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CONFORMALLY FLAT HYPERSURFACES IN A CONFORMALLY FLAT RIEMANNIAN MANIFOLD

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Introduction. A Riemannian manifold is called *conformally flat* if it is locally conformally equivalent to a Euclidean space, i.e., if each point of the manifold has a neighborhood where there exists a conformal diffeomorphism onto an open subset in a Euclidean space. Well-known examples of such manifolds which are also hypersurfaces in a Euclidean $(n+1)$-space E^{n+1} are the following: a Euclidean n-space E^n, a Euclidean n-sphere S^n, a right circular cylinder $E^{n-1}\times S^1$ and a Riemannian product manifold $S^{n-1}\times E^1$. It will then be natural to ask: Is there any other conformally flat hypersurface in E^{n+1} which is not conformally diffeomorphic to any of the above examples? Generalizing this question, we can pose the following problem: Classify the conformally flat hypersurfaces in a conformally flat Riemannian manifold up to conformal equivalence. This problem is attractive in conformal geometry.

As a first step to the above problem, we shall classify up to isometry local structures of conformally flat hypersurfaces, especially in a Riemannian manifold of constant curvature, and it is the main purpose of this paper. In fact, our study goes as follows. After preparing some basic definitions and formulas in §1, we shall determine, in §2, the types of the second fundamental forms of these hypersurfaces. The result is that *a hypersurface in a conformally flat Riemannian $(n+1)$-manifold $(n>3)$ is conformally flat if and only if at each point, at least $(n-1)$ eigenvalues of the second fundamental form are identical* (Theorem 3). Making use of this fact, in §3, we shall classify local structures of conformally flat hypersurfaces in a Riemannian manifold of constant curvature. The result is summarized as follows: *Let $M^n (n>3)$ be a conformally flat hypersurface of a Riemannian $(n+1)$-manifold of constant curvature. Then there exist four types (for details, see* §3*) of the local structures of M^n, all of which are loci of a moving $(n-1)$-submanifold $M^{n-1}(v)$ which is of constant curvature for each value of a parameter v* (Theorem 4).

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Throughout this paper manifolds are always assumed to be connected and C^∞. The authors would like to express sincere thanks to Professors M. Obata and H. Omori for their many valuable suggestions and helps during the preparation of this paper.

1. Preliminaries. Let $f: M^n \to \tilde{M}^{n+1}$ be an isometric immersion of a Riemannian n-manifold M^n into a Riemannian $(n+1)$-manifold $\tilde{M}^{n+1}$ with metric g, i.e., M^n be a hypersurface in $\tilde{M}^{n+1}$. For all local formulas and computations, we may consider f as an imbedding and thus identify $x \in M^n$ with $f(x) \in \tilde{M}^{n+1}$. The tangent space $T_x(M)$ of M^n at x is identified with a subspace of the tangent space $T_x(\tilde{M})$ of $\tilde{M}^{n+1}$ at x. In a usual way we regard the second fundamental form A as a symmetric linear transformation on the tangent space $T_x(M)$. For the basic definitions, notations and formulas concerning differential geometry of submanifolds, we mainly follow Chapter VII of Kobayashi-Nomizu [1].

The relationship between the curvature tensor R of M^n and the curvature tensor $\tilde{R}$ of $\tilde{M}^{n+1}$ is expressed by the Gauss equation

$$(1) \qquad R(X, Y) = \tilde{R}^T(X, Y) + AX \wedge AY, \quad X, Y \in T_x(M),$$

where $X \wedge Y$ denotes the skew-symmetric linear transformation of $T_x(M)$ defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

and the superscript T means the orthogonal projection into $T_x(M)$, i.e., $\tilde{R}^T(X, Y)Z$ is the tangential component of $\tilde{R}(X, Y)Z$ for $X, Y, Z \in T_x(M)$.

It is well-known that if $n > 3$, then M^n is conformally flat if and only if the curvature tensor R of M^n splits into the following form

$$(2) \qquad R(X, Y) = \frac{1}{n-2}(SX \wedge Y + X \wedge SY) - \frac{\operatorname{tr} S}{(n-1)(n-2)} X \wedge Y, \quad X, Y \in T_x(M),$$

where S denotes the Ricci tensor of type (1, 1), i.e.,

$$g(SX, Y) = \operatorname{Ric}(X, Y) = \operatorname{tr}\{Z \to R(Z, X)Y\},$$

and tr abbreviates the trace.

2. Principal curvatures of conformally flat hypersurfaces in a conformally flat space. In this section, we always assume that M^n and $\tilde{M}^{n+1}$ are both conformally flat, and $n > 3$.

Applying (2) to $\tilde{M}^{n+1}$, we have

$$\tilde{R}^T(Z, X)Y = \frac{1}{n-1}\{g(X, Y)\tilde{S}^T Z - g(\tilde{S}^T Z, Y)X + g(\tilde{S}^T X, Y)Z - g(Z, Y)\tilde{S}^T X\} - \frac{\operatorname{tr}\tilde{S}}{n(n-1)}\{g(X, Y)Z - g(Z, Y)X\}, \quad X, Y, Z \in T_x(M),$$

and then

$$\operatorname{tr}\{Z \to \tilde{R}^T(Z, X)Y\} = g\Big(\frac{\operatorname{tr}\tilde{S}^T}{n-1}X + \frac{n-2}{n-1}\tilde{S}^T X - \frac{\operatorname{tr}\tilde{S}}{n}X, Y\Big).$$

On the other hand,

$$\operatorname{tr}\{Z \to (AZ \wedge AX)Y\} = g(\operatorname{tr}A \cdot AX - A^2 X, Y).$$

Since (1) implies

$$\operatorname{Ric}(X, Y) = \operatorname{tr}\{Z \to \tilde{R}^T(Z, X)Y\} + \operatorname{tr}\{Z \to (AZ \wedge AX)Y\},$$

we obtain

$$S = \frac{\operatorname{tr}\tilde{S}^T}{n-1}I + \frac{n-2}{n-1}\tilde{S}^T - \frac{\operatorname{tr}\tilde{S}}{n}I + \operatorname{tr}A \cdot A - A^2, \tag{3}$$

where I denotes the identity transformation, and then

$$\operatorname{tr}S = 2\operatorname{tr}\tilde{S}^T - \operatorname{tr}\tilde{S} + (\operatorname{tr}A)^2 - \operatorname{tr}A^2. \tag{4}$$

Now, at a point $x \in M^n$, we take an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_x(M)$ such that each e_i is an eigenvector of A, i.e.,

$$Ae_i = \lambda_i e_i, \quad i = 1, \cdots, n.$$

Then (1) implies for $i \neq j$ that

$$R(e_i, e_j) = \frac{1}{n-1}(\tilde{S}^T e_i \wedge e_j + e_i \wedge \tilde{S}^T e_j) - \frac{\operatorname{tr}\tilde{S}}{n(n-1)}e_i \wedge e_j + \lambda_i\lambda_j e_i \wedge e_j. \tag{5}$$

On the other hand, from (2), (3) and (4) we have

$$R(e_i, e_j) = \frac{1}{n-1}(\tilde{S}^T e_i \wedge e_j + e_i \wedge \tilde{S}^T e_j) - \frac{\operatorname{tr}\tilde{S}}{n(n-1)}e_i \wedge e_j + \frac{1}{n-2}\{(\lambda_i + \lambda_j)\operatorname{tr}A - (\lambda_i^2 + \lambda_j^2)\}e_i \wedge e_j - \frac{1}{(n-1)(n-2)}\{(\operatorname{tr}A)^2 - \operatorname{tr}A^2\}e_i \wedge e_j. \tag{6}$$

Thus, (5)-(6) gives

$$(7) \qquad (n-1)(n-2)\lambda_i\lambda_j - (n-1)\operatorname{tr} A(\lambda_i + \lambda_j) + (n-1)(\lambda_i^2 + \lambda_j^2) + (\operatorname{tr} A)^2 - \operatorname{tr} A^2 = 0 .$$

Consequently, for mutually distinct indices i, j, k, we get

$$(8) \qquad (\lambda_j - \lambda_k)\{(n-2)\lambda_i + \lambda_j + \lambda_k - \operatorname{tr} A\} = 0 .$$

From this equation, we can prove the following

PROPOSITION 1. *The number of distinct principal curvatures is at most two at each point.*

PROOF. Let $\lambda_1, \lambda_2, \lambda_3$ be distinct principal curvatures. Then from (8), we have the following two relations:

$$(n-2)\lambda_3 + \lambda_1 + \lambda_2 - \operatorname{tr} A = 0 ,$$

$$(n-2)\lambda_2 + \lambda_1 + \lambda_3 - \operatorname{tr} A = 0 .$$

Hence we get

$$(n-3)(\lambda_3 - \lambda_2) = 0 .$$

This is a contradiction. q.e.d.

PROPOSITION 2. *The multiplicity of a principal curvature λ is 1, $n-1$ or n at each point.*

PROOF. Suppose that the multiplicity of λ is p with $1 < p < n-1$. Then from Proposition 1, there exists exactly one principal curvature μ such that $\lambda \neq \mu$. Thus, (8) implies that

$$(n-2)\lambda + \lambda + \mu - \{p\lambda + (n-p)\mu\} = 0, \text{ i.e.,}$$
$$(n-p-1)(\lambda - \mu) = 0 .$$

This contradicts our assumption, so $p = 1, n-1$ or n. q.e.d.

Now the "only if" part of the following theorem is clear.

THEOREM 3. *Let M^n be a hypersurface of a conformally flat Riemannian $(n+1)$-manifold $\widetilde{M}^{n+1}$, $n > 3$. Then M^n is conformally flat if and only if at each point of M^n, the second fundamental form A of M^n is one of the following types*:

(I) $A = \lambda I$, $I =$ *the identity transformation,*

(II) *A has two distinct eigenvalues of multiplicity $n-1$ and 1 respectively.*

PROOF. We prove the "if" part. Let λ and μ be the (possibly equal) eigenvalues of multiplicity $n-1$ and 1 respectively. It is sufficient to prove that λ, μ and $\operatorname{tr} A = (n-1)\lambda + \mu$ satisfy the equation (7). But it is a straightforward calculation, so we omit it here. q.e.d.

REMARK 1. The second fundamental form A does not have the conformal nature in the sense that A is not invariant by a conformal change of metric. However, the property that the tensor $\mathscr{A} = A - (\operatorname{tr} A/n)I$ is equal to 0 or not equal to 0 is unchanged by any conformal change of metric. So total-umbilicalness is of conformal nature.

REMARK 2. Similar results in E^{n+1} can be seen in [3].

3. Conformally flat hypersurfaces in a space of constant curvature. Throughout this section, we assume that the ambient manifold $\tilde{M}^{n+1} (n > 3)$ is a Riemannian manifold of constant (sectional) curvature $\tilde{c}$.

REMARK. Of course every Riemannian manifold of constant curvature is conformally flat. Moreover, our assumption is reasonable in the following sense: Firstly, by a theorem of Kuiper [2], a conformally flat simply connected Riemannian n-manifold can be conformally developed over a Euclidean n-sphere. Secondly, conformal-flatness is of conformal nature. So, if necessary, we have only to take the simply connected Riemannian covering manifold.

Now we classify conformally flat hypersurfaces in a Riemannian manifold of constant curvature.

Case I. $A = \lambda I$ on M^n.

In this case, from the Gauss equation (1), we have

$$R(X, Y) = (\lambda^2 + \tilde{c}) X \wedge Y, \quad X, Y \in T_x(M) .$$

This shows that M^n is a totally umbilical hypersurface of constant curvature $\lambda^2 + \tilde{c}$.

Case II. A has two eigenvalues λ and μ, which are distinct at every point of M^n, of multiplicity $n-1$ and 1 respectively.

In this case it is convenient for our purpose to use moving frames. Furthermore we agree on the following ranges of indices unless otherwise stated:

$$1 \leqq A, B, C, \cdots \leqq n+1 ,$$
$$1 \leqq i, j, k, \cdots \leqq n-1 .$$

We choose a local field of orthonormal frames e_A in $\tilde{M}^{n+1}$ such that, restricted to M^n, the vectors $e_1, \cdots, e_n$ are tangent to M^n (and, consequently, e_{n+1} is normal to M^n). With respect to the frame field of $\tilde{M}^{n+1}$ chosen above, let ω_A and ω_{AB} be the field of dual frames and connection forms respectively. We restrict these forms to M^n. Then we have

$$\omega_{n+1} = 0 . \tag{9}$$

Moreover, by our assumption, we may choose the above frame field e_A in such a way that

$$\omega_{i,n+1} = \lambda\omega_i , \tag{10}$$

$$\omega_{n,n+1} = \mu\omega_n . \tag{11}$$

Now we put

$$\omega_{in} = \sum_j B_{ij}\omega_j + B_i\omega_n . \tag{12}$$

Then, taking exterior differentiation of (10), we get

$$d\omega_{i,n+1} = \lambda \sum_j \omega_{ij} \wedge \omega_j + \mu \sum_j B_{ij}\omega_j \wedge \omega_n ,$$

$$d(\lambda\omega_i) = \sum_j \lambda_{,j}\omega_j \wedge \omega_i + \lambda_{,n}\omega_n \wedge \omega_i + \lambda \sum_j \omega_{ij} \wedge \omega_j + \lambda \sum_j B_{ij}\omega_j \wedge \omega_n ,$$

where we have put

$$d\lambda = \sum_i \lambda_{,i}\omega_i + \lambda_{,n}\omega_n .$$

Hence we get

$$\sum_j \lambda_{,j}\omega_j \wedge \omega_i + (\lambda - \mu) \sum_j B_{ij}\omega_j \wedge \omega_n - \lambda_{,n}\omega_i \wedge \omega_n = 0 ,$$

from which we obtain

$$\lambda_{,j} = 0 \quad \text{and} \quad B_{ij} = \frac{\lambda_{,n}}{\lambda - \mu}\delta_{ij} . \tag{13}$$

In a similar way, from (11), (12) and (13), we get

$$d\omega_{n,n+1} = \lambda \sum_i B_i\omega_i \wedge \omega_n ,$$

$$d(\mu\omega_n) = \sum_i \mu_{,i}\omega_i \wedge \omega_n + \mu \sum_i B_i\omega_i \wedge \omega_n ,$$

where we have put

$$d\mu = \sum_i \mu_{,i}\omega_i + \mu_{,n}\omega_n .$$

Therefore we get

$$B_i = \frac{\mu_{,i}}{\lambda - \mu} . \tag{14}$$

Consequently, combining (13) and (14), we obtain

$$\omega_{in} = \frac{1}{\lambda - \mu}(\lambda_{,n}\omega_i + \mu_{,i}\omega_n) . \tag{15}$$

It follows easily from (15) that the distribution of the space spanned by principal vectors corresponding to each principal curvature is involutive. In fact, (15) shows that the Pfaff equation $\omega_n = 0$ is completely integrable.

On the other hand, in (13), $\lambda_{,j} = 0$ means that λ is constant on each integral submanifold of the distribution corresponding to λ. So we may consider that λ is a function of a parameter v, for example the arc length of an orthogonal trajectory, which is an integral submanifold corresponding to μ, of the family of these integral submanifolds. Furthermore, taking exterior differentiation of (15) and comparing the coefficients of the term $\omega_i \wedge \omega_j$, we obtain

$$\lambda_{,n}\mu_{,j} = 0 . \tag{16}$$

From now on we restrict our discussion to a neighborhood U, diffeomorphic with an open ball, of a point $x \in M^n$ where one of the following cases, Case II-A, Case II-B and Case II-C, occurs.

Case II-A. $\lambda_{,n} \neq 0$ on U, hence $\mu_{,j} = 0$ for all j on U.

In this case, with respect to an adapted frame field, the connection form (ω_{AB}) of M, restricted to U, is given by

$$\left[\begin{array}{ccc|c|c}
\omega_{11}, & \cdots\cdots\cdots & ,\ \omega_{1,n-1} & (\lambda_{,n}/(\lambda-\mu))\omega_1 & \lambda\omega_1 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\omega_{n-1,1}, & \cdots\cdots\cdots & ,\ \omega_{n-1,n-1} & (\lambda_{,n}/(\lambda-\mu))\omega_{n-1} & \lambda\omega_{n-1} \\
\hline
-(\lambda_{,n}/(\lambda-\mu))\omega_1, & \cdots & ,\ -(\lambda_{,n}/(\lambda-\mu))\omega_{n-1} & 0 & \mu\omega_n \\
\hline
-\lambda\omega_1, & \cdots\cdots\cdots & ,\ -\lambda\omega_{n-1} & -\mu\omega_n & 0
\end{array}\right] . \tag{17}$$

Substituting (17) into

$$d\omega_{ij} = \textstyle\sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{in} \wedge \omega_{nj} + \omega_{i,n+1} \wedge \omega_{n+1,j} - \tilde{c}\omega_i \wedge \omega_j ,$$

we have the curvature form

$$\Omega_{ij} = d\omega_{ij} - \textstyle\sum_k \omega_{ik} \wedge \omega_{kj} = -\left(\left(\dfrac{\lambda_{,n}}{\lambda-\mu}\right)^2 + \lambda^2 + \tilde{c}\right)\omega_i \wedge \omega_j . \tag{18}$$

Furthermore, from (17), we get $d\omega_n = 0$ on U. This implies that we may put $\omega_n = dv$ on U. (17) also shows that the integral submanifold $M^{n-1}(v)$ (restricted to U) corresponding to λ (and so v) is totally umbilical in M^n.

Thus, on U, M^n is a locus of a moving $(n-1)$-submanifold $M^{n-1}(v)$ along which the principal curvature λ of multiplicity $n-1$ is constant and which is umbilical in M^n and of constant curvature

$$\left(\frac{1}{\lambda-\mu}\frac{d\lambda}{dv}\right)^2 + \lambda^2 + \tilde{c} ,$$

where v is the arc length of an orthogonal trajectory of the family $M^{n-1}(v)$.

Moreover, differentiating $\omega_{in} = (\lambda_{,n}/(\lambda - \mu))\omega_i$, we get

$$\left(\frac{\lambda'}{\lambda-\mu}\right)' - \left(\frac{\lambda'}{\lambda-\mu}\right)^2 - (\lambda\mu + \tilde{c}) = 0\,, \tag{19}$$

where the prime denotes the differentiation with respect to v.

Case II-B. $\mu_{,j} \neq 0$ on U for some j, hence $\lambda_{,n} = 0$ on U.

In this case, the connection form (ω_{AB}) above is given by

$$\left[\begin{array}{c|c|c}
\omega_{11}, \cdots\cdots\cdots, \omega_{1,n-1} & (\mu_{,1}/(\lambda-\mu))\omega_n & \lambda\omega_1 \\
\vdots \qquad \ddots \qquad \vdots & \vdots & \vdots \\
\omega_{n-1,1}, \cdots\cdots\cdots, \omega_{n-1,n-1} & (\mu_{,n-1}/(\lambda-\mu))\omega_n & \lambda\omega_{n-1} \\ \hline
-(\mu_{,1}/(\lambda-\mu))\omega_n, \cdots, -(\mu_{,n-1}/(\lambda-\mu))\omega_n & 0 & \mu\omega_n \\ \hline
-\lambda\omega_1, \cdots\cdots\cdots, -\lambda\omega_{n-1} & -\mu\omega_n & 0
\end{array}\right], \tag{20}$$

and the curvature form Ω_{ij} is

$$\Omega_{ij} = -(\lambda^2 + \tilde{c})\omega_i \wedge \omega_j\,. \tag{21}$$

(20) shows that the integral submanifold $M^{n-1}(v)$ is totally geodesic in M^n, but this is not of conformal nature.

Consequently, on U, M^n is a locus of a moving $(n-1)$-submanifold $M^{n-1}(v)$ which is totally geodesic in M^n and of constant curvature $\lambda^2 + \tilde{c}$.

Case II-C. $\lambda_{,n} = 0$ and $\mu_{,j} = 0$ for all j on U.

This case can be considered as a particular one of Case II-A as well as Case II-B. The connection form (ω_{AB}) above becomes

$$\left[\begin{array}{c|c|c}
\omega_{11}, \cdots\cdots\cdots, \omega_{1,n-1} & 0 & \lambda\omega_1 \\
\vdots \qquad \ddots \qquad \vdots & \vdots & \vdots \\
\omega_{n-1,1}, \cdots\cdots\cdots, \omega_{n-1,n-1} & 0 & \lambda\omega_{n-1} \\ \hline
0, \cdots\cdots\cdots, 0 & 0 & \mu\omega_n \\ \hline
-\lambda\omega_1, \cdots\cdots\cdots, -\lambda\omega_{n-1} & -\mu\omega_n & 0
\end{array}\right], \tag{22}$$

and the curvature form Ω_{ij} is

$$\Omega_{ij} = -(\lambda^2 + \tilde{c})\omega_i \wedge \omega_j . \tag{23}$$

Moreover, differentiating $\omega_{in} = 0$, we have

$$\lambda\mu + \tilde{c} = 0 . \tag{24}$$

From this and our assumption, λ and μ are both constant on U if $\tilde{c} \neq 0$. Hence, in case $\tilde{c} \neq 0$, M^n is locally (i.e. on U) a Riemannian product manifold $M^{n-1}(\lambda^2 + \tilde{c}) \times M^1$, where $M^{n-1}(\lambda^2 + \tilde{c})$ is a Riemannian $(n-1)$-manifold of constant curvature $\lambda^2 + \tilde{c}$ and M^1 is a curve in M^n whose first curvature is constant along M^1.

REMARK. Note that the point at which $\lambda_{,n} = 0$ and $\mu_{,j} = 0$ for all j may occur as a boundary point of neighborhood of Case II-A or Case II-B. The structure of M^n at that point is also determined by (22).

After all, M^n is covered, except for the set of measure zero, by a family of neighborhoods discussed above.

Case III. *A* has possibly equal eigenvalues λ and μ of multiplicity $n-1$ and 1 respectively.

In this case, remark that an umbilical point comes also as a boundary point of a neighborhood of Case II. However, we can see that M^n is covered, except for the set of measure zero, by a family of neighborhoods discussed in the preceding cases [i.e., the local version of Case I, Case II-A, Case II-B, Case II-C].

Summarizing the above, we arrive at the following

THEOREM 4. *Let $M^n(n > 3)$ be a conformally flat hypersurface of a Riemannian $(n+1)$-manifold $\tilde{M}^{n+1}$ of constant curvature $\tilde{c}$. Then M^n is locally a locus of a moving $(n-1)$-submanifold $M^{n-1}(v)$ which is of constant curvature for each value of a parameter v. The local structures of M^n is determined, except for the set of measure zero, by Case* II-A, *Case* II-B, *Case* II-C *and the local version of Case* I.

REMARK 1. An umbilical hypersurface of constant curvature $\lambda^2 + \tilde{c}$, where λ is the principal curvature, is a particular case of this locus (Case I).

REMARK 2. If, in particular, $\tilde{M}^{n+1}$ is a Euclidean $(n+1)$-space E^{n+1}, i.e. $\tilde{c} = 0$, then locally M^n is one of the following:

(Case I) a totally umbilical hypersurface (hence of constant curvature).

(Case II-A) a surface of revolution—Let $(x^1, \cdots, x^{n+1})$ be a canonical coordinate system of E^{n+1} and γ a curve in the (x^1, x^2)-plane defined by $x^1 = \gamma(x^2)$, $x^2 > 0$. Rotating γ about x^2-axis, we get a surface of revolution

$G\cdot\gamma$ where G is the rotation group $G = SO(n) = SO(x^1, x^3, \cdots, x^{n+1})$.

(Case II-B) a tube—Let γ be a curve in E^{n+1}. The normal sphere bundle of γ with (sufficiently small) fixed radius is, by definition, a tube.

(Case II-C) a product manifold $S^{n-1} \times E^1$ or a cylinder $E^{n-1} \times \gamma$ built over a plane curve γ.

The second assertion is obtained by observing the differential equation (19).

REFERENCES

[1] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, vol. I, vol. II, Interscience, New York, 1963, 1969.

[2] N. H. KUIPER, On conformally flat-spaces in the large, Ann. of Math., 50 (1949), 916-924.

[3] J. A. SCHOUTEN, Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Maßbestimmung auf eine Mannigfaltigkeit mit euklidischer Maßbestimmung, Math. Z., 11 (1921), 58-88.

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