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AN INEQUALITY BETWEEN SQUARE NORMS ON DUAL GROUPS

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Abstract. The Plancherel Theorem asserts the equality of the L^2-norms (with respect to Haar measure) of a function f on a locally compact abelian group G and of its Fourier transform $\hat{f}$. The Hausdorff-Young inequality gives conditions on p and q under which $\|\hat{f}\|_q \leqq \|f\|_p$. We consider a different variant: we place a measure μ on $\hat{G}$, a measure w on G, and examine

$$\int_{\hat{G}} |\hat{f}|^2 d\mu \leqq \int_G |f|^2 dw .$$

Our main results show that it is enough to consider the case in which w is equivalent to Haar measure, and we give a condition on w which is necessary and sufficient for the inequality to hold for every $\mu \geqq 0$ with $\|\mu\| \leqq 1$.

Let G be a locally compact abelian group and let $\hat{G}$ be its dual group. We denote the Fourier transform of a function f on G by $\hat{f}$. In this note we shall consider the inequality

$$\int_{\hat{G}} |\hat{f}|^2 d\mu \leqq \int_G |f|^2 dw \tag{1}$$

which we require to hold for all functions f in the space $\mathscr{K}(G)$ of continuous functions of compact support on G, for some positive measures μ on $\hat{G}$ and w on G.

Inequalities of this kind have a long history (see, for example [2]). They have appeared more recently because of their importance in the solution of multiplier problems for weighted L^p-spaces ([5], in particular a remark on page 50, and [6], especially Lemmas 2.1 and 2.2). These authors usually consider cases in which one of the groups G and $\hat{G}$ is the circle group and the other the integers, though in his Theorem 3b in [5], Hirschman quotes a result for R^n. Work on general groups has usually yielded only abstract characterizations of multipliers [1], and we hope that a study of the inequality (1) might be a first step to some more concrete representations.

Our principal results are as follows. First, if the inequality (1) holds for some non-zero measure μ, then the Haar measure m of G must be absolutely continuous with respect to w. Moreover, the inequality remains valid if we replace w by its absolutely continuous part: the singular part of w can be omitted. We may therefore write $dw(x) = v(x)\,dm(x)$ for some measurable function v. Then the inequality (1) holds for every positive bounded μ with $\|\mu\| \leqq 1$ if and only if $1/v \in L^1(G)$, and in this case G must be σ-compact. We write

$$P = \{(\mu, w) \colon \mu \geqq 0,\ w \geqq 0, \text{ and } \int_{\hat{G}} |\hat{f}(\gamma)|^2\, d\mu(\gamma) \leqq \int_G |f(x)|^2\, dw(x) \text{ for all } f \in \mathscr{K}(G)\}\,,$$

$${}_\mu P = \{w \colon (\mu, w) \in P\} \text{ and } P_w = \{\mu \colon (\mu, w) \in P\}\,,$$

and we obtain some elementary properties of these sets.

All the facts from harmonic analysis we use can be found in Hewitt and Ross [3].

We denote by m (resp. λ) the Haar measure on G (resp. $\hat{G}$). If w is a measure on G, w_x denotes the translate of w by $x \in G$, $\int_G f(y)\, dw_x(y) = \int_G f(y - x)\, dw(y)$.

The value of the character determined by the element γ of $\hat{G}$ at the element x of G will be denoted by $\langle x, \gamma\rangle$. Then the Fourier transform $\hat{f}$ of $f \in \mathscr{K}(G)$ is given by $\hat{f}(\gamma) = \int_G f(x)\langle x, -\gamma\rangle dm(x)$.

PROPOSITION 1.1. (i) $(\lambda, m) \in P$.

(ii) *If* $(\mu, w) \in P$, $0 \leqq \mu' \leqq \mu$ *and* $w \leqq w'$, *then* $(\mu', w') \in P$.

(iii) *If* $(\mu, w) \in P$, $\gamma \in \hat{G}$ *and* $x \in G$, *then* $(\mu_\gamma, w_x) \in P$.

(iv) *For each* μ *and each* w, ${}_\mu P$ *and* P_w *are convex.*

(v) ${}_\mu P$ *is a weak*-closed subset of the dual* $M(G)$ *of* $\mathscr{K}(G)$.

PROOF. Part (i) is immediate from the Plancherel Theorem, and (ii) is obvious. For (iii) we have, when $(\mu, w) \in P$ and $f \in \mathscr{K}(G)$

$$\int_{\hat{G}} |\hat{f}(\xi)|^2\, d\mu_\gamma(\xi) = \int_{\hat{G}} |\hat{f}(\xi - \gamma)|^2\, d\mu(\xi) \leqq \int_G |f(x)\langle x, \gamma\rangle|^2\, dw(x) = \int_G |f(x)|^2\, dw(x)\,,$$

using the facts that the Fourier transform of $f(\cdot)\ \langle\cdot, \gamma\rangle$ is the mapping $\xi \to \hat{f}(\xi - \gamma)$ and that the modulus of a character is 1. Thus, from $(\mu, w) \in P$ follows $(\mu_\gamma, w) \in P$. A similar argument now proves that $(\mu_\gamma, w_x) \in P$. The convexity of ${}_\mu P$ and P_w is easy to see, which deals with (iv). Finally, for $f \in \mathscr{K}(G)$, the map $w \to \int_G |f(x)|^2\, dw(x)$ is a weak* continuous linear functional on $M(G)$ for each $f \in \mathscr{K}(G)$;

$$_\mu P = \bigcap_{f \in \mathscr{K}(G)} \left\{ w \colon \int_{\hat{G}} |\hat{f}(\gamma)|^2 \, d\mu(\gamma) \leqq \int_G |f(x)|^2 \, dw(x) \right\}$$

is therefore an intersection of weak*-closed half-spaces, and so is closed (and convex).

REMARK 1.2. The functions $\hat{f}$ for $f \in \mathscr{K}(G)$ do not in general lie in $\mathscr{K}(\hat{G})$, and the result of part (v) therefore will not hold for P_w. However, the argument does show that P_w is weak*-closed when it is regarded as a subset of the dual of any space containing all functions $\hat{f}$ for $f \in \mathscr{K}(G)$.

We shall now improve part (iii) of Proposition 1.1. We denote the convolution product of two measures μ and ν by $\mu * \nu$. A convolution product is always defined if one of the measures has compact support or if both measures are bounded.

PROPOSITION 1.3. *Let* $(\mu, w) \in P$. *If* μ *is bounded, let* ν *be a bounded measure and if* μ *is not bounded, let* ν *have compact support, and in both cases, let* ν *be positive with* $\|\nu\| \leqq 1$. *Let* u *be a positive measure of compact support with* $\|u\| \geqq 1$. *Then* $(\nu * \mu, u * w) \in P$.

PROOF. If δ_γ is the unit point mass at γ, then $\mu_\gamma = \delta_{-\gamma} * \mu$. By (iii) and (iv) of Proposition 1.1, if π is any convex combination of δ_γ's and 0 (for it is obvious that $(0, w) \in P$, where 0 is the zero measure) then $\pi * \mu \in P_w$. Given ν as in the statement of the proposition, we can find a net (π_α) with support $\pi_\alpha \subseteqq$ support ν for every α, and with $\pi_\alpha \to \nu$ in any weak* topology of the kind mentioned in Remark 1.2. Since $\pi_\alpha * \mu \to \nu * \mu$ in the same weak* topology under either of the given conditions, it follows from the fact that P_w is closed that $(\nu * \mu, w) \in P$. The rest of this proposition is proved in a similar way.

We next prove a technical lemma which helps to simplify many arguments.

LEMMA 1.4. *If* (1) *holds for all* $f \in \mathscr{K}(G)$ *then it also holds when* $f \in L^2(m + w)$, *when either* f *or* (*if we allow the possibility of infinite values*) $\hat{f}$ *is the characteristic function of a compact set, and when* $\hat{f} \in \mathscr{K}(\hat{G})$.

PROOF. Inequality (1) and the Plancherel Theorem give $\int_{\hat{G}} |\hat{f}|^2 \, d(\lambda + \mu) \leqq \int_G |f|^2 \, d(m + w)$ for $f \in \mathscr{K}(G)$. Let $g \in L^2(m + w)$. Then $g \in L^2(m)$, so $\hat{g}$ is well-defined as an element of $L^2(\lambda)$. Let (g_n) be a sequence in $\mathscr{K}(G)$ with $g_n \to g$ in $L^2(m + w)$. Then (g_n) is Cauchy in $L^2(m + w)$ and the inequality above shows that $(\hat{g}_n)$ is Cauchy in $L^2(\lambda + \mu)$. Thus $(\hat{g}_n)$ converges in $L^2(\lambda + \mu)$ to a limit which may obviously be identified with $\hat{g}$. We may therefore take limits in the inequality for g_n to get $\int_{\hat{G}} |\hat{g}|^2 d(\lambda + \mu) \leqq \int_G |g|^2 d(m + w)$, from which the inequality (1) for g follows by the Plancherel Theorem. The case in which f is the characteristic function of a compact set is included in this. If $\hat{f}$ is the characteristic function of a compact set or if $\hat{f} \in \mathscr{K}(\hat{G})$, then either $\int_G |f|^2 dw = \infty$ and (1) is trivial or $f \in L^2(m + w)$ (note that $\hat{f} \in L^2(\lambda)$ so $f \in L^2(m)$) and (1) for f has already been established.

Our next result gives some special properties of Haar measure in this context.

PROPOSITION 1.5. (i) *Let* (λ, m) *be a normalized pair of Haar measures. Then if* $(\mu, m) \in P$, $\mu \leqq \lambda$.

(ii) *Suppose that for some measure* ν *on* $\hat{G}$, $\mu \leqq \nu$ *for all* $\mu \in P_w$. *Then there is a Haar measure* λ *on* $\hat{G}$ *such that* $\mu \leqq \lambda$ *for* $\mu \in P_w$. *Moreover, if* P_w *is a lattice (for the usual ordering in the space of measures),* λ *can be chosen so that* $P_w = \{\mu\colon 0 \leqq \mu \leqq \lambda\}$. *Dual results hold when the roles of* G *and* $\hat{G}$ *are interchanged.*

PROOF. (i) Suppose $\mu \leqq \lambda$ is false. Then we can find a compact set $K \subseteqq \hat{G}$ with $\mu(K) > \lambda(K)$. By Lemma 1.4 we may take f so that $\hat{f}$ is the characteristic function of K. Then

$$\int_{\hat{G}} |\hat{f}|^2 d\mu = \mu(K) > \lambda(K) = \int_{\hat{G}} |\hat{f}|^2 d\lambda = \int_G |f|^2 dm$$

(by the Plancherel Theorem) so that $(\mu, m) \notin P$.

(ii) Since P_w is bounded above, by ν, it has a supremum ([3] Vol. I B. 35) which we can again denote by ν, so that $\nu = \sup\{\mu\colon \mu \in P_w\}$. For $\gamma \in \hat{G}$ the map $\mu \to \mu_\gamma$ is a bijection of P_w on to itself, for it maps P_w into itself by Proposition 1.1 (iii) and has an inverse $\mu \to \mu_{-\gamma}$. Therefore

$$\nu_\gamma = \sup\{\mu_\gamma\colon \mu \in P_w\} = \sup\{\mu\colon \mu \in P_w\} = \nu\ .$$

Thus, ν is translation invariant and so is a Haar measure. If now P_w is a lattice we can find an increasing net (μ_α) in P_w with $\mu_\alpha \to \nu$. It follows that

$$\int_{\hat{G}} |\hat{f}|^2 d\nu = \operatorname*{Lim}_\alpha \int_{\hat{G}} |\hat{f}|^2 d\mu_\alpha \leqq \int_G |f|^2 dw\ ,$$

i.e. that $\nu \in P_w$. The proof of (ii) is now complete.

The hypothesis that P_w should be a lattice in the last part of (ii) is necessary. For consider the case in which G (and so also $\hat{G}$) is finite.

Then as we shall see later (Proposition 2.6 (ii)), P_w is always bounded, but if $\mu \in P_w$ the measure sup $\{\mu_\gamma : \gamma \in \hat{G}\}$ is represented by the function whose constant value is sup $\{\mu(\xi) : \xi \in \hat{G}\}$, and this may not be in P_w (Proposition 2.8).

2. The main theorem. We now prove the first of the theorems mentioned in the introduction.

THEOREM 2.1. *If $w \in {}_\mu P$ for some $\mu > 0$, then the Haar measure m is absolutely continuous with respect to w.*

PROOF. Suppose that the conclusion is false. Then we can find a measurable set E such that $w(E) = 0$ but $m(E) > 0$, and hence a compact set $K \subseteqq E$ such that $w(K) = 0$ but $m(K) > 0$. Take f to be the characteristic function of K (see Lemma 1.4). Since f is in $L^1(m)$, $\hat{f}$ is continuous and as $m(K) \neq 0$, $\hat{f} \neq 0$. As $\mu \neq 0$, we can therefore find $\gamma \in \hat{G}$ such that $\int_{\hat{G}} |\hat{f}|^2 d\mu_\gamma > 0$. However, $\int_G |f|^2 dw = 0$, and therefore $(\mu_\gamma, w) \notin P$. Proposition 1.1 (iii) shows that this is contrary to hypothesis.

Since the support of m is the whole of G we have the following corollary.

COROLLARY 2.2. *If $w \in {}_\mu P$, the support of w is G.*

Theorem 2.1 in conjunction with the next result shows that it is enough to consider measures w equivalent to Haar measure.

THEOREM 2.3. *Let $(\mu, w) \in P$. Write $w = u + s$ where u is absolutely continuous with respect to Haar measure m, and s is singular with respect to m. Then $(\mu, u) \in P$.*

PROOF. Let E be a Borel set with $s(E) = 0$, $m(G \backslash E) = 0$. If χ_E is the characteristic function of E and $f \in \mathscr{K}(G)$ then both $f\chi_E$ and $f(1 - \chi_E)$ belong to $L^2(m + w)$, and so we may apply Lemma 1.4. Moreover, $\hat{f} - (f\chi_E)^\wedge = (f(1 - \chi_E))^\wedge = 0$. Hence

$$\int_{\hat{G}} |\hat{f}|^2 d\mu = \int_{\hat{G}} |(f\chi_E)^\wedge|^2 d\mu \leqq \int_G |f\chi_E|^2 dw = \int_G |f|^2 \chi_E dw$$
$$= \int_G |f|^2 du .$$

The theorem is proved.

We next show that a proof given by Hirschman ([4]) works in a more general context with only minor modifications. We shall use the same symbol for a non-negative measurable function v on G (which we allow to take the value $+\infty$) and the measure $v(x)\, dm(x)$ associated with it.

We shall also write

$$\tilde{P} = \left\{(w, \mu): \int_G |f(x)|^2 dw(x) \leqq \int_{\hat{G}} |\hat{f}(\gamma)|^2 d\mu(\gamma) \text{ for } \hat{f} \in \mathscr{K}(\hat{G})\right\} .$$

THEOREM 2.4. *Let v (resp. φ) be a non-negative measurable function on G (resp. $\hat{G}$). If $(\varphi, 1/v) \in P$, then $(v, 1/\varphi) \in \tilde{P}$.*

PROOF. If φ is identically zero, the result is trivial. Otherwise, by Theorem 2.1, $1/v$ vanishes only on a set of Haar measure zero, and so we may assume $v(x) < \infty$ for all x. Using Lemma 1.4, we may suppose that the inequality represented by the statement $(\varphi, 1/v) \in P$ holds for $f \in L^2(m + v)$ and not merely for $f \in \mathscr{K}(G)$.

Let $K \subseteqq G$ be compact and such that v is essentially bounded on K. Let $\hat{f} \in \mathscr{K}(\hat{G})$. Then $\chi_K f v \in L^2(m + v^{-1})$ (where χ_K is the characteristic function of K) because K is compact and $\chi_K v$ is essentially bounded. Put $F_K = (\chi_K f v)^\wedge$. Since $(\varphi, 1/v) \in P$, we have

$$\int_{\hat{G}} |F_K|^2 \varphi d\lambda \leqq \int_G |\chi_K f v|^2 \cdot \frac{1}{v} \cdot dm = \int_G \chi_K |f|^2 v dm .$$

Hence, using the Parseval identity,

$$\begin{aligned} \int_G \chi_K |f|^2 v dm &= \int_G \chi_K f v \bar{f} dm = \int_{\hat{G}} F_K \bar{\hat{f}} d\lambda \\ &\leqq \left(\int_{\hat{G}} |F_K|^2 \varphi d\lambda\right)^{1/2} \left(\int_{\hat{G}} |\hat{f}|^2 \cdot \frac{1}{\varphi} \cdot d\lambda\right)^{1/2} \\ &\leqq \left(\int_G \chi_K |f|^2 v\, dm\right)^{1/2} \left(\int_{\hat{G}} |\hat{f}|^2 \cdot \frac{1}{\varphi} \cdot d\lambda\right)^{1/2} . \end{aligned}$$

Since the left-hand integral is finite we conclude that

$$\int_G \chi_K |f|^2 v dm \leqq \int_{\hat{G}} |\hat{f}|^2 \cdot \frac{1}{\varphi} \cdot d\lambda .$$

Since $f \in L^2(m + v)$ its support is σ-compact. Since $v(x) < \infty$ for all x, we can find a sequence (K_n) of compact sets such that $K_n \uparrow$ (support f) and v is essentially bounded on each K_n. Replacing K by K_n and taking the limit we see that $(v, 1/\varphi) \in \tilde{P}$.

Our next theorem was also promised in the introduction.

THEOREM 2.5. *Let w be an absolutely continuous measure given by $dw(x) = v(x)\, dm(x)$. Then the following are equivalent.*

(i) *There is an element γ of $\hat{G}$ and a neighbourhood V of γ such that, for all $\varphi \in L^1(\lambda)$ with support $\varphi \subseteqq V$, $\varphi \geqq 0$, and $\|\varphi\| \leqq 1$, $(\varphi, w) \in P$.*

(ii) $\delta_0 \in P_w$.

(iii) *Every* $\mu \geqq 0$ *with* $\|\mu\| \leqq 1$ *belongs to* P_w.
(iv) $1/v \in L^1(m)$, *and* $\|1/v\|_1 \leqq 1$.
If any of these assertions holds, then G *is* σ*-compact.*

PROOF. The standard construction of an approximate identity in $L^1(\lambda)$ shows how to find a net (φ_α) in $L^1(\lambda)$ with support in V and $\|\varphi_\alpha\| \leqq 1$ which has the property that $\varphi_\alpha \to \delta_\gamma$ in the weak* topology. From Remark 1.2 we conclude that (i) implies that $\delta_\gamma \in P_w$, and then Proposition 1.1 (iii) shows that $\delta_0 \in P_w$. Proposition 1.3 gives (ii) implies (iii), and it is trivial that (iii) is stronger than (i). We shall prove that (ii) is equivalent to (iv).

Now (ii) represents the inequality

$$|\hat{f}(0)|^2 \leqq \int_G |f(x)|^2 dw(x) \qquad (f \in \mathscr{K}(G)) .$$

or, replacing the Fourier transform, and recognizing that $\langle x, 0\rangle = 1$ for all x,

$$\left|\int_G f(x)dm(x)\right| \leqq \left(\int_G |f(x)|^2 dw(x)\right)^{1/2} \qquad (f \in \mathscr{K}(G)) .$$

Obviously this inequality holds for all f if and only if it holds when f is positive and so if and only if

$$\|f\|_1 \leqq \|f\|_{2,w} \qquad (f \in \mathscr{K}(G)) ,$$

where the expression on the right denotes the norm of f as an element of $L^2(w)$.

Now assume that this norm inequality holds. Then each continuous linear functional on $L^1(m)$ is also continuous on $L^2(w)$, or in other words, for each $g \in L^\infty(m)$ there is $Wg \in L^2(w)$ such that

$$\int_G g(x)f(x)dm(x) = \int_G Wg(x)f(x)dw(x) \qquad (f \in \mathscr{K}(G)) .$$

Since $dw = vdm$, this gives $vWg = g$ almost everywhere, so that $g/v = Wg \in L^2(w)$. In particular, we may take $g = 1$ to find

$$\int_G \frac{1}{v(x)} dm(x) = \int_G \frac{1}{v(x)^2} v(x)dm(x) = \|W1\|_{2,w}^2 < \infty .$$

Moreover, since $\|f\|_1 \leqq \|f\|_{2,w}$, W, being the adjoint of a contraction, is a contraction too, and so $\|W1\|_{2,w} \leqq 1$.

Finally, assume (iv). Define Wg for $g \in L^\infty(m)$ by $Wg = g/v$. Then

$$\|Wg\|_{2,w}^2 = \int_G \frac{g^2}{v^2} vdm = \int_G g^2 \cdot \frac{1}{v} \cdot dm \leqq \|g\|_\infty^2 .$$

So for $f \in \mathscr{K}(G)$,

$$\left|\int_G gfdm\right| = \left|\int_G (Wg)fvdm\right| \leqq \| Wg \|_{2,w} \|f\|_{2,w} \leqq \|g\|_\infty \|f\|_{2,w} .$$

Taking suprema over $\|g\|_\infty \leqq 1$ gives

$$\|f\|_1 \leqq \|f\|_{2,w}$$

which we have seen is equivalent to (ii).

Finally, as $1/v \in L^1(m)$, support $1/v =$ Closure of $\{x: v(x) < \infty\}$ is σ-compact. But as w is a measure (in the dual of $\mathscr{K}(G)$) v is finite almost everywhere, so G is σ-compact.

The inequalities represented by Theorem 2.4 are by no means the best possible. For if G is the circle group and $\hat{G}$ the integers, the pair (μ, w) with $\mu(n) = (|n| + 1)^{-\alpha}$ for $n \in \hat{G}$ and $dw(x) = x^{-\alpha}dm(x)$ for $x \in G$, belongs to P if $0 < \alpha < 1$ (see [6]) and in this case $\mu \notin L^1(\lambda)$ and so certainly does not satisfy $\|\mu\| \leqq 1$. If we interchange the roles of G and $\hat{G}$, the pair (μ', w') with $d\mu'(x) = x^\alpha dm(x)$ and $w'(n) = (|n| + 1)^\alpha$ belongs to P for $0 < \alpha < 1$ (see [3] again, or use Theorem 2.4). This shows that in the notation of Theorem 2.4, if we know only that $(\varphi, w) \in P$ for *one* $\varphi \in L^1(m)$, then we cannot conclude that $1/v \in L^1(m)$.

In the case in which $\hat{G}$ is discrete, we can say more.

PROPOSITION 2.6. *Let G be compact, and let $dw(x) = v(x)\, dm(x)$.*
(i) *If $(\mu, w) \in P$ and $\mu \neq 0$ then $1/v \in L^1(m)$.*
(ii) *For each $\gamma \in \hat{G}$, $|\mu(\gamma)|^s \leqq w(G)$.*

PROOF. (i) Since G is discrete and $\mu \neq 0$, for some constant $k > 0$ and some $\gamma \in \hat{G}$, $k\delta_\gamma \leqq \mu$. Hence $(k\delta_\gamma, w) \in P$. Since P_w is translation invariant, $(k\delta_0, w) \in P$. It is easy to see that $(\delta_0, 1/k \cdot w) \in P$, and so from Theorem 2.4, $1/kv \in L^1(m)$, whence $1/v \in L^1(m)$.

(ii) In the inequality

$$\int_{\hat{G}} |\hat{f}|^2 d\mu \leqq \int_G |f|^2 dw ,$$

we simply take f to be the character $\langle x, \gamma \rangle$.

If we interchange the roles of G and $\hat{G}$, the same proof as for part (ii) gives the following.

PROPOSITION 2.7. *Let G be discrete. Then if $(\mu, w) \in P$, $\mu(\hat{G}) \leqq |w(x)|^2$ for each $x \in G$.*

We saw in Proposition 1.1 that if (λ, m) is a normalized pair of Haar measures, $\mu \leqq \lambda$ and $m \leqq w$, then $(\mu, w) \in P$. We would like to observe that this does not cover all cases.

PROPOSITION 2.8. *Suppose that G is σ-compact. Then there exists $(\mu, w) \in P$ such that for no normalized pair (λ, m) of Haar measures is it true that $\mu \leqq \lambda$, $m \leqq w$.*

PROOF. If G is not compact, all we need do is take an atomic measure for μ in using Theorem 2.5. If μ is a measure with $\|\mu\| \leqq 1$ on discrete dual $\hat{G}$ of a compact group G, then the smallest Haar measure λ which dominates μ is defined by $\lambda(\gamma) = \sup\{\mu(\xi): \xi \in \hat{G}\}$ for each $\gamma \in \hat{G}$. Put $k = \lambda(\gamma)$ for any γ. The dual Haar measure m assigns mass $1/k$ to the whole group G. According to Theorem 2.5, (μ, w) will belong to P if w is given by $dw(x) = v(x)\,dm(x)$ and $\|1/v\|_1 \leqq 1$. Thus to prove our proposition, we need only find v such that $\int_G 1/v \cdot dm \leqq 1$ and yet $v(x) \geqq 1/k$ a.e. is false. This is clearly always possible.

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