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SURGERY ON 1-CONNECTED HOMOLOGY MANIFOLDS

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1. Introduction and statement of a result. In this paper, we show that the analogue of the surgery technique of Browder and Novikov holds for 1-connected homology manifolds.

Our theorem is as follows.

THEOREM. *Let X be an 1-connected Poincaré complex with* $\dim X = n \geqq 5$ *and let ξ be a homology cobordism bundle (cf. Martin-Maunder [6]) over X with* $\dim \xi = k$. *Let* $\alpha \in \lim_{j\to\infty} \pi_{n+k+j}(T(\xi^k \oplus \theta^j))$ *be such that* $h(\alpha) = \Phi(g)$, *where* $h: \pi_{n+k+j}(T(\xi^k \oplus \theta^j)) \to H_{n+k+j}(T(\xi^k \oplus \theta^j))$ *is the Hurewicz homomorphism,* $\Phi: H_n(X) \to H_{n+k+j}(T(\xi^k \oplus \theta^j))$ *is the Thom isomorphism, and* $g \in H_n(X)$ *is a generator.*

Then there exists an obstruction

$$c(\alpha) \in \begin{cases} Z & \text{for} \quad n = 4m \\ Z_2 & \text{for} \quad n = 4m+2 \\ 0 & \text{for} \quad n = \text{odd} \end{cases}$$

If $c(\alpha) = 0$, *there exists a homology manifold* M^n *such that* $f: M \to X$ *is homotopy equivalent and the normal bundle* $\mathcal{N}(M)$ *for M embedded in* S^N *is equivalent to* $f^*(\xi \oplus \theta^{N-k})$.

2. Proof of the theorem. The essential parts of the proof are the stability of $\pi_i(BHML(n))$ and the embeddability of spheres.

Sato [10] showed the next theorem.

THEOREM. *Let M be a homology manifold with the dimension* $n \geqq 5$. *Assume that* ∂M *is a PL-manifold or* $\partial M = \emptyset$. *If the obstruction class*

$$\{\lambda(M)\} \in H_{n-4}(M, \mathscr{H}^3)$$

is zero, then there exists a PL-manifold N with a pseudo homology cell decomposition which is cellularly equivalent to M. Furthermore $N \to M$ *is a resolution.*

Matumoto [5] and Martin [7] showed the next theorem by this theorem.

THEOREM. *The next sequences are exact.*

$$0 \to \pi_i(BPL(n)) \to \pi_i(BHML(n)) \to 0 \quad \textit{for } i \neq 3, 4; i + n \geqq 7$$

$$0 \to \pi_4(BPL(n)) \to \pi_4(BHML(n)) \to \mathscr{H}^3 \to 0 \quad \textit{for } n \geqq 3$$

$$\pi_3(BPL(n)) \to \pi_3(BHML(n)) \to 0 \quad \textit{for } n \geqq 3$$

$$0 \to \pi_i(BHML(n)) \to \pi_i(BHML(n+1)) \to 0 \quad \textit{for } i \leqq n-1, i+n \geqq 7.$$

We need the next lemma.

LEMMA. *Let M be a compact 1-connected homology manifold with* $\dim M = m \geqq 5$. *For $\alpha \in \pi_i(M^m)$, $1 \leqq m/2$, there exists an embedding $\bar{\alpha}: S^i \to M'^m$ such that there exists an h-cobordism $(W; M, M')$ such that*

$$i_*(\alpha) = j_*(\bar{\alpha}) \in \pi_i(W)$$

where $i: M \subset W$, $j: M' \subset W$ are inclusions.

PROOF. Let $\alpha: S^i \to M$ be a continuous map, and let $*$ be a base point of S^i. Let $\varDelta^m$ be an m-simplex in M^m. We choose an embedding $\beta: S^i \to \operatorname{Int} \varDelta^m$ and an embedded path $\gamma: I \to M^m$ such that $\gamma(0) = \alpha(*)$, $\gamma(1) = \beta(*)$ and $\gamma(I) \cap \beta(S^i) = \gamma(1)$. We divide S^i as $S^i = P \cup Q \cup R$, where

$$P = \left\{(x_0, \cdots, x_i) \middle| x_0^2 + \cdots + x_i^2 = 1,\ x_0 \geqq \frac{1}{2}\right\}$$

$$Q = \left\{(x_0, \cdots, x_i) \middle| x_0^2 + \cdots + x_i^2 = 1,\ -\frac{1}{2} \leqq x_0 \leqq \frac{1}{2}\right\}$$

$$R = \left\{(x_0, \cdots, x_i) \middle| x_0^2 + \cdots + x_i^2 = 1,\ x_0 \leqq -\frac{1}{2}\right\}$$

$$R \supset \bar{R} = \left\{(x_0, \cdots, x_i) \middle| x_0^2 + \cdots + x_i^2 = 1,\ x_0 \leqq -\frac{3}{4}\right\}.$$

There exist continuous maps $\alpha': P \to S^i$ such that $\alpha'(P \cap \{x_0 = 1/2\}) = *$ and $\alpha' | (P - \{x_0 = 1/2\})$ is a homeomorphism, $\gamma': Q \to I$ defined by $\gamma'(x_0, \cdots, x_i) = x_0 + 1/2$, and $\beta': R \to S^i$ such that $\beta'(R \cap \{x_0 = -1/2\}) = *$ and $\beta' | (R - \{x_0 = -1/2\})$ is a homeomorphism. We define $\alpha'': S^i \to M$ by $\alpha'' | P = \alpha \circ \alpha'$, $\alpha'' | Q = \gamma \circ \gamma'$ and $\alpha'' | R = \beta \circ \beta'$. Then α'' is homotopic to α.

We say that a map f from a polyhedron X to a polyhedron Y is a resolution if $f^{-1}(y)$, for any $y \in Y$, is acyclic. By the theorem of Maunder ([8], Corollary 3.6), we have a resolution $f: \Sigma^i \to S^i$ and a map $\tilde{\alpha}: \Sigma^i \to M$ such that $\tilde{\alpha} | \bar{R} = \alpha'' \circ f | R'$, $R' = f^{-1}(\bar{R})$, $\alpha'' \circ f$ is homotopic to $\tilde{\alpha}$ and $\tilde{\alpha}$ satisfies the general position property for homology manifolds. Σ^i is a polyhedron but is not necessarily a homology manifold.

(i) The case where $i < m/2$.

$\tilde{\alpha}$ is an embedding such that $H_*(\tilde{\alpha}(\Sigma^i - \operatorname{Int} R')) = 0$. Let $\mathscr{R}$ be the derived neighbourhood of $\tilde{\alpha}(\Sigma^i - \operatorname{Int} R')$. Then we have an embedding

$\bar{\alpha}: S^i \to M' = (M - \operatorname{Int} \mathscr{R}) \cup$ cone of $\partial \mathscr{R}$.

(ii) The case where $i = m/2$.

Let Λ be $\{x \mid \tilde{\alpha}^{-1}(x)$ is two points $\{x_1, x_2\}\}$. For $x \in \Lambda$, there exists an embedding $s_x: I \to (\Sigma^i - \operatorname{Int} R')$ such that $s_x(0) = x_1$ and $s_x(1) = x_2$. The map $\alpha' \circ s_x: S^1 \to M$ is null homotopic. There exists a continuous map $g: I^2 \to M$ such that $g \mid S^1 = \alpha' \circ s_x$. We have a resolution $p: D \to I^2$ and a map $h: D \to M$ such that $p \mid p^{-1}(S^1)$ is homeomorphic, $h \mid (D - p^{-1}(S^1))$ is injective, $g \circ p$ is homotopic to h, and $h(D - p^{-1}(S^1)) \cap \tilde{\alpha}(\Sigma^i) = \varnothing$. Then we have an embedding $\tilde{\tilde{\alpha}}: \tilde{\alpha}(\Sigma^i) \bigcup_{x \in \Lambda} (\bigcup_{\tilde{\alpha} \circ s_x} h(D)) \equiv \Sigma' \to M$ such that $H_* \tilde{\tilde{\alpha}}((\Sigma' - \tilde{\alpha}(R'))) = 0$. Let $\mathscr{R}$ be the derived neighbourhood of $\tilde{\tilde{\alpha}}(\Sigma' - \operatorname{Int} \tilde{\alpha}(R'))$. Then we define an embedding by $\bar{\alpha}: S^i \to M' = (M - \operatorname{Int} \mathscr{R}) \cup$ cone of $\partial \mathscr{R}$. Thus we proved the lemma.

REMARK. Let Σ^n be a homology sphere such that $\pi_1(\Sigma^n) \neq 0$. Then there is not a map from S^n to Σ^n which is homology equivalence. But S^n is equivalent to Σ^n as a homology cobordism bundle over a point.

LEMMA. *Let* $(W; M, N)$ *be an h-cobordism and* ξ *be a homotopy cobordism bundle (cf. definition of* [4]) *over* W. *Then there exists a bundle map* $f: \xi \mid M \to \xi \mid N$.

PROOF. Let γ be a deformation map from W to N. Then $(\gamma^{\#}(\xi \mid N)) \mid M$ is equivalent to $\xi \mid M$ and there exists a bundle map from $(\gamma^{\#}(\xi \mid N)) \mid M$ to $\xi \mid N$. Then there exists a bundle map $f: \xi \mid M \to \xi \mid N$.

PROOF OF THE THEOREM. We can assume that $\xi^k \oplus \theta^1$ is a homotopy cobordism bundle over X. Let $\alpha: S^{n+k+1} \to T(\xi^k \oplus \theta^1)$ be a map such that $h(\alpha) = \Phi(g)$. We can embed X in R^j, for large j. Let $\mathscr{R}(X)$ be a regular neighbourhood of X in R^j. Let ξ' be a homotopy cobordism bundle over $\mathscr{R}(X)$ induced from $\xi^k \oplus \theta^1$. Then the inclusion map $i: T(\xi \oplus \theta^1) \subset T(\xi')$ is a homotopy equivalence. By the transversality theorem of homology manifolds (Martin [4]), for large p, we have a map $\beta: S^{n+k+1} \to T(\xi') \times I^p$ which is transversal to $\mathscr{R}(X) \times I^p$ and is homotopic to $(i \circ \alpha) \times 0$. Then we have a normal map $\alpha': M \to X$ with degree 1.

The other part of the surgery is the same to the PL cases by Matumoto's theorem and by the lemmas.

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