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# APPLICATIONS OF A RICCATI TYPE DIFFERENTIAL EQUATION TO RIEMANNIAN MANIFOLDS WITH TOTALLY GEODESIC DISTRIBUTIONS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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**0. Introduction.** In this paper we study the class of Riemannian manifolds $M$ equipped with tangent subbundles which are invariant under covariant differrentiation. These subbundles are called totally geodesic distributions. Indeed, they are integrable distributions in the usual sense and their integral submanifolds are totally geodesic submanifolds of $M$. Although it is, no doubt, of interest to study such distributions in general, we are particularly motivated by the example of Chern and Kuiper [3]. Their example comes up naturally in the study of submanifolds of space forms, and is called the relative nullity distribution of the submanifold. Since its introduction, the study of this kind of distributions has provided us with a good deal of information on the submanifolds of space forms. Roughly speaking, those results concern the following two questions in connection with the relative nullity distribution. One is whether or not a submanifold of the flat space form has a natural Riemannian product structure with respect to the distribution. The other is the problem as to what kind of estimate on the dimension of the relative nullity distribution, i.e., the index of relative nullity, can be made if $M$ is a submanifold of $S^N$ or $CP^N$. The results in the first category are called the cylinder theorems after Hartman and Nirenberg and may be found in [2], [6], [7], [10] and [13]. As for the latter, an estimate of the index was first given by Nomizu [9] for compact complex hypersurfaces of $CP^N$, and the same estimate was later extended to all complete submanifolds of $S^N$ and $CP^N$ [1]. This estimate is, however, quite crude.

More recently, Ferus has improved this estimate for submanifolds of $S^N$ by making use of a Riccati type differential equation [4]. From consideration of known examples, his results seems to be the best possible. Although it was known in special cases that totally geodesic distributions satisfy the Riccati equation, Ferus has shown in a clear way that any such distributions satisfy the same equation.

In § 1, we rewrite this differential equation to suit our purpose. We obtain in this section some results on non-integrability of the orthogonal distributions and on a semi-global product structure of manifolds with totally geodesic distributions under a certain condition. As special cases, we get existence of zeros of certain vector fields over complete surfaces of positive curvature and a comprehensive description of the totally geodesic distributions over complete surfaces of non-negative curvature.

In § 2 we find a new estimate of the index. Our main result is that the index of relative nullity of a complete Kählerian submanifold $M$ of $CP^N$ is either 0 or the dimension of $M$. This is a partial answer to the conjecture made in [5].

In § 3 further use of the Riccati type equation is made to prove a cylinder theorem on Kählerian submanifolds of $C^N$. We also give an intrinsic splitting of certain Kählerian manifolds.

**1. Preparations and some direct applications of the Riccati type equation.** Let $M$ be an $n$-dimensional Riemannian manifold. We denote by $D$ a $\mu$-dimensional smooth distribution over $G$, i.e., a $\mu$-dimensional smooth subbundle of the tangent bundle $TM$ of $M$ given over an open subset $G$ of $M$. Furthermore, let us assume that the distribution $D$ is totally geodesic, i.e., for any two smooth vector fields $X$ and $Y$ of $D$, $\nabla_X Y$ is also a vector field of $D$, where $\nabla$ is the Riemannian connection of $M$. Since the equation $\nabla_X Y - \nabla_Y X = [X, Y]$ holds, $D$ is integrable in the usual sense; therefore, we can speak of the maximum integral submanifolds of $D$. Those submanifolds are often refered to as the leaves of $D$. Let $D_x$ be the subspace of the tangent space $TM_x$ of $M$ at $x$ given by the distribution $D$. We denote by $D'_x$ the orthogonal complement of $D_x$ in $TM_x$ with respect to the Riemannian metric $g$ of $M$. Then the orthogonal distribution $D'$ over $G$ is defined to be the mapping that assigns to each $x \in G$ the orthogonal complement $D'_x$. Since $D'$ determines a unique subbundle of $TM$, it is clear that the Riemannian connection $\nabla$ induces a canonical metric connection in $D'$ as follows: If $X$ and $Y$ are smooth vector fields in $TM$ and $D'$, respectively, define the connection $\nabla'$ in $D'$ by $\nabla'_X Y = P\nabla_X Y$, where $P$ is the projection of the tangent bundle onto $D'$. Now let $X$ and $Y$ be smooth vector fields of $D$ and $D'$, respectively. Define a linear operator $A(x, X): D'_x \to D'_x$ by $A(x, X)(Y) = -P(\nabla_Y X)_x$, where the subscript $x$ means the restriction to the point $x$. $A(x, X)$ is called the conullity operator at $x$ in the direction

$X$ [13]. For the sake of convenience, we shall denote $A(x, X)(Y)$ by $A(x, X, Y)$.

PROPOSITION 1. *$A(x, X, Y)$ depends on the values of $X$ and $Y$ at $x$ alone, and the mapping that assigns to each $x$ the operator $A(x, \ )$ is a smooth section of the bundle* Hom $(D, \text{End } D')$, *where* End $D'$ *is the set of all endomorphisms of $D'$.*

PROOF. The last half of the statement is clear. In order to show the first half, it is sufficient to verify that $A(x, fX, Y)$ equals $f(x)A(x, X, Y)$, where $f$ is a smooth function in a neighborhood of $x$. Now by the definition of the operator, for any $Y \in D'_x$ $A(x, fX, Y) = -P(\nabla_Y fX)_x = -P(Yf \cdot X + f\nabla_Y X)_x = -P(f\nabla_Y X)_x = -f(x)P(\nabla_Y X)_x = f(x)A(x, X, Y)$

q.e.d.

Let $\lambda(t)$ be a unit speed geodesic in one of the leaves of $D$. Since these leaves are totally godesic submanifolds of $M$, we might as well consider $\lambda$ to be a geodesic of $M$. Denote by $\dot{\lambda}(t)$ the velocity vectors of $\lambda$ at $t$. Then by mimicing the computation by Ferus [4], one can reach the following differential equation along $\lambda$:

$$(\nabla'_{\dot{\lambda}(t)} A(\lambda(t), \dot{\lambda}(t)))Y = A^2(\lambda(t), \dot{\lambda}(t))Y - PR(\dot{\lambda}(t), Y)\dot{\lambda}(t) . \tag{1}$$

Here $R$ is the curvature tensor of $M$. In order to have a better look at the equation (1), let $Y_1(t), \cdots, Y_{n-\mu}(t)$ be a parallel frame field of $D'$ along $\lambda$. Let $[\alpha_{ij}(t)]$, $1 \leqq i, j \leqq n-\mu$, be the matrix representation of $A(\lambda(t), \dot{\lambda}(t))$ relative to the frame field along $\lambda$. If we take it into account that $Y_i$, $1 \leqq i \leqq n-\mu$, is parallel along $\lambda$ and $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$, we have reached the following lemma:

LEMMA 1. *$[\alpha_{ij}(t)]$ satisfies the following Riccati type differential equation:*

$$[\alpha'_{ij}(t)] = [\alpha_{ij}(t)]^2 + [K_{ij}(t)] , \tag{2}$$

*Here the superscript 2 means the usual matrix product and $K_{ij}(t)$ is given by $K_{ij}(t) = g(R(\dot{\lambda}(t), Y_i(t))Y_j(t), \dot{\lambda}(t))$.*

REMARK. A well known equation $g(R(X, Y)V, U) = g(R(U, V)Y, X)$ tells us that $[K_{ij}(t)]$ is a symmetric matrix. Also note that $K_{ii}(t)$ is the sectional curvature of the plane spanned by $\dot{\lambda}(t)$ and $Y_i(t)$,

We shall now show a property of the differential equation which will be used throughout this section.

PROPOSITION 2. *Let $[\alpha'_{ij}] = [\alpha_{ij}]^2 + [K_{ij}(t)]$ be the Riccati type equation as in Lemma 1. Assume that either* (1) *$K$ is positive semidefinite* *for all $t$, or (2) $K$ is constant and the trace of $K$ is non-negative. Then $A(t)=0$ is the only possible global solution with an initial condition $A(0)$ which is symmetric and has non-negative eigenvalues. Consequently, $K(t)=0$ for all $t$ if the differential equation has a global solution under the given conditions.*

PROOF. First of all, we shall show that any solution to our differential equation with a symmetric initial condition must be symmetric as long as the solution exists. Now let $[\alpha_{ij}(t)]$ be a solution with a symmetric initial condition $[\alpha_{ij}(0)]$. Consider the transposed equation ${}^t[\alpha'_{ij}] = {}^t[\alpha_{ij}]^2 + {}^t[K_{ij}(t)]$, where the superscript $t$ means the transpose. Then it is easy to see that the transpose ${}^t[\alpha_{ij}(t)]$ of the original solution is a solution to it with the initial condition ${}^t[\alpha_{ij}(0)] = [\alpha_{ij}(0)]$. Because of symmetry of $K$, those two equations turn out to be the same as systems of differential equations. Thus uniqueness of the solutions implies that ${}^t[\alpha_{ij}(t)] = [\alpha_{ij}(t)]$.

From now on, let us assume that the initial condition has a diagonal form. Note here that we do not lose any generality by assuming so.

Now writing down the equation componentwisely, we have for a solution $[\alpha_{ij}(t)]$,

$$\alpha'_{ij}(t)=\sum_k \alpha_{ik}(t)\alpha_{kj}(t) + K_{ij}(t) \,. \tag{3}$$

In particular, if $i=j$, since $[\alpha_{ij}(t)]$ being symmetric, we have

$$\alpha'_{ii}(t) = \sum_k \alpha^2_{ik}(t) + K_{ii}(t) \quad \text{for all} \quad 1 \leqq i \leqq n-\mu \,. \tag{4}$$

First let $K_{ii}(t)=0$ for all $t$. Since (4) can be decomposed as $\alpha'_{ii}(t) = \alpha^2_{ii}(t) +$ non-negative term, and since $\alpha_{ii}(0)$ is non-negative by the condition, one can conclude that the solution to (4) grows more rapidly than that to the differential equation $\beta' = \beta^2$ with the same initial condition $\alpha_{ii}(0)$. Because of uniqueness of the solution, the latter has $\beta(t) = \alpha_{ii}(0)/(1 - t\alpha_{ii}(0))$ as its solution. This solution is global if and only if $\alpha_{ii}(0)=0$, since it would otherwise blow up at $t = 1/\alpha_{ii}(0)$. Thus we have shown all the eigenvalues of $[\alpha_{ij}(0)]$ are 0, i.e., $[\alpha_{ij}(0)]=0$. Suppose that $\alpha_{hh}(s) \neq 0$ for some $h$ and at some $s$. It is clear that $\alpha_{hh}(s)$ is positive. Then taking a new variable $t' = t - s$ makes it possible to apply the above argument to the new equation with $\alpha_{kk}(s)$ as its initial condition. This provides a contradiction. Therefore, $\alpha_{kk}(t)=0$ for all $t$ and for all $1 \leqq k \leqq n-\mu$. Consequently, $\alpha_{ij}(t)=0$ for all $t$ and for all $i$ and $j$. Hence, $[\alpha_{ij}(t)]=0$ and $[K_{ij}(t)]=0$ for all $t$. Now suppose that $K_{ii}(t)$ is non-negative, but not identically 0 for some $i$. It is clear that $\alpha_{ii}(s)$

is positive for some $s$. This implies that one can assume that $\alpha_{ii}(0)$ is positive without loss of generality. Then we see that the solution to (4) with $\alpha_{ii}(0)$ as its initial condition blows up, since the solution to $\beta' = \beta^2$ with the initial condition $\alpha_{ii}(0)$ does. This is a contradiction. Therefore, $K_{ii}$ must be 0 for all $i$. Hence, $[\alpha_{ij}(t)] = 0$ and $[K_{ij}(t)] = 0$ for all $t$.

So the first half of Lemma 2 has been proven. As for the second half, one of the diagonal elements of $K$, say $K_{ii}$ must be non-negative by the condition $\sum_i K_{ii}$ being non-negative. By applying the argument of the first half, we get $K_{ii} = 0$ and $\alpha_{ij}(t) = 0$ for all $t$ and all $j$. This forces us to conclude that there is a non-negative diagonal element of $K$ other than $K_{ii}$, and consequently 0. Repeating this process leads us to the conclusion that $K = 0$ and $[\alpha_{ij}(t)] = 0$ for all $t$. q.e.d.

Before stating our first theorem, we shall introduce a convention. Let $M$ be a Riemannian manifold of dimension $n$ as before. We say a distribution $D$ is involutive at a point $x$ if for any two smooth vector fields $X$ and $Y$ of $D$, the vector $[X, Y]_x$ given by the bracket $[X, Y]$ at $x$ belongs to $D_x$. In particular, $D$ is involutive everywhere means that it is integrable.

THEOREM 1. *Let $M$ be an $n$-dimensional Riemannian manifold with a totally geodesic distribution $D$ over an open subset $G$ of $M$. Let $\mu$ be the dimension of $D$. Furthermore, assume that $D$ is complete,* i.e., *all the leaves are complete manifolds. Then the orthogonal distribution $D'$ is nowhere involutive over $G$ if $M$ has positive sectional curvatures. In particular, $D'$ cannot be integrable.*

PROOF. Suppose $D'$ is involutive at $x \in G$. Let $\lambda(t)$ be a unit speed geodesic in the leaf containing $x$ such that $\lambda(0) = x$. Let $X \in D$ be a local extension of $\dot{\lambda}(t)$ in a neighborhood of $x$. For any two vector fields $Y$ and $Z$ of $D'$ around $x$, we have $g(A(x, X, Y), Z) = -g(P\nabla_Y X, Z)_x = -g(\nabla_Y X, Z)_x$. On the other hand, $g(\nabla_Y X, Z) + g(X, \nabla_Y Z) = Y \cdot g(X, Z) = 0$ implies that $g(A(x,X,Y),Z)_x = g(X,\nabla_Y Z)_x$. Similarly, we have $g(A(x,X,Z),Y) = g(X, \nabla_Z Y)_x$. Thus $g(A(x, X, Y), Z) - g(A(x, X, Z), Y) = g(\dot{\lambda}(t), [Y, Z])_x = 0$, because $D'$ is involutive at $x$ by the assumption.

We have shown, therefore, that $A(\lambda(0), \dot{\lambda}(0))$ is a symmetric operator. It is clear from the assumption of Theorem 1 and the remark below Lemma 1 that $[K_{ij}(t)]$ is positive definite at every $t$. Suppose that $A(\lambda(0), \dot{\lambda}(0))$ has a non-negative eigenvalue. Then the argument concerning the equation (4) in the proof of Proposition 2 tells us that $A(\lambda(t), \dot{\lambda}(t))$ blows up at some $t$. This is a contradiction, since $A(\lambda(t), \dot{\lambda}(t))$

is globally defined and differentiable at all $t$. Now if $A(\lambda(0), \dot{\lambda}(0))$ has only negative eigenvalues, take $A(\lambda(0), -\dot{\lambda}(0))$ which obviously has non-negative eigenvalues. Again we face a contradiction by Proposition 2. q.e.d.

COROLLARY 1. *Let $M$ be a 2-dimensional complete Riemannian manifold with positive Gaussian curvature. Then any totally geodesic distributions over $M$ must be trivial ones. In particular, any vector field of $M$ whose trajectories are geodesics must have at least one zero.*

PROOF. Since any 1-dimensional distribution is integrable, the first half is clear by Theorem 1. If the vector field is non-vanishing, it gives rise to a totally geodesic ditribution over $M$. This is impossible.

Next we shall display some results which can be shown easily by applying the argument in the proof of Proposition 2.

THEOREM 2. *Let $D$ be a totally geodesic distribution over a complete Riemannian manifold $M$ with non-negative sectional curvatures such that the orthogonal distribution $D'$ is integrable. Then for each point $x$ in $M$ there is an open neighborhood $U$ of $x$ which is a Riemannian product of an open neighborhood $U_1$ of $x$ in the leaf of $D$ passing through $x$ and an open neighborhood $U_2$ of $x$ in the leaf of $D'$ passing through $x$. Furthermore, if $M$ is simply connected, $M$ is the global Riemannian product of a leaf of $D$ and a leaf of $D'$.*

PROOF. Since $D'$ is integrable, our linear operators $A$ are nothing but the second fundamental tensors of the leaves of $D'$. Therefore, all $A$ are symmetric operators. It is easy to see that $[K_{ij}(t)]$ are positive semi-definite. Thus by Proposition 2, $A = 0$ and $K = 0$. Note here that the eigenvalues of $A$ can be assumed non-negative without loss of generality. Hence, the definition of $A$ implies that $D$ and $D'$ are parallel. The local decomposition theorem of de Rham gives us the first half. The last half is also a mere consequence of the global decomposition theorem of de Rham. For the detail, see [8]. A less general result of this type has been shown independently in [14]. q.e.d.

COROLLARY 2. *Let $M$ be a complete Riemannian manifold with non-negative sectional curvatures. Let $D$ be a totally geodesic distribution of codimension 1 over $M$. Then $M$ is locally the Riemannian product in the sense of Theorem 2. In particular, if $M$ is simply connected, the above product is global.*

PROOF. Obvious because the orthogonal distribution is integrable.

COROLLARY 3. *Let $M$ be a 2-dimensional complete Riemannian manifold with non-negative Gaussian curvature. If there is a non-trivial totally geodesic distribution over $M$, $M$ is flat. Moreover, any such distribution is covered by a totally geodesic distribution of $R^2$ given by the family of parallel lines defined by $y = cx + d$, where $c$ is a constant and $d$ varies over all real numbers. In fact $M$ must be one of the following: the Euclidean space, the cylinders, the tori, the Möbius bands and the Klein bottles.*

PROOF. The first half is clear. As for the last half, it suffices to say that any totally geodesic distribution is lifted to the universal covering $R^2$ as a totally geodesic distribution, which is given by the above form. For the last remark, see Theorem 2.2.5 [15]. q.e.d.

THEOREM 3. *Let $M$ be a complete locally symmetric space with a totally geodesic distribution $D$ such that $D'$ is integrable. Assume that $[K_{ij}(t)]$ of Lemma 1 has non-negative trace for each direction at each point. Then $M$ is locally a Riemannian product in the sense of Theorem 2. If $M$ is, in addition, simply connected, it is actually a global product.*

PROOF. All we have to show is that $K$ is constant. Since $M$ is locally symmetric, we have $\nabla R = 0$, where $R$ is the curvature tensor. Thus by the definition of $K$ in Lemma 1, $K$ is constant along any geodesic. Now we can reduce Theorem 3 to (2) of Proposition 2 without loss of generality. q.e.d.

COROLLARY 4. *Let $M$ be a complete locally symmetric space with non-negative Ricci curvatures. If there is any 1-dimensional totally geodesic distribution $D$ such that $D'$ is integrable, then $M$ is locally a product in the sense of Theorem 2. In particular, if $M$ is simply connected, $M$ is a product of the real line $R$ and a locally symmetric space.*

PROOF. It suffices to see that the trace of $K$ is a Ricci curvature and any 1-dimensional complete and simply connected manifold must be the real line. q.e.d.

Having observed a few direct results of Proposition 2, we ask whether or not we can get a somewhat more global aspect for the product structure without simply connectedness. To this end, let us start with two well known examples. Let $T = R^2/\Gamma$ be the flat torus, where $\Gamma$ is the set of all integral pairs in $R^2$. Then the family of parallel lines in $R^2$ given in Corollary 3 gives rise to a totally geodesic distribution over $T$. If the slope $c$ is irrational, any leaf of the distribution is dense in $T$. So it is impossible to obtain a product form without certain conditions. On the other hand, if $c$ is rational, each leaf is a circle which winds around $T$ a finite number of times. This example suggests a sort of semi-global product structure: Any leaf is isometric to each other and there is a neighborhood of any leaf which is isometric to the product of the leaf and an open subset of a leaf of the orthogonal distribution. We shall pursue this semiglobal structure in a little more general case.

LEMMA 2. *Let $D$ be a totally geodesic distribution $D$ over a complete Riemannian manifold $M$. Assume that $D$ provides a local product structure in the sense of Theorem 2. Let $Y$ be a unit normal vector to a regular leaf $L$ of $D$. Then the parallel displacement of $Y$ along $L$ is independent of the path in $L$.*

PROOF. For this special case, Lemma 2 turns out to be equivalent to the fact $H(L) = 1$ for a regular leaf, where $H(L)$ is the holonomy group of $L$ with respect to $D$. Let $\lambda: [0, s] \to L$ be a closed simple path starting and ending at $x \in L$. Let $Y(t)$ be the parallel displacement of $Y$ along $\lambda(t)$. Then it is easy to show that for sufficiently small $\varepsilon > 0$, $\mathrm{Exp}_{\lambda(t)}\varepsilon Y(t)$ stays in the same leaf for all $t \in [0, s]$, because of compactness of the image of $\lambda$ and the product structure, where $\mathrm{Exp}_x$ is the exponential map of $M$ at $x$. Thus $\mathrm{Exp}_{\lambda(0)}\varepsilon Y(0)$ and $\mathrm{Exp}_{\lambda(s)}\varepsilon Y(s)$ belong to the same leaf. Since $L$ is regular, they must agree. q.e.d.

Note here that the choice of $\varepsilon$ seems to depend on the path. However, we shall show in the following lemma that the choice of $\varepsilon$ is independent of the path.

LEMMA 3. *Let $M$ be as in Lemma 2. Let $L$ be a regular leaf. Then there is an $\varepsilon$ such that all the leaves in the saturation of a Frobenius neighborhood of breadth $\varepsilon$ at $x \in L$ are isometric to $L$.*

PROOF. Let $U = U_1 \times U_2$ be a local product Frobenius neighborhood of $x$ such that the breadth of $U$ is $\varepsilon$, $U_2$ is a normal neighborhood of $x$ in the leaf of $D'$ passing through $x$ and finally each leaf of $D$ meets a slice of $U$ at most once. Let $L'$ be a leaf in the saturation of $U$. Then there is a unique unit normal $X_x$ to $L$ at $x$ such that $\mathrm{Exp}_x\,\delta X_x$ belongs to $L'$ for some $\delta > 0$. Denote by $X_y$ the parallel displacement of $X_x$ to $y$. By Lemma 2, $X_y$ is well defined. We shall sketch that the mapping $y \mapsto \mathrm{Exp}_y\,\delta X_y$ gives rise to the desired isometry between $L$ and $L'$. Let $\lambda: [0, a] \to L$ be a minimal geodesic such that $\lambda(0) = x$ and $\lambda(s) = y$. Let $Y(s)$ be the parallel displacement of $\dot\lambda(0)$ along $\mathrm{Exp}_{\lambda(0)}sX_{\lambda(0)}$. Now define a geodesic variation $V$: $[0, a] \times [0, \varepsilon]$ by $V(t, s) = \mathrm{Exp}_{\mathrm{Exp}_x sX_x} tY(s)$. Since the rectangle $[0, a] \times [0, \varepsilon]$ is compact, we can find a grid consisting of parallel lines to the sides of the rectangle in such a way that the image of each subrectangle of the grid is mapped by $V$ into a local product Frobenius neighborhood. Using the similar process to the diagram chasing argument on the sides of the subrectangles, one can show that the vector field $V_*(\partial/\partial s)$ is parallel along any $s$-curves. Thus one can concludes that each $s$-curve is actually the curve of the form $\mathrm{Exp}_{\lambda(t)} sX_{\lambda(t)}$ for some fixed $t$. Therefore, $\mathrm{Exp}_{\lambda(t)}\delta X_{\lambda(t)} = V(t, \delta) = \mathrm{Exp}_{\mathrm{Exp}_x \delta X_x} tY(\delta)$ is in $L'$, since $L'$ is totally geodesic. Thus $\mathrm{Exp}_y \delta X_y \in L'$ for all $y \in L'$. It is easy to show that this mapping is one to one and onto. In order for the mapping to be an isometry, it suffices to see the mapping is locally isometric. The detail of this proof is left to the reader. q.e.d.

THEOREM 4. *Let $M$ be a connected complete Riemannian manifold. Let $D$ be a regular totally geodesic distribution over $M$ whose $D'$ is integrable. Then $M$ is the total space of a Riemannian fibre space provided* (1) *$M$ has non-negative sectional curvatures, or* (2) *$M$ is locally symmetric and the matrices $K$ of Lemma* 1 *have non-negative traces.*

PROOF. It is clear that a regular integrable distribution gives a natural quotient space $B$ and a natural projection $\pi: M \to B$. See [11] for the details. Let $U = U_1 \times U_2$ be a Frobenius neighborhood as in the proof of Lemma 3. Then define a mapping $\mathrm{Exp}: L \times U_2 \to M$ as follows: First since $U_2$ is a normal neighborhood of $x$ in the leaf of $D'$ through $x$, we can identify any point $u$ in $U_2$ with a unique vector $Y_x$ in $D'_x$ via $\mathrm{Exp}_x$ restricted to $D'_x$. Then the correspondence $(y, u) \mapsto \mathrm{Exp}_y Y_y$ gives us the desired mapping Exp, where $Y_y$ is the parallel displacement of $Y_x$ to $y$. The fibration $(M, \pi, B)$ has $L$ as its typical fibre. q.e.d.

**2. Application of the Riccati type equation I. Estimate of the index of relative nullity.** Our next aim is to make use of the Riccati equation to investigate complete totally geodesic distributions on $CP^N$, i.e., $N$-dimensional complex projective space with constant holomorphic sectional curvature $c(>0)$. As usual we denote the homogeneous coordinates of $CP^N$ by $(Z_0, \cdots, Z_N)$, where $Z_i = x_i + iy_i$, $0 \leqq i \leqq N$. We start this section by determining all the possible complete totally geodesic submanifolds of $CP^N$.

LEMMA 4. *Let $CP^N$ be as above. A complete totally geodesic submanifold $M$ of $CP^N$ is either a complex projective subspace or a real* *projective subspace described as follows: If $M$ is a complex submanifold of dimension $\mu$, then $M$ is given by the natural imbedding of $C^{\mu+1}$ into $C^{N+1}$ via a proper choice of homogeneous coordinates, i.e.,* $(z_0, \cdots, z_\mu) \mapsto (z_0, \cdots, z_\mu, 0, \cdots, 0)$ *in $C^{N+1}$. If $M$ is a real projective space of real dimension $\mu$, by taking real and complex homogeneous coordinates properly in $M$ and $CP^N$, respectively, the imbedding of $M$ into $CP^N$ is given as follows. Let $(x_0, \cdots, x_\mu)$ and $(z_0, \cdots, z_N)$ be the homogeneous coordinates of $M$ and $CP^N$, respectively. Our imbedding is given by the natural imbedding of $R^{\mu+1}$ into $C^{N+1}$,* i.e., $(x_0, \cdots, x_\mu) \mapsto (x_0, \cdots, x_\mu, 0, \cdots, 0)$. *In this case, $M$ has $c/4$ as its constant sectional curvature.*

PROOF. Let $\tilde{\nabla}$ be the standard connection of $CP^N$ and let $\nabla$ be the induced connection of $M$. If $\tilde{R}$ and $R$ are the curvature tensors of $CP^N$ and $M$, respectively, we have $\tilde{R}(X, Y)Z = c/4[(X \wedge Y)Z + (JX \wedge JY)Z + 2g(X, JY)JZ]$, where $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$ and $J$ is the complex structure of $CP^N$. On the other hand, $\tilde{R}(X, Y)Z = [\tilde{\nabla}_X, \tilde{\nabla}_Y]Z - \tilde{\nabla}_{[X,Y]}Z = \tilde{\nabla}_X(\nabla_Y Z + \alpha(Y, Z)) - \tilde{\nabla}_Y(\nabla_X Z + \alpha(X, Z)) - (\nabla_{[X,Y]}Z + \alpha([X, Y], Z)) = \tilde{\nabla}_X(\nabla_Y Z) - \tilde{\nabla}_Y(\nabla_X Z) - \nabla_{[X,Y]}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R(X, Y)Z$. In particular, let $X$, $Y$, $Z$ be orthonormal tangent vectors to $M$. Then by a simple computation one can easily verify that $(X\wedge Y)Z+(JX\wedge JY)Z+2g(X, JY)JZ = \tilde{R}(X, Y)Z$ is tangential to $M$ if and only if either $TM$ is $J$-invariant or $J(TM)$ is orthogonal to $TM$. In the former case, $M$ is a complex projective space; for the detail, see [8]. In the latter case, we can actually construct a real projective space having a point in common with $M$ and being tangent to $M$ at the point. Furthermore, such a real projective space is imbedded in $CP^N$ as described in Lemma 4. Since such a real projective space is totally geodesic, it must be $M$ itself. q.e.d.

THEOREM 5. *Let $D$ be a complete totally geodesic distribution of dimention $\mu$ defined in an open subset $G$ of $CP^N$. If there is a point $x$ in $G$ where $D_x$ is $J$-invariant, then $\mu \leqq \mu_{2N}$ unless $\mu = 2N$. Here $\mu_N$ will be given below. Furthermore, if there is an $A(y, X)$ such that $A(y, X)$ commutes with the complex structure $J$ for some point $y$ in the leaf that contains $x$, then either $\mu = 0$ or $\mu = 2N$. We define $\mu_n$ in the following way. Let $r(k)$ be the largest integer such that the standard fibration $V_{k,r(k)} \to V_{k,1}$ of Stiefel manifolds has a global cross-section, where $V_{s,t}$ denotes the set of all ordered $t$-frames in $s$-dimensional real Euclidean space $R^s$. For any integer $n$, set $\mu_n$ to be the largest integer such that $r(n - \mu_n) \geqq \mu_n + 1$. For example, $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 1$, $\mu_4 = 0$, $\mu_5 = 1$, $\mu_6 = 2$, $\mu_7 = 3$, $\mu_8 = 0$, $\mu_9 = 1$, $\mu_{10} = 2$ etc.*

PROOF. Let $\lambda(t)$ be a unit speed geodesic in the leaf containing $x$

such that $\lambda(0) = x$. In Lemma 1 we have shown that the matrix representation of $A(\lambda(t), \dot{\lambda}(t))$ with respect to a parallel orthonormal frame field satisfies the Riccati differential equation. Since the leaves are totally geodesic and since $J$ is parallel, we know that $D$ is $J$-invariant along the leaf containing $x$. Consider the last term of the equation in Lemma 1, which is given by $[K_{ij}(t)] = g(R(\dot{\lambda}(t), Y_i(t))Y_j(t), \dot{\lambda}(t))$. As is well known, the curvature of $CP^N$ is given by $\tilde{R}(X, Y) = c/4[X \wedge Y + JX \wedge JY + 2g(X, JY)J]$. Using the fact that $D$ is $J$-invariant along the leaf, we easily see that $[K_{ij}(t)] = (c/4)\cdot I$, where $I$ is the $(2N - \mu) \times (2N - \mu)$ identity matrix. So our equation turns out to be

$$[\alpha'_{ij}(t)] = [\alpha_{ij}(t)]^2 + (c/4)\cdot I\,. \tag{5}$$

Next we shall show that the solution to (5) with the initial condition $[\alpha_{ij}(0)]$ which has a real eigenvalue $\varphi$ must blow up at a finite value of $t$. Passing to conjugation and complexification, if necessary, we can assume without loss of generality that the initial condition $[\alpha_{ij}(0)]$ has the upper triangular form,

$$\text{i.e.}\,,\quad [\alpha_{ij}(0)] = \begin{pmatrix} \varphi & \cdots & * \\ \vdots & * \ddots & \vdots \\ 0 & \cdots & * \end{pmatrix}.$$

Applying the uniqueness theorem of solutions to ordinary differential equations which satisfy the Lipschitz's condition to our equation (5), we conclude that the solution matrix to (5) with the given initial condition has also the upper triangular form at every $t$, and that the $(1, 1)$-component of the solution is the solution to the scalar Riccati equation $\beta'(t) = \beta^2(t) + c/4$ with the initial condition $\varphi$. A simple computation yields that the solution $\beta(t)$ is given by $\beta(t) = \sqrt{c}/2\ \tan(\sqrt{c}/2t + \arctan 2\varphi/\sqrt{c})$. Thus $\alpha_{11}(t)$ blows up when $t = 2/\sqrt{c}\,(\pm(2n+1)\pi - \arctan 2\varphi/\sqrt{c})$, where $n = 0, 1, 2, \cdots$. Hence we have shown that $A(x, \ )$ cannot have any real eigenvalues. Now we shall prove the first half of Theorem 2. We follow the Ferus' idea for this part. By virture of Lemma 3, the leaf that contains $x$ must be complex projective space of complex dimension $\mu/2$. Suppose that $\mu \neq 2N$. Let $X_i$, $1 \leqq i \leqq \mu$, be an orthonormal base for $D_x$. Define a mapping from $D'_x$, i.e., the orthogonal complement of $D_x$ in $TCP^N_x$, into $(D'_x)^{\mu+1}$, i.e., the set of all $(\mu+1)$-tuples of vectors in $D'_x$, by assigning $(Y, A(x, X_1, Y), \cdots, A(x, X_\mu, Y))$ to each $Y$ in $D'_x$.

Then these $\mu + 1$ vectors are linearly independent. For $\beta_0 Y + \sum_1^\mu \beta_i A(x, X_i, Y) = 0$ implies that $A(x, \sum \beta_i X_i, Y) = -\beta_0 Y$ by linearity of $A(x, X)$ in $X$. Thus $-\beta_0$ is a real eigenvalue of $A(x, \sum \beta_i X_i)$. As is shown above, this is impossible unless $\sum \beta_i X_i = 0$. Since $X_i$ are linearly independent, we have $\beta_0 = \beta_1 = \cdots = \beta_\mu = 0$. So the fibration $V_{2N-\mu,\mu+1} \to V_{2N-\mu,1}$ has a cross-section defined by the above mapping. Hence $\mu \leqq \mu_{2N}$ by the definition of $\mu_{2N}$. In order to show the last half of Theorem 5, we first make sure that $D$ is $J$-invariant everywhere in the leaf that contains $x$, and we can, therefore, assume without loss of generality that there is an $A(x, X)$ which commutes with $J$. Suppose that $0 < \mu < 2N$. As is shown above, $A(x, X)$ cannot have any real eigenvalues. Now let $\alpha + \beta i$ be a complex eigenvalue of $A(x, X)$. Since $JA(x, X) = A(x, X)J$, we can find a vector $Y$ in $D'_x$ such that $A(x, X, Y) = \alpha Y + \beta JY$. Take the unit vector $Z$ in $D_x$ given by $Z = (\alpha X - \beta JX)/\sqrt{\alpha^2+\beta^2}$. By using linearity of $A(x, X)$ and the fact that $CP^N$ is Kählerian, we have $A(x, Z, Y) = \sqrt{\alpha^2+\beta^2}Y$. Thus $A(x, Z)$ has a real eigenvalue. This is impossible, so only possible case is either $\mu = 0$ or $\mu = 2N$. In the latter case, $D_x = TCP^N$ for all $x$ in $G$. This completes the proof of Theorem 5.

We shall now apply Theorem 5 to the relative nullity distribution of submanifolds of $CP^N$ to get the result mentioned in the introduction. Before going to the specific cases, we shall review some fundamental notions and notations on submanifolds.

Let $f: M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold $M$ into a Riemannian manifold $\tilde{M}$. Let $\tilde{\nabla}$ be the Riemannian connection of $M$, and let $\nabla$ be the induced connection of $M$. Then for any two tangent vector fields $X$ and $Y$ to $M$, we have that $\alpha(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$. This $\alpha$ is called the second fundamental form of the submanifold $(M, f)$. According to Chern and Kuiper [3], the subspace $RN_x$ of $TM_x$ defind by $RN_x = [X \in TM_x: \alpha(X, Y) = 0,$ for all $Y \in TM_x]$ is called the relative nullity space of $(M, f)$ at $x$.

The dimension $\mu(x)$ of $RN_x$ is called the nullity of $(M, f)$ at $x$. It is well known that the subset $G$ of $M$ where $\mu(x)$ assumes the minimum, say $\mu$, is open in $M$, and $\mu$ is called the index of relative nullity of $(M, f)$. The distribution on $G$ which assigns $RN_x$ to each $x \in G$ is called the relative nullity distribution of $(M, f)$. It is known that the distribution, say $RN$, is totally geodesic, hence integrable. It has also been shown that if $M$ is complete and if the ambient space $M$ is a space form, then all the leaves of $RN$ are complete as Riemannian manifolds [1].

COROLLARY 5. *Let $(M, f)$ be a Kählerian submanifold of complex $n$-dimension of $CP^N$. If $M$ is complete, then the index of relative nullity is either* 0 *or* $2n$. *In particular, if $\mu > 0$, then $M = CP^n$ and*

*$M$ is imbedded in $CP^N$ canonically as described in Lemma* 4.

PROOF. Our proof will be derived from Theorem 5, Lemma 4 and the following propositions, which will also be used further to show Lemma 6 in Section 3.

PROPOSITION 3. *Let $(M, f)$ be a Kählerian submanifold of a complex space form $\tilde{M}$. Then $RN_x$ and $RN'_x$ are $J$-invariant subspaces of $TM_x$, where $J$ is the complex structure of $M$.*

PROOF. For any $X \in RN_x$ and for any $Y \in TM_x$, we have $\alpha(JX, Y) = \alpha(X, JY) = J\alpha(X, Y)$, since $J$ commutes with $\alpha$. Thus if $X$ belongs to $RN_x$, so does $JX$. If $Y$ is in $RN'$, we have $g(X, JY) = g(JX, J^2Y) = -g(JX, Y) = 0$, because $JX \in RN_x$. Thus $RN$ and $RN'$ are $J$-invariant. q.e.d.

PROPOSITION 4. *Let $(M, f)$ be as above in Proposition* 2. *Then we have $\alpha(X, A(x, Z)Y) = \alpha(A(x, Z)X, Y)$ for all $X$, $Y \in RN'_x$ and $Z \in RN_x$.*

PROOF. Let $X$, $Y$ and $Z$ be as in the statement of Proposition 3. $\tilde{R}(X, Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z = \tilde{\nabla}_X(\nabla_Y Z + \alpha(Y, Z)) - \tilde{\nabla}_Y(\nabla_X Z + \alpha(X, Z)) - \nabla_{[X,Y]}Z - \alpha([X, Y], Z) = \tilde{\nabla}_X\nabla_Y Z - \tilde{\nabla}_Y\nabla_X Z - \nabla_{[X,Y]}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \alpha(X, \nabla_Y Z) - \alpha(Y, \nabla_X Z) - \nabla_{[X,Y]}Z = R(X, Y)Z + \alpha(X, \nabla_Y Z) - \alpha(Y, \nabla_X Z)$. Since $\tilde{R}(X, Y)Z = c/4((X \wedge Y)Z + (JX \wedge JY)Z + 2g(X, JY)JZ)$ is tangent to $M$, the normal component $\alpha(X, \nabla_Y Z) - \alpha(Y, \nabla_X Z) = 0$. This equality together with the definition of $A$ gives us the desired result. q.e.d.

PROPOSITION 5. *$A(x, Z)$ is a complex linear map of $RN'_x$ for all $x \in G$ and all $Z \in RN_x$.*

PROOF. Let $J$ be the complex structure of $M$. As is seen in Proposition 3, $RN_x$ and $RN'_x$ are $J$-invariant. So by Proposition 4 we get $\alpha(X, A(x, Z)JY) = \alpha(JY, A(x, Z)X) = J\alpha(Y, A(x, Z)X)$. On the other hand, $\alpha(X, JA(x, Z)Y) = J\alpha(X, A(x, Z)Y)$. So we have $\alpha(X, A(x, Z)JY) - \alpha(X, JA(x, Z)Y) = \alpha(X, [A(x, Z)J - JA(x, Z)]Y) = 0$. Now suppose that $JA(x, Z) - A(x, Z)J \neq 0$. Then there exists $Y' \in RN'_x$ such that $[JA(x, Z) - A(x, Z)J]Y' \neq 0$. However, $[JA(x, Z) - A(x, Z)J]Y'$ belongs to $RN'_x$, so there must exist $X'$ in $RN'_x$ such that $\alpha(X', [A(x, Z)J - JA(x, Z)]Y') \neq 0$. This is a contradiction. Thus $A$ and $J$ commute. q.e.d.

Next we shall prove a somewhat more artificial result on the relative nullity distribution. Let $f$ be an isometric immersion of a Riemannian manifold $M$ of dimension $n$ into $CP^N$. Then the submanifold $(M, f)$ of $CP^N$ is called totally real if $J(f_*TM)$ is orthogonal to $f_*TM$ in $TCP^N$, where $f_*$ is the Jacobian map of $f$. For example, the imbeddings of real projective spaces in Lemma 4 are totally real imbeddings.

In this case, despite the facts that $CP^N$ is not a real space form and that $(M, f)$ is a real submanifold of $CP^N$, we have

LEMMA 5. *If $(M, f)$ is a totally real submanifold of $CP^N$, then the relative nullity distribution of $(M, f)$ is totally geodesic, and complete provided $M$ is complete.*

PROOF. We shall omit the proof of differentiability of $RN$. Let $\tilde{R}$ and $R$ be as before. Then for $Z \in TM$ and $X$ and $Y \in RN$ we have $\tilde{R}(X, Z)Y = R(X, Z)Y - \alpha(Z, \nabla_X Y)$ by a routine computation and by using that $X$ and $Y$ are in $RN$. On the other hand, $\tilde{R}(X, Z)Y = c/4[(X \wedge Z)Y + (JX \wedge JZ)Y + 2g(X, JZ)JY]$ implies that actually $\tilde{R}(X, Z)Y = c/4(X \wedge Z)Y$, because $(M, f)$ is totally real. Therefore $\tilde{R}(X, Z)Y$ is tangential along $M$, hence $\alpha(Z, \nabla_X Y) = 0$. This implies $\nabla_X Y \in RN$, and consequently, $RN$ is totally geodesic. In order to show completeness of the leaves, we have to appeal to a long argument. Since, keeping $R(X, Z)Y = c/4(X \wedge Z)Y$ in mind, it is possible to follow every step of the relative nullity case in a real space form, we just refer to [1] for the proof. See Theorem 1.8.1. q.e.d.

THEOREM 6. *Let $(M, f)$ be a totally real submanifold of $CP^N$. If the index of relative nullity satisfies the inequality $\mu > \mu_n$, then $\mu = n \leqq N$, and $M$ is the real projective space $RP^n$ of dimension $n$ which is canonically imbedded as a totally geodesic submanifold described in Lemma 4.*

PROOF. By virture of Lemma 4 the leaves of $RN$ must be real projective space of dimension $\mu$, which are also totally geodesic in $CP^N$. Therefore it is clear that $\mu \leqq N$. Let $X_i$, $1 \leqq i \leqq \mu$, be an orthonormal base for $RN_x$ at a point where $\mu(x) = \mu$. Define a mapping from $RN'_x$, i.e., the orthogonal complement of $RN_x$ in $TM_x$, into $(RN'_x)^{\mu+1}$, i.e., the set of all $(\mu+1)$-tuples of vectors in $RN'_x$, by assigning $(Y, A(x, X_1, Y), \cdots, A(x, X_\mu, Y))$ to each $Y \in RN'_x$. These $\mu + 1$ vectors are linearly independent as observed in the proof of Theorem 5. So the fibration $V_{n-\mu,\mu+1} \rightarrow V_{n-\mu,1}$ has a cross-section defined by the above mapping. Hence $\mu \leqq \mu_n$ by the definition of $\mu_n$. Thus if $\mu > \mu_n$, the only possible case is $\mu = n$, and again by Lemma 4 $M = RP^n(c/4)$, i.e., the real projective space of constant sectional curvature $c/4$ and of dimension $n$. q.e.d.

## 3. Application of the Riccati type equation II. Cylinder theorems.

In this section we shall prove a cylinder theorem of Kählerian submanifolds of the complex Euclidean space of complex dimension $N$. This result will further extend to an intrinsic case under rather strong conditions. For the sake of convenience, from now on our dimension means the complex dimension unless otherwise specified.

THEOREM 7. *Let $(M, f)$ be a Kählerian submanifold of $C^N$ which has $\mu$ ($>0$) as its index of relative nullity. Let $M$ be complete. Then $M$ is cylindrical if there exists a point $x$ in $M$ where $TM_x$ contains an $(n-\mu)$-dimensional subspace in which the holomorphic sectional curvatures never vanish. Here $M$ is cylindrical means that there exists a $\mu$-dimensional Kählerian manifold $M_1$ and an $(n-\mu)$-dimensional Kählerian manifold $M_2$ such that there is a holomorphic isometry $g: M_1 \times M_2 \to M$ whose composition with $f$,* i.e., *$f \cdot g$, maps $M_1 \times M_2$ into $C^N$ as follows: The restriction of $f \cdot g$ to $M_1 \times \{y\}$, $y \in M_2$, maps $M_1 \times \{y\}$ holomorphically and isometrically onto a $\mu$-dimensional plane which is parallel to each other in $C^N$, and its restriction to $\{y\} \times M_2$, $y \in M_1$, maps $\{y\} \times M_2$ into an $(N-\mu)$-plane orthogonal to the relative nullity plane in $C^N$, and furthermore, those images are all parallel in $C^N$.*

COROLLARY 6. *Let $(M, f)$ be a complete Kählerian submanifold of $C^N$. If the nullity is greater than or equal to $n-1$, $M$ is $(n-1)$-cylindrical.*

From now on, let us assume that $M$ is simply connected.

LEMMA 6. *Let $x$ be the point where $TM_x$ contains a $\mu$-dimensional subspace in which the holomorphic sectional curvatures never vanish. Then there is a neighborhood $N$ of $x$ such that $A(y, Z) = 0$ for all $y \in N$ and all $Z \in RN_y$.*

PROOF. By Theorem 2.3.1. of [1], the leaves of the distribution are complete. So any geodesic $\lambda(t)$ with $\lambda(0) = x$ is extendable infinitely long in the leaf containing $x$. Recalling that $R(X, Y) = 0$, for $X \in RN$ we know from Lemma 1 that $A(\lambda(t), \dot{\lambda}(t))$ satisfies the Riccati type equation $[\alpha'_{ij}(t)] = [\alpha_{ij}(t)]^2$ with respect to any orthonomal parallel frame field. Now suppose that $\varphi$ be a real eigenvalue of $A(\lambda(0), \dot{\lambda}(0))$. By taking a parallel orthonormal frame field $Y_i$, $1 \leqq i \leqq 2n-\mu$, as described in the proof of Theorem 5, and making use of uniqueness of solutions, one can easily see that the $(1, 1)$-component of $[\alpha_{ij}(t)]$ has the form $\alpha_{11}(t) = \varphi/(1-\varphi t)$. This cannot be a global smooth solution unless $\varphi = 0$. Therefore we conclude that the all real eigenvalues of $A(\lambda(0), \dot{\lambda}(0))$ must be 0. But applying the same argument as in the proof of Theorem 5 to the complex eigenvalues of $A(\lambda(0), \dot{\lambda}(0))$, we actually see that all the eigenvalues are 0. Thus $A(\lambda(0), \dot{\lambda}(0))$ is nilpotent for all geodesics and for all points. Suppose that $A(x, Z) \neq 0$ for some unit vector $Z$ in $RN_x$. Then there exist $Y$ and $X$ in $RN'_x$ such that $A(x, Z, X) = 0$ and $A(x, Z, Y) = X$. This can be easily shown by considering the Jordan canonical form of the nilpotent transformation $A(x, Z)$. By Proposition 4, $\alpha(X, X) = \alpha(X, A(x, Z, Y)) = \alpha(A(x, Z, X), Y) = 0$. So together with the Gauss equation, we have $g(R(X, JX)JX, X) = -g(\alpha(X, JX), \alpha(X, JX)) + g(\alpha(JX, JX), \alpha(X, X)) = -2g(\alpha(X, X), \alpha(X, X)) = 0$. Since $g(R(X, JX)JX, X)$ is a scalar multiple of the holomorphic sectional curvature of the plane $X \wedge JX$, it cannot be 0 by our assumption. This is a contradiction. Therefore $A(x, Z) = 0$ for all $Z \in RN_x$. Finally if we consider the holomorphic sectional curvature as a continuous function from the quotient bundle of $TM$ whose typical fiber is $(n-1)$-dimensional complex project space, it is clear that there is a neighborhood $N$ of $x$ such that any point of $N$ has a $\mu$-dimensional subspace in its tangent space where the holomorphic sectional curvatures never vanish. Applying the above argument to the all points in $N$, we have shown the desired result.

q.e.d.

LEMMA 7. *The distribution $RN$ and $RN'$ restricted to $N$ are parallel, therefore we have a local Kählerian product structure of a neighborhood of $x$.*

PROOF. Obvious. q.e.d.

LEMMA 8. *Let $M$ be the manifold described in Theorem 7, and let $x$ be the point given in Theorem 7. Then at any point $y$ in $M$, there exists a unique $\mu$-dimensional subspace $P_y$ of the tangent space $TM_y$ such that (8.1) $f_*(P_y)$'s are parallel to each other in $C^N$ if they are regarded as $\mu$-planes in $C^N$, and (8.2) $P_y$ is contained in $RN_y$ for all $y \in M$ and in particular, for $y \in G$ $P_y = RN_y$ and $f_*(P_y)$ is exactly $f$-image of the leaf of $RN$ passing through $y$.*

In order to prove Lemma 8, we need the following proposition. For the proof, see Proposition 2.3.2. in [1].

PROPOSITION 6. *Let $M$ be as in Lemma 8. $RN_y$ coincides with the subspace $N_y$ of $TM_y$ defined by $N_y = [X \in TM_y: R(X, Y) = 0$, for all $Y \in TM_y]$ for all $y \in M$. $N_y$ is called the nullity space of $M$ at $y$.*

PROOF OF LEMMA 8. Let $G$ be the open set given by $G = [x \in M: \mu(x) = \mu]$. For $y$ define $P_y$ to be the subspace of $TM_y$ which is the image space of $RN_x = P_x$ by the parallel translation along a minimal geodesic between $x$ and $y$. We shall show that these $P_y$ satisfy (8.1) and (8.2). Let $\lambda: [0, s] \rightarrow M$ be a unit speed minimal geodesic between $x$ and $y$ such that

$\lambda(0) = x$ and $\lambda(s) = y$. Note that $\lambda$ is a real analytic mapping. Let $X_x$ be a vector in $RN_x$, and let $Y_x$, $Z_x$ and $W_x$ be vectors in $TN_x$. Translate them parallelly along $\lambda$, and denote the resulting vector fields by $X_t$, $Y_t$, $Z_t$ and $W_t$, respectively. Consider the real-valued function $k(t)$ defined by $k(t) = g(R(X_t, Y_t)Z_t, W_t)$, $0 \leqq t \leqq s$. It is easy to check that $k$ is a real analytic function.

By Lemma 7 and Proposition 6, $X_t$ must stay in $RN_{\lambda(t)}$ in a neighborhood of $x$. Therefore $k(t) = 0$ in $[0, s]$ by real analyticity of $k(t)$. The definition of $N_y$ and Proposition 6 tell us that $P_y$ is contained in $N_y = RN_y$. This proves the first half of (8.2). The last half follows from the fact that the leaves are totally geodesic not only in $M$ but also in $C^N$. In order to show (8.1), consider the composition of $\lambda$ and $f$, say $f\cdot\lambda = \bar{\lambda}$. Since $\lambda$ and $f$ are real analytic, so is $\bar{\lambda}$. Define $\bar{X}_t$ to be $f_*(X_t)$. Let $N_0$ be a vector at $f(x)$ which is orthogonal to $f_*(P_x)$. Translate it parallelly to get a parallel vector field $N_t$ along $\bar{\lambda}$. Consider a new function $h(t)$ given by $h(\mathrm{t}) = \tilde{g}(\bar{X}_t, N_t)$.

Then $dh(t)/dt = \dot{\bar{\lambda}}_t\tilde{g}(\bar{X}_t, N_t) = \tilde{g}(\tilde{\nabla}_{\dot{\bar{\lambda}}_t}\bar{X}_t, N_t) + \tilde{g}(\bar{X}_t, \tilde{\nabla}_{\dot{\bar{\lambda}}_t}N_t) = \tilde{g}(f(\nabla_{\dot{\lambda}_t}X_t) + \alpha(\dot{\lambda}(t), X_t), N_t) = 0$, because $X_t$ is parallel along $\lambda$ and $X_t$ is in $RN_{\lambda(t)}$. Here $\dot{\bar{\lambda}}(t)$ denotes the velocity vector of $\bar{\lambda}(t)$. Since $h(0) = 0$, $h(t) = 0$. By (8.2) and by the way $P_y$ is defined, we see that $f_*(P_{\lambda(t)})$ is parallel to each other along $\lambda(t)$. But the holonomy group of $C^N$ is trivial, so we actually have that $P_{\lambda(t)}$ is parallel to each other in $C^N$. Since this argument holds for all points, we have shown (8.1). q.e.d.

Now we can conclude our proof of Theorem 7. Define a $\mu$-dimensional distribution $D$ on the whole $M$ by assigning to each $x$ the subspace $P_x$ of $TM_x$. Let $D'$ be the orthogonal distribution of $D$. Then it is easy to check that $D$ and $D'$ have the following properties:

(6) $D$ and $D'$ are $J$-invariant and differentiable,

(7) $D$ and $D'$ are parallel, in particular, they are integrable, and

(8) $D$ and $D'$ are orthogonal to each other, and the restriction of $D$ and $D'$ to $G$ coincide with $RN$ and $RN'$, respectively.

Applying the de Rham decomposition theorem for Kählerian manifolds to these distributions, we have the global product structure stated in Theorem 7. Finally if $M$ is not simply connected, the well known argument via its universal covering manifold will give us the desired result. q.e.d.

As for Corollary 6, all we have to show is that either $M$ is flat or $M$ has a point where the tangent space contains a 1-dimensional subspace whose holomorphic sectional curvature never vanishes. Suppose that $M$ is not flat. Then there must exist a vector whose holomorphic sectional curvature does not vanish. If $M$ is flat, then by Proposition 6, $M$ is totally geodesic. So $M$ is an $n$-complex plane of $C^N$. Finally we shall state another immediate result of Theorem 7.

COROLLARY 7. *Let $M$ be as in Theorem 7. If the dimension of $M - G$ is strictly less than the index $\mu$, there does not exist any $\mu$-dimensional subspace in which the holomorphic sectional curvatures never vanish.*

PROOF. Because of the product structure the dimension of $M - G$ must be greater than or equal to $\mu$. q.e.d.

Having obtained some extrinsic results, it would be quite natural to ask whether or not the intrinsic analogue holds. The following is the intrinsic version of Corollary 6.

THEOREM 8. *Let $M$ be a simply connected complete complex $n$-dimensional Kählerian manifold with a real analytic Riemannian metric. If the index of nullity is $n - 1$, then $M$ is a Kählerian product of a Riemann surface $M_1$ and a flat Kählerian manifold $M_2$ of dimension $n - 1$.*

PROOF. Assume that $M$ is not flat. Then there is a point $x$ in $M$ where $TM_x$ contains a vector whose holomorphic sectional curvature does not vanish. First of all we shall show that there exists a neighborhood $V$ of $x$ which has a Kählerian product structure. To this end, as before we have to show that all $A(y, X)$ vanish in $V$. We can actually assume that there is a neighborhood $U$ of $x$ at each point of which there is a vector whose holomorphic sectional curvature does not vanish. As is shown in the proof of Lemma 6, all real eigenvalues of $A(y, \ )$ are 0 in $M$. Let $X$ and $JX$ be an orthonormal base for $N'_y$ at $y \in U$, where $N'_y$ is the orthogonal complement of the nullity space $N_y$ at $y$. Suppose that $A(y, Z, X) = aX + bJX$. Then taking $W = aZ - bJZ$ in $N_y$, we have that $A(y, W, X) = (a^2 + b^2)X$. Thus $a^2 + b^2 = 0$ implies $a = b = 0$. Similarly, we can show $A(y, Z, JX) = 0$. So $A(y, Z) = 0$ for all $y \in U$ and for all $Z \in N_y$. Following the proof of Lemma 6, we have a local product structure in $V$.

Now we shall extend the nullity distribution to the whole $M$.

Let $x$ be a point which belongs to $G$ given by $G = [x \in M: \mu(x) = n - 1]$. For any point $y$ in $M$ define $P_y$ to be the parallel displacement of $N_x$ along a path between $x$ and $y$. First of all we shall show that definition of $P_y$ does not depend on the choice of the path. Let $\alpha$ and

$\beta$ be two paths between $x$ and $y$. Suppose that the parallel displacements of $N_x$ along $\alpha$ and $\beta$ do not coincide at $y$. Then the parallel displacement of $N_x$ along the loop $\beta^{-1}\alpha$ at $x$ would not be $N_x$. On the other hand, it has been known that for a real analytic connection in a real analytic manifold, the restricted holonomy group, the local holonomy group and the infinitesimal holonomy group all agree. For the proof, see Proposition 10.5 and Theorem 10.8 in [8]. Because of the local product structure around $x$, we see that $N_x$ is invariant by the local holonomy group, therefore by the restricted holonomy group. This presents a contradiction. Thus the parallel displacements of $N_x$ along $\alpha$ and $\beta$ coincide at $y$. This tells us that $P_y$ is independent of the path. Define a new distribution $D$ on $M$ by assigning to each $y$ the subspace $P_y$ of $TM_y$. Then it is easy to show that $D$ is real analytic and $J$-invariant. Since it follows from the definition of $D$ that $D$ is a parallel distribution, our global Kählerian product structure is a simple result of the de Rham decomposition theorem with respect to the distribution $D$. q.e.d.

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