N. Without loss of generality, we may assume that the multiplicity of λ is equal to 1 on N. Let G be the set of all points of N at which the gradient of λ is a non-zero vector. It is evident that G is open in N, so that it is open in M. If the set G is empty, then λ and μ are constant on each component of N. Since $\lambda \neq \mu$ on N and λ and μ are continuous on M, we have the same property on the boundary of N. Accordingly N is closed in M. It follows from the fact that M is connected that N must coincide with M itself, so this implies that λ and μ are constant on M. Therefore, we may consider the other case $G \neq \emptyset$. Then, on a neighborhood V of a point $q \in G$ in G, there exist frame fields in B satisfying the condition (1.3). Using the structure equations (1.1), we obtain

$$
\begin{cases} d\lambda \wedge \omega_1 + (\lambda - \mu) d\omega_1 = 0 \,, \\ d\mu \wedge \omega_\alpha - (\lambda - \mu)\omega_1 \wedge \omega_{1\alpha} = 0 \quad \text{for} \quad \alpha = 2, 3, \cdots, m \,. \end{cases} \tag{2.1}
$$

From the second equation of (2.1) we get

$$
d\mu = \mu_1 \omega_1 \,, \tag{2.2}
$$

because of the assumption of dimension and E. Cartan's lemma. By the condition $\lambda\mu = -\bar{c} =$ constant, we have

$$
d\lambda = \lambda_1 \omega_1 \,. \tag{2.3}
$$

It follows from the equations (2.1), (2.2) and (2.3) that we have

$$
d\omega_1 = 0 \,, \tag{2.4}
$$

which implies that there exists a function u on a neighborhood W in V such that $\omega_1 = du$. In W, u may be considered as a distance from the integral submanifold through q corresponding to μ. Since the multiplicities of λ and μ are constant on N and M is complete, we may consider u as a function defined on N. In particular, there exists a geodesic $\Gamma = \{\gamma(u)\}$ parametrized by arc length u in such a way that

$$
\gamma(0) = q \,, \quad \gamma'(0) = e_1 \,, \quad \omega_1 = du \,. \tag{2.5}
$$

Since M is complete, we can extend this geodesic Γ infinitely in both directions. We denote the extended geodesic by the same symbol $\Gamma = \{\gamma(u)\}$. It follows from the equations (2.1) $\sim$ (2.4) that the principal curvature μ is constant along the integral submanifold corresponding to μ, and μ is a function of u. From (2.1) we get

$$\omega_{1\beta} = (\mu'/(\lambda - \mu))\omega_\beta \,, \qquad \beta = 2, 3, \cdots, m \,, \tag{2.6}$$

where $\mu' = d\mu/du = \mu_1$. Using (1.1) and (1.2), from (2.6) we have

$$(\mu'/(\lambda - \mu))' + (\mu'/(\lambda - \mu))^2 = 0 \,. \tag{2.7}$$

Since $\mu' = \operatorname{grad} \mu \neq 0$ on G, it follows from (2.7) on G that the following equation

$$\mu'/(\lambda - \mu) = 1/(u + c_1) \,, \qquad c_1 = \text{constant} \neq 0 \tag{2.8}$$

is obtained, so that on G we have the solution

$$\mu^2 + \bar{c} = c_2/(u + c_1)^2 \,, \qquad c_2 = \text{constant} \neq 0 \,. \tag{2.9}$$

Now, we may consider the following two cases:

$$\text{Case (1):} \quad \partial G \cap \Gamma = \varnothing \,, \qquad \text{Case (2):} \quad \partial G \cap \Gamma \neq \varnothing \,.$$

In the first case, since $\gamma(u)$ is defined on $-\infty < u < \infty$, there exists $u_0 = -c_1 \neq 0$, so that we have

$$\lim_{u \to u_0} (\mu^2 + \bar{c}) = +\infty \ \text{ or } \ -\infty \,,$$

which contradicts the continuity of $\mu^2 + \bar{c}$.

In the case (2), let $q_1 = \gamma(u_1)$ be the first point at which the geodesic $\Gamma = \{\gamma(u)\}$ meets the boundary of G. In this case we may suppose that $0 < u_1$, $\gamma(u) \in G$ for $0 < {}^\forall u < u_1$ and $\gamma(u_1) \notin G$. Then we can consider the following two cases:

$$\text{Case (a)} \quad q_1 = \gamma(u_1) \in N \,, \qquad \text{Case (b)} \quad q_1 = \gamma(u_1) \in \partial N \,.$$

In the case (a), we can see that

$$\lim_{u \to u_1 - 0} \operatorname{grad} \mu \neq 0 \,,$$

which contradicts $\operatorname{grad} \mu = 0$ at q_1. In the case (b), we have

$$\lim_{u \to u_1 - 0} (\mu^2 + \bar{c}) \neq 0 \,,$$

which contradicts $\mu^2 + \bar{c} = 0$ at q_1. Thus G must be empty. This completes the proof.

REMARK. Using Theorem in this section, we see that Theorem of S. Tanno and T. Takahashi [4] holds for $m = 3$.

3. Hypersurfaces in a hyperbolic space. Let M be an m-dimensional connected and complete Riemannian manifold isometrically immersed in an $(m+1)$-dimensional hyperbolic space $H^{m+1}(c)$ of constant curvature c. In this section, we assume that M is also *of constant curvature c*. Then we shall give many examples of such hypersurfaces which are not totally geodesic.

Let $g = (g_{ij})$, $A = (A_{ij})$ and $\left\{ {i \atop jk} \right\}$ be the metric tensor of M, the second fundamental tensor of M and the Christoffel's symbols of g. Then, in our situation, as is well known, the Gauss' and the Codazzi's equations are written by

$$A_{ik}A_{lj} - A_{il}A_{jk} = 0 , \tag{3.1}$$

$$A_{ij,k} + \sum_l \left\{ {l \atop ij} \right\} A_{lk} = A_{ik,j} + \sum_l \left\{ {l \atop ik} \right\} A_{lj} , \tag{3.2}$$

where $A_{ij,k} = \partial A_{ij}/\partial x^k$ when $(x^1, x^2, \cdots, x^m)$ is a local coordinate system of M.

Now, as a model space of an m-dimensional hyperbolic space $H^m(c)$ of constant curvature c, we take the upper half space $R^m_+ = \{(x^1, x^2, \cdots, x^m) \in R^m \mid x^m > 0\}$ with the metric tensor given by

$$g_{ij} = -\delta_{ij}/\{c(x^m)^2\} . \tag{3.3}$$

In this case, Christoffel's symbols of g are

$$\left\{ {k \atop ij} \right\} = -(\delta_{kj}\delta_{im} + \delta_{ki}\delta_{jm} - \delta_{ij}\delta_{km})/x^m . \tag{3.4}$$

By Sasaki's Theorem 1 in [3], if we give a non-trivial symmetric tensor A of type (0, 2) on R^m_+ satisfying (3.1) and (3.2), then we obtain a hypersurface in $H^{m+1}(c)$ which has A as its second fundamental tensor and is not totally geodesic. Hence, in order to find a complete hypersurface of constant curvature c in $H^{m+1}(c)$ which is not totally geodesic, it is sufficient to find only a non-trivial symmetric tensor A of type (0, 2) on R^m_+ satisfying (3.1) and (3.2). From (3.1), we see that the matrix $A = (A_{ij})$ is at most of rank 1, so that we may consider the special case

$$A_{11} = \lambda \text{ and } A_{ij} = 0 \text{ if } (i, j) \neq (1, 1) , \tag{3.5}$$

where λ is a differentiable function on R^m_+. Then, it is clear that (3.1) holds. Using (3.4) and (3.5), we see that (3.2) is equivalent to the following differential equations:

$$\begin{cases} \lambda_{,k} = 0 & \text{for } k = 2, 3, \cdots, m-1\,, \\ \lambda_{,m} = -\lambda/x^m\,. \end{cases} \tag{3.6}$$

We easily see that a function

$$\lambda = \lambda(x^1, x^m) = g(x^1)/x^m \tag{3.7}$$

is a solution of (3.6), where $g(x^1)$ is an arbitrary non-zero differentiable function on R_+^m of x^1. For two distinct functions $g(x^1)$ and $\tilde{g}(x^1)$, we have two distinct solutions λ and $\tilde{\lambda}$ of (3.6), that is, two distinct non-trivial symmetric tensors A and $\tilde{A}$ on R_+^m satisfying (3.1) and (3.2). On the other hand, by Sasaki's Theorem 2 in [3], we see that there exist two hypersurfaces of constant curvature c in $H^{m+1}(c)$ which have A and $\tilde{A}$ as their fundamental tensors respectively and they are not congruent in the large under the group of motions of $H^{m+1}(c)$. Thus, we obtain many examples of hypersurfaces of constant curvature c in $H^{m+1}(c)$ which are not totally geodesic.

By the above argument, we have shown that there exist many examples of complete hypersurfaces with type number 1 in a hyperbolic space $H^{m+1}(c)$ of constant curvature c which are not congruent in the large under the group of motions of $H^{m+1}(c)$.

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