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MEAN CURVATURES FOR HOLOMORPHIC $2p$-PLANES IN KÄHLERIAN MANIFOLDS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. Introduction. Let (M, g) be an m-dimensional Riemannian manifold with (positive definite) metric tensor g. By $K(X, Y)$ we denote the sectional curvature for a 2-plane spanned by X and Y. Let x be a point of M and let π be a q-plane at x. Let $(e_i, i = 1, \cdots, m)$ be an orthonormal basis at x such that $e_1, \cdots, e_q$ span π (such a basis is called an adapted basis for π). S. Tachibana [3] defined the mean curvature $\rho(\pi)$ for π by

$$\rho(\pi) = \frac{1}{q(m-q)} \sum_{a=q+1}^{m} \sum_{\alpha=1}^{q} K(e_\alpha, e_a) ,$$

which is well-defined, i.e., independent of the choice of adapted basis for π. He obtained the following

PROPOSITION A. (S. Tachibana [3].) *In an m (> 2)-dimensional Riemannian manifold (M, g), if mean curvature for q-plane is independent of the choice of q-planes at each point, then*

(i) *for $q = 1$ or $m - 1$, (M, g) is an Einstein space,*

(ii) *for $1 < q < m - 1$ and $2q \neq m$, (M, g) is of constant curvature,*

(iii) *for $2q = m$, (M, g) is conformally flat.*

The converse is true.

Taking holomorphic $2p$-planes, instead of q-planes, an analogous result in Kählerian manifolds is also obtained.

PROPOSITION B. (S. Tachibana [4].) *In a Kählerian manifold (M, g, J), $m = 2n \geqq 8$, if mean curvature for holomorphic $2p$-plane is independent of the choice of holomorphic $2p$-planes at each point, then*

(i) *for $1 < p < n - 1$ and $2p \neq n$, (M, g, J) is of constant holomorphic sectional curvature,*

(ii) *for $2p = n$, the Bochner curvature tensor vanishes.*

The converse is true.

Proposition A is the best possible. However, in Proposition B the case $m = 4$, 6 and the case $p = 1$ or $n - 1$ are excluded. We prove for

$p=1$ (or $n-1$) and $m \neq 4$ in Lemma 1, and for $p=1$ and $m=4$ in Lemma 3. Consequently we have a generalization of Proposition B:

THEOREM. *In a Kählerian manifold (M, g, J), $m=2n \geqq 4$, if mean curvature for holomorphic $2p$-plane is independent of the choice of holomorphic $2p$-planes at a point x, then*

(i) for $1 \leqq p \leqq n-1$ and $2p \neq n$, (M, g, J) is of constant holomorphic sectional curvature at x,

(ii) for $2p=n$, the Bochner curvature tensor vanishes at x.
The converse is true.

If mean curvature for holomorphic $2p$-plane is independent of the choice of holomorphic $2p$-planes on M, then (i) (M, g, J) is of constant holomorphic sectional curvature, or (ii) the Bochner curvature tensor vanishes. For (i) the converse is true. For (ii) the converse is true, if the scalar curvature S is constant.

Since the case $m=2$ is trivial, the above theorem is the best possible.

2. **The case $p=1$ and $n \neq 2$.** Let (M, g, J) be a Kählerian manifold with an almost complex structure tensor J and the Kählerian metric tensor g. Then we have

$$g(X, Y)=g(JX, JY), \quad JJX=-X, \quad \nabla J=0\,, \tag{2.1}$$

where X and Y denote vector fields on M (or tangent vectors at a point) and ∇ denotes the Riemannian connection with respect to g. By $R=(R^i{}_{jkl})$, $R_1=(R_{jk}=R^r{}_{jkr})$ and $S=(R_{jk}g^{jk})$ we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature. Then

$$R(X, Y)Z=R(JX, JY)Z\,, \tag{2.2}$$

$$R_1(X, Y)=R_1(JX, JY)\,. \tag{2.3}$$

For a J-basis $(e_\lambda, Je_\lambda=e_{\lambda^*}, \lambda=1, \cdots, n)$ we have

$$R_1(e_1, e_1)=K(e_1, Je_1)+\sum_{q=2}^{n}[K(e_1, e_q)+K(e_1, e_{q^*})]\,. \tag{2.4}$$

Let (X, Y) be an orthonormal pair at x such that $g(X, JY)=0$. Then, we have (cf. for example, R. L. Bishop and S. I. Goldberg [1], p. 517)

$$\begin{aligned} &K(X, Y)+K(X, JY) \\ &=\frac{1}{4}[H(X+JY)+H(X-JY)+H(X+Y)+H(X-Y)-H(X)-H(Y)]\,, \end{aligned} \tag{2.5}$$

where $H(X)=K(X, JX)$ denotes the holomorphic sectional curvature for X. An adapted basis for a holomorphic $2p$-plane π is of the form:

$$(e_1, \cdots, e_p, e_{1^*}, \cdots, e_{p^*}, e_{p+1}, \cdots, e_n, e_{(p+1)^*}, \cdots, e_{n^*})$$

such that $e_1, \cdots, e_p, e_{1^*}, \cdots, e_{p^*}$ span π. The mean curvature $\rho(\pi)$ for π is

$$\rho(\pi) = \frac{1}{2p(n-p)} \sum_{q=p+1}^{n} \sum_{\alpha=1}^{p} [K(e_\alpha, e_q) + K(e_\alpha, e_{q^*})] . \tag{2.6}$$

LEMMA 1. *If $n \neq 2$ and if mean curvature for holomorphic 2-plane is independent of the choice of holomorphic 2-planes at x, then (M, g, J) is of constant holomorphic sectional curvature at x.*

PROOF. By (2.6) and $p = 1$ we have

$$\frac{1}{2(n-1)} \sum_{q=2}^{n} [K(e_1, e_q) + K(e_1, e_{q^*})] = \rho(\pi) = \rho . \tag{2.7}$$

By (2.4) and (2.7), we have

$$2(n-1)\rho = R_1(e_1, e_1) - K(e_1, Je_1) . \tag{2.8}$$

Since $K(e_1, Je_1) = H(e_1)$ and since $\rho(\pi) = \rho$ is independent of π, we have for any unit vector X at x

$$2(n-1)\rho = R_1(X, X) - H(X) . \tag{2.9}$$

Since R_1 satisfies (2.3), we have a J-basis (e_λ, Je_λ) such that R_1 is diagonal with respect to (e_λ, Je_λ). Putting $X = \sin\theta e_1 + \cos\theta e_2$ in (2.9), we have

$$2(n-1)\rho = \sin^2\theta R_1(e_1, e_1) + \cos^2\theta R_1(e_2, e_2) - H(\sin\theta e_1 + \cos\theta e_2) .$$

Applying (2.9) again to e_1 and e_2, we have

$$H(\sin\theta e_1 + \cos\theta e_2) = \sin^2\theta H(e_1) + \cos^2\theta H(e_2) . \tag{2.10}$$

More generally, for $X = \sum(A_\lambda e_\lambda + B_\lambda Je_\lambda)$, $\sum(A_\lambda^2 + B_\lambda^2) = 1$, we have

$$H(X) = \sum_{\lambda=1}^{n} (A_\lambda^2 + B_\lambda^2) H(e_\lambda) , \tag{2.11}$$

where we have used $H(e_\lambda) = H(Je_\lambda)$. By (2.10) or (2.11), we have

$$H(e_1 + Je_q) = \frac{1}{2} H(e_1) + \frac{1}{2} H(e_q) , \qquad \text{etc.}$$

Therefore, (2.5) gives

$$K(e_1, e_q) + K(e_1, e_{q^*}) = \frac{1}{4}[H(e_1) + H(e_q)] . \tag{2.12}$$

Putting (2.12) into (2.4), we have

$$R_1(e_1, e_1) = H(e_1) + \frac{1}{4}[(n-1)H(e_1) + \sum_{q=2}^{n} H(e_q)] .$$

The last equation and (2.8) give

$$8(n-1)\rho = (n-2)H(e_1) + \sum_{\lambda=1}^{n} H(e_\lambda) . \tag{2.13}$$

(2.13) holds for $e_2, \cdots, e_n$. Thus, $n \neq 2$ implies

$$H(e_1) = H(e_2) = \cdots = H(e_n) .$$

Then (2.11) shows that (M, g, J) is of constant holomorphic sectional curvature at x.

3. **The case $p = 1$ and $n = 2$.** Most part of the proof of Lemma 1 is valid in this case too. Let $(e_1, e_2, e_{1^*}, e_{2^*})$ be a J-basis at x which diagonalizes R_1. By (2.3) if we change $(e_2, e_{2^*}) \to (\bar{e}_2, \bar{e}_{2^*})$ by $\bar{e}_2 = \alpha e_2 + \beta e_{2^*}$, $\alpha^2 + \beta^2 = 1$, R_1 is still diagonalized with respect to $(e_1, e_{1^*}, \bar{e}_2, \bar{e}_{2^*})$. We choose α so that $K(e_1, \alpha' e_2 + \beta' e_{2^*})$ takes the maximum for $\alpha' = \alpha$. We write $(e_1, e_{1^*}, \bar{e}_2, \bar{e}_{2^*})$ again by $(e_1, e_{1^*}, e_2, e_{2^*})$. We need the following well known lemma (cf. for example, R. L. Bishop and S. I. Goldberg [1], p. 512).

LEMMA 2. *Let e_i, e_j, e_k be part of an orthonormal basis. If $K(e_i, e_j)$ is critical in $K(e_i, \cos\theta e_j + \sin\theta e_k)$, then $R_{ijik} = 0$ (where $R_{ijkl} = g(R(e_k, e_l)e_j, e_i)$).*

By our choice of the J-basis and by Lemma 2, we have

$$R_{1212^*} = 0 . \tag{3.1}$$

By (2.2) and (3.1), we have

$$R_{121^*2} = R_{1^*2^*12^*} = R_{1^*2^*1^*2} = 0 . \tag{3.2}$$

Next for an orthonormal pair (X, Y), we have

$$\begin{aligned} K(X, Y) = \frac{1}{8}[3(1+\cos\theta)^2 H(X+JY) + 3(1-\cos\theta)^2 H(X-JY) \\ - H(X+Y) - H(X-Y) - H(X) - H(Y)] , \end{aligned} \tag{3.3}$$

where $\cos\theta = g(X, JY)$ (cf. R. L. Bishop and S. I. Goldberg [1], p. 516). Putting $X = e_2$ and $Y = \cos te_{2^*} + \sin te_1$ in (3.3), we have

$$K(e_2, \cos te_{2^*} + \sin te_1) = \frac{1}{8}[(1 + 7\cos^2 t)H(e_2) + \sin^2 tH(e_1)] , \tag{3.4}$$

where we have used (cf. (2.11))

$$\begin{aligned} &H(e_2 - \cos te_2 + \sin te_{1^*}) \\ &= \frac{1}{2(1-\cos t)}[(1-\cos t)^2 H(e_2) + \sin^2 tH(e_1)] , \qquad \text{etc.} \end{aligned}$$

By (3.4) we see that $K(e_2, e_{2^*})$ is critical in $K(e_2, \cos te_{2^*} + \sin te_1)$. Hence, Lemma 2 gives

$$R_{22^*21} = 0 . \tag{3.5}$$

Similarly we have

$$\begin{aligned} R_{22^*21^*} &= R_{11^*12} = R_{11^*12^*} = 0 , \\ R_{22^*2^*1^*} = R_{22^*2^*1} &= R_{11^*1^*2^*} = R_{11^*1^*2} = 0 . \end{aligned} \tag{3.6}$$

By (3.3) and (2.11) we have

$$-R_{1212} = \frac{1}{8}(H(e_1) + H(e_2)) = -R_{12^*12^*} = -R_{1^*21^*2} . \tag{3.7}$$

Of course, $H(e_1) = -R_{11^*11^*}$ and $H(e_2) = -R_{22^*22^*}$. By (2.13), we have $8\rho = H(e_1) + H(e_2)$. (2.9) gives

$$H(e_1) + H(e_2) = 4R_1(e_1, e_1) - 4H(e_1) . \tag{3.8}$$

We write $R_1(e_i, e_j) = R_{ij}$. Then the scalar curvature S is given by

$$\begin{aligned} S &= R_{11} + R_{22} + R_{1^*1^*} + R_{2^*2^*} \\ &= 3H(e_1) + 3H(e_2) \end{aligned} \tag{3.9}$$

by (3.8) and (2.3). Hence, by (2.9) again, we have

$$\begin{aligned} R_{11} &= R_{1^*1^*} = H(e_1) + \frac{1}{12}S , \\ R_{22} &= R_{2^*2^*} = H(e_2) + \frac{1}{12}S . \end{aligned} \tag{3.10}$$

On the other hand, g and J have components:

$$g_{ij} = \delta_{ij} , \tag{3.11}$$

$$\begin{aligned} J^{\lambda}_{\lambda^*} &= J_{\lambda\lambda^*} = -J_{\lambda^*\lambda} = -1 , \\ J_{\lambda i} &= 0 , \qquad i \neq \lambda^* . \end{aligned} \tag{3.12}$$

The Bochner curvature tensor $(B^i{}_{jkl})$ is given by (cf. S. Tachibana [2], K. Yano and S. Bochner [6])

$$\begin{aligned} B_{ijkl} = R_{ijkl} &- \frac{1}{m+4}(R_{jk}g_{il} - R_{jl}g_{ik} + g_{jk}R_{il} - g_{jl}R_{ik} \\ &+ R_{jr}J^r_kJ_{il} - R_{jr}J^r_lJ_{ik} + J_{jk}R_{ir}J^r_l - J_{jl}R_{ir}J^r_k \\ &- 2R_{kr}J^r_lJ_{ij} - 2R_{ir}J^r_jJ_{kl}) \\ &+ \frac{S}{(m+2)(m+4)}(g_{jk}g_{il} - g_{jl}g_{ik} + J_{jk}J_{il} - J_{jl}J_{ik} - 2J_{kl}J_{ij}) . \end{aligned} \tag{3.13}$$

LEMMA 3. *Let $m = 2n = 4$. If mean curvature for holomorphic 2-plane is independent of the choice of holomorphic 2-planes at x, then $B_{ijkl} = 0$ at x.*

PROOF. Since B_{ijkl} is a K-curvature-like tensor in the sense of S. Tachibana [4] and indeces 1 and 2 are reversible, it suffices to show $B_{ijkl} = 0$ for the following six cases:

$$(ijkl) = (11^*11^*),\ (11^*12^*),\ (11^*12),\ (1212),\ (1212^*),\ (12^*12^*)\ ,$$

because, for example, (11^*22^*) is verified by (1212) and (12^*12^*) by the Bianchi identity and (2.2). Verification is made applying (3.1), $\cdots$, (3.12) to (3.13) with $m = 4$.

4. **The converse.** If a Kählerian manifold (M, g, J) is of constant holomorphic sectional curvature H (at x, or on M, resp.), for any J-basis (e_λ, Je_λ), we have

$$K(e_\lambda, e_{\lambda^*}) = H\ ,$$
$$K(e_\lambda, e_j) = \frac{1}{4}H \qquad \text{for } i \neq \lambda^*\ . \tag{4.1}$$

Hence $\rho(\pi)$ is constant (at x, or on M, resp.).

Next, assume that (M, g, J) has the vanishing Bochner curvature tensor at x. Take any J-basis (e_λ, Je_λ). Then (3.13) gives

$$\begin{aligned} R_{\lambda\mu\lambda\mu} &= R_{\lambda\mu^*\lambda\mu^*} \\ &= -\frac{1}{m+4}(R_{\lambda\lambda} + R_{\mu\mu}) + \frac{1}{(m+2)(m+4)}S\ , \end{aligned} \tag{4.2}$$

where $\lambda \neq \mu$. Let π be a holomorphic $2p$-plane spanned by $(e_1, \cdots, e_p, e_{1^*}, \cdots, e_{p^*})$. Since $K(e_\lambda, e_\mu) = -R_{\lambda\mu\lambda\mu}$, by (2.6) and (4.2) we have

$$\begin{aligned} \rho(\pi) &= \frac{1}{2p(n-p)} \sum_{q=p+1}^{n} \sum_{\alpha=1}^{p} 2[\frac{1}{m+4}(R_{\alpha\alpha} + R_{qq}) - \frac{1}{(m+2)(m+4)}S] \\ &= \frac{1}{p(n-p)}\Big[\frac{1}{m+4}\Big((n-p)\sum_{\alpha=1}^{p} R_{\alpha\alpha} + p\sum_{q=p+1}^{n} R_{qq}\Big) \\ &\qquad - \frac{p(n-p)}{(m+2)(m+4)}S\Big]\ . \end{aligned}$$

Therefore, if $n - p = p$, i.e., $2p = n$, we have

$$\rho(\pi) = \frac{m+2-2p}{2(m+2)(m+4)p}S = \frac{1}{4n(n+1)}S\ . \tag{4.3}$$

Thus, $\rho(\pi)$ is constant at x. $\rho(\pi)$ is independent of the choice of points

x and the choice of holomorphic $2p$-planes π at x, if and only if S is constant.

REMARK. Typical examples of Kählerian manifolds with vanishing Bochner curvature tensor are

(1) (M, g, J) of constant holomorphic sectional curvature,

(2) product manifold $(M_1, g_1, J_1) \times (M_2, g_2, J_2)$ of two Kählerian manifolds of constant holomorphic sectional curvature $H_1 = H$ and $H_2 = -H$ (cf. S. Tachibana and R. C. Liu [5]).

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