Tôhoku Math. Journ.
26 (1974), 11-24.

SOME QUASI-HAUSDORFF TRANSFORMATIONS

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(Received December 5, 1972)

1. Let $\{\nu_n\}$ be any given sequence of complex numbers. The quasi-Hausdorff transformation (H^*, ν_n) is defined by

$$(1)\qquad t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n}\nu_n) s_k$$

whenever this series converges. We will use (H^*, ν_n) also to denote the matrix of the transformation (1), and write s, t for the sequences $\{s_k\}$, $\{t_n\}$; thus (1) may be written

$$t = (H^*, \nu_n)s\ .$$

We say that the (H^*, ν_n) method is applicable to s if (1) converges for all n, so that t is defined; we say that s is summable (H^*, ν_n) to l if, further $t_n \to l$ as $n \to \infty$. We use a similar terminology for other transformations.

The matrix (H^*, ν_n) is the transpose of the matrix of the Hausdorff transformation[†] (H, ν_n). It is familiar that, given two sequences $\{\nu_n\}$, $\{\omega_n\}$ (say), we have

$$(H, \nu_n)(H, \omega_n) = (H, \nu_n\omega_n)\ .$$

Taking the transpose of this result (with ν, ω interchanged) we have, as is familiar

$$(2)\qquad (H^*, \nu_n)(H^*, \omega_n) = (H^*, \nu_n\omega_n)\ .$$

But the matrices considered are not, in general, row finite, so that their multiplication is not necessarily associative; thus we cannot assert that

$$(3)\qquad (H^*, \nu_n)[(H^*, \omega_n)s] = [(H^*, \nu_n)(H^*, \omega_n)]s\ .$$

Thus the situation differs from that which applies for the corresponding Hausdorff transformations in that, notwithstanding (2), we cannot assert that the result of applying first the (H^*, ω_n) and then the (H^*, ν_n) transformation is the same as that of applying the $(H^*, \nu_n\omega_n)$ transformation.

It has been shown by Ramanujan [4] that there is a close connection between Hausdorff summability (H, μ_n) and quasi-Hausdorff summability

† For those properties of Hausdorff transformations to which reference is made, see, e.g. [1, Chapter XI].

(H^*, μ_{n+1}); in particular, whenever (H, μ_n) is regular then so is (H^*, μ_{n+1}). When

$$\mu_n = \frac{1}{\binom{n+r}{n}}, \tag{4}$$

(H, μ_n) reduces to the Cesàro transformation (C, r); thus it is natural to describe the quasi-Hausdorff transformation (H^*, μ_{n+1}) with μ_n given by (4) as the quasi-Cesàro transformation (C^*, r). The properties of (C^*, r) have been investigated by me [2], [3]; a more general transformation was investigated independently by A. J. White [5].

When

$$\omega_n = \frac{1}{(n+1)^r} \tag{5}$$

(H, ω_n) reduces to the Hölder transformation (H, r); we will therefore describe the (H^*, ω_{n+1}) transformation with ω_n given by (5) as the quasi-Hölder transformation (H^*, r).

It is known (e.g. [1]) that Cesàro and Hölder summabilities (C, r), (H, r) are equivalent. Thus if for a given r, μ_n, ω_n are given by (4), (5) we have $\mu_n = \nu_n \omega_n$ where (H, ν_n) is regular. Hence, by what has already been said

$$(H^*, \mu_{n+1}) = (H^*, \nu_{n+1})(H^*, \omega_{n+1}),$$

and (H^*, ν_{n+1}) is regular. But, since we cannot assert (3), we cannot deduce from this that summability (C^*, r) is implied by summability (H^*, r). Similar remarks apply with the roles of (C^*, r), (H^*, r) interchanged.

When r is an integer, the Hölder transformation (H, r) is the same as the transformation obtained by r iterations of the $(C, 1)$ transformation; and we can deduce that

$$(H^*, r) = [(C^*, 1)]^r. \tag{6}$$

But although (6) holds as a relation between matrices, we cannot deduce that the result of r iterations of the $(C^*, 1)$ transformation is the same as (H^*, r).

We will restrict consideration to integer values of r; accordingly, it will be assumed throughout from now on that r is a positive integer. On this understanding, we investigate the relations between (C^*, r), $(C^*, 1)^r$, (H^*, r). Here $(C^*, 1)^r$ is used to denote the result of r iterations of the $(C^*, 1)$ transformation.

The results to be proved are as follows.

THEOREM 1. *(C^*, r) and $(C^*, 1)^r$ are equivalent.*

THEOREM 2. *If s is summable (H^*, r) to l, then it is summable (C^*, r) to l. If s is summable (C^*, r) to l, and if (H^*, r) is applicable, then s is summable (H^*, r) to l. However, except in the trivial case $r = 1$, the applicability of (H^*, r) is not implied by (C^*, r) summability.*

Let now $r_1 > r$ (where r_1 is also an integer). It is known [3, Theorem 1; 5, Theorems 2,3] that, if s is summable (C^*, r) to l then it is summable (C^*, r_1) to l. It therefore follows at once from Theorem 2 that, if s is summable (H^*, r) to l and if (H^*, r_1) is applicable, then s is summable (H^*, r_1) to l. However, the hypothesis that (H^*, r_1) is applicable cannot in general be omitted.

THEOREM 3. *Let $r_1 > r$ (r_1 an integer). Let s be summable (H^*, r) to l. If $r = 1$, then (H^*, r_1) is applicable. This result becomes false if $r > 1$.*

It follows at once from Theorem 3 and the remarks made above that summability (H^*, r) implies summability (H^*, r_1) without any supplementary "applicability condition" when $r = 1$, but not when $r > 1$.

2. We require some lemmas.

LEMMA 1. *Let*

$$F(k, x) = \sum_{\rho=0}^{r} (-1)^\rho P_\rho(k) x^\rho ;$$

$$G(k, x) = \sum_{\rho=0}^{r} (-1)^\rho P_\rho(k - \rho) x^\rho ,$$

where, for each ρ, $P_\rho(k)$ is a polynomial in k of degree not exceeding r. Suppose that $F(k, x)$ has the property that, when expressed as a polynomial in k, the coefficient of k^q is divisible by $(1 - x)^q$ $(q = 1, 2, \cdots, r)$. Then $G(k, x)$ also has this property.

Write

$$F(k, x) = \sum_{q=0}^{r} \phi_q(x) k^q . \tag{7}$$

It is enough to consider the contribution to $G(k, x)$ of one term in the sum (7), since the general result can then be obtained by addition. Taking, then, q as fixed, let a_ρ be the coefficient of k^q in $(-1)^\rho P_\rho(k)$; thus

$$\phi_q(x) = \sum_{\rho=0}^{r} a_\rho x^\rho .$$

The contribution of this term to $G(k, x)$ is

$$\sum_{\rho=0}^{r} a_\rho (k-\rho)^q x^\rho \ . \tag{8}$$

We can write (8) as $L^q \phi_q(x)$, where the operator L is defined by

$$Lf(x) = kf(x) - xf'(x) \ .$$

Since $\phi_q(x)$ is divisible by $(1-x)^q$, it follows by induction on t that $L^t\phi_q(x)$ is a polynomial in k of degree t, the coefficient of k^σ being divisible by $(1-x)^{q+\sigma-t}$. Applying this result with $t=q$, the lemma follows.

LEMMA 2. *Suppose that*

$$\psi(x) = \sum_{\rho=0}^{r} (-1)^\rho a_\rho x^\rho$$

is divisible by $(1-x)^q$. *Let* $Q(x)$ *be a polynomial in* x *of degree* ν. *Then*

$$\sum_{\rho=0}^{r} (-1)^\rho a_\rho Q(k-\rho) \tag{9}$$

is a polynomial in k *of degree at most* $\nu - q$. *In the case* $q=\nu$, *the conclusion is to be interpreted as meaning that* (9) *is constant; in the case* $q>\nu$, *it is to be interpreted as meaning that* (9) *is identically zero.*

It is slightly more convenient to prove a similar result, but with (9) replaced by

$$\sum_{\rho=0}^{r} (-1)^\rho a_\rho Q(k+\rho) \ ; \tag{10}$$

this will give the conclusion, for we can apply this result with $Q(x)$ replaced by $Q(-x)$ and with k replaced by $-k$.

Write

$$\psi(x) = (1-x)^q \psi_1(x) \ ,$$

and write E for the "shift operator" defined by $EQ(k) = Q(k+1)$. Then we can write (10) as

$$\left(\sum_{\rho=0}^{r} (-1)^\rho a_\rho E^\rho\right) Q(k) = ((1-E)^q \psi_1(E))Q(k) = \Delta^q(\psi_1(E)Q(k)) \ .$$

The operator $\psi_1(E)$ operating on a polynomial cannot increase its degree; the operator Δ^q decreases its degree by q (with the same conventions as in the statement of the lemma). Hence the conclusion.

LEMMA 3. *Let* $F(k, x)$, $P_\rho(k)$ *satisfy the conditions of Lemma* 1. *Let* $Q(k, n)$ *be a polynomial in* k, n *of degree* ν. *Then*

$$\sum_{\rho=0}^{r} (-1)^\rho Q(k-\rho, n) P_\rho(k-\rho) \tag{11}$$

is a polynomial in k, n of degree at most ν.

Write

$$Q(k, n) = \sum_{\mu=0}^{\nu} n^{\mu} Q_{\mu}(k) ;$$

thus, for each μ, $Q_{\mu}(k)$ is a polynomial of degree at most $\nu - \mu$. By Lemma 1, we can write

$$P_{\rho}(k - \rho) = \sum_{q=0}^{r} a_{q,\rho} k^{q}$$

where, for each q,

$$\sum_{\rho=0}^{r} (-1)^{\rho} a_{q,\rho} x^{\rho}$$

is divisible by $(1 - x)^{q}$. Hence, by Lemma 2

$$\sum_{\rho=0}^{r} (-1)^{\rho} a_{q,\rho} Q_{\mu}(k - \rho)$$

is a polynomial in k of degree at most $\nu - \mu - q$. Multiplying by $k^{q} n^{\mu}$ and summing with respect to q, μ, we obtain the conclusion.

LEMMA 4. *Suppose that the* $(C^*, 1)^r$ *transformation is applicable to* s; *let the* $(C^*, 1)^r$ *transform be denoted by* $\{t_n^{(r)}\}$. *Then*

$$s_k = \sum_{\rho=0}^{r} (-1)^{\rho} P_{\rho}^{(r)}(k) t_{k+\rho}^{(r)} \tag{12}$$

where, for each ρ, $P_{\rho}^{(r)}(k)$ *is a polynomial in* k *of degree* r, *and where*

(i) *For* $\rho = 1, 2, \cdots r$, $P_{\rho}^{(r)}(k)$ *is divisible by*

$$(k + 1)(k + 2) \cdots (k + \rho) ;$$

(ii) *The coefficient of* k^{q} *in*

$$f^{(r)}(k, x) = \sum_{\rho=0}^{r} (-1)^{\rho} P_{\rho}^{(r)}(k) x^{\rho}$$

is divisible by $(1 - x)^{q}$.

Since the $(C^*, 1)$ transformation is defined by

$$t_n^{(1)} = (n + 1) \sum_{k=n}^{\infty} \frac{s_k}{(k + 1)(k + 2)} , \tag{13}$$

it is clear that, whenever (13) converges,

$$s_k = (k + 2) t_k^{(1)} - (k + 1) t_{k+1}^{(1)} ; \tag{14}$$

thus the conclusion of the lemma holds when $r = 1$. Assume now that the result is true for $r - 1$ (where $r \geqq 2$). Since

$$t_{k+\rho}^{(r-1)} = (k + \rho + 2)t_{k+\rho}^{(r)} - (k + \rho + 1)t_{k+\rho+1}^{(r)} ,$$

it follows that

$$\begin{aligned} s_k &= \sum_{\rho=0}^{r-1} (-1)^\rho P_\rho^{(r-1)}(k)[(k + \rho + 2)t_{k+\rho}^{(r)} - (k + \rho + 1)t_{k+\rho+1}^{(r)}] \\ &= \sum_{\rho=0}^{r} (-1)^\rho P_\rho^{(r)}(k) t_{k+\rho}^{(r)} , \end{aligned}$$

where

$$P_\rho^{(r)}(k) = (k + \rho + 2)P_\rho^{(r-1)}(k) + (k + \rho)P_{\rho-1}^{(r-1)}(k) . \tag{15}$$

Here we adopt the convention that $P_r^{(r-1)}(k)$, $P_{-1}^{(r-1)}(k)$ are taken to mean 0. It follows at once from (15) and the induction hypothesis that $P^{(r)}(k)$ is a polynomial of degree r, and that (i) holds. To prove (ii), we deduce from (15) that

$$f^{(r)}(k, x) = x(1 - x)\frac{d}{dx}f^{(r-1)}(k, x) + k(1 - x)f^{(r-1)}(k, x) + (2 - x)f^{(r-1)}(k, x) ,$$

and (ii) now follows from the induction hypothesis.

It may be remarked that the transformation (14), giving s in terms of $\{t_k^{(1)}\}$, is the $(H^*, n + 2)$ transformation. The transformation (12) is obtained by r iterations of this and thus (since we are now considering row finite matrices) it is the $(H^*, (n + 2)^r)$ transformation. Hence

$$P_\rho^{(r)}(k) = (-1)^\rho \binom{k + \rho}{k} \Delta^\rho (k + 2)^r .$$

But this result does not appear to be of any help in proving (ii).

We now define $S_n^{(r)}$ inductively by

$$S_n^{(0)} = s_n; \qquad S_n^{(r)} = S_0^{(r-1)} + S_1^{(r-1)} + \cdots + S_n^{(r-1)} \ (r \geqq 1) .$$

As is familiar, this is equivalent to the definition

$$S_n^{(r)} = \sum_{k=0}^{n} \binom{n - k + r - 1}{n - k} s_k .$$

LEMMA 5. *If $\lambda > 0$, and if*

$$\sum^{\infty} \frac{s_n}{n^\lambda}$$

converges, then

$$\sum^{\infty} \frac{S_n^{(1)}}{n^{1+\lambda}}$$

converges.

We take the hypothesis and conclusion in the equivalent forms that

$$\sum_0^\infty \frac{s_n}{\binom{n+\lambda}{n}}, \qquad \sum_0^\infty \frac{S_n^{(1)}}{\binom{n+\lambda+1}{n}}$$

converge respectively. Write

$$T_n = \sum_{\nu=n}^\infty \frac{s_\nu}{\binom{\nu+\lambda}{\nu}},$$

so that $T_n \to 0$ as $n \to \infty$. Then

$$\begin{aligned}
&\sum_{n=0}^N \frac{S_n^{(1)}}{\binom{n+\lambda+1}{n}} \\
&= \sum_{n=0}^N \frac{1}{\binom{n+\lambda+1}{n}} \sum_{\nu=0}^n \binom{\nu+\lambda}{\nu}(T_\nu - T_{\nu+1}) \\
&= \sum_{\nu=0}^N \binom{\nu+\lambda}{\nu}(T_\nu - T_{\nu+1}) \sum_{n=\nu}^N \frac{1}{\binom{n+\lambda+1}{n}} \\
&= \frac{\lambda+1}{\lambda}\left\{\sum_{\nu=0}^N (T_\nu - T_{\nu+1}) - \frac{1}{\binom{N+\lambda+1}{N+1}} \sum_{\nu=0}^N \binom{\nu+\lambda}{\nu}(T_\nu - T_{\nu+1})\right\}.
\end{aligned}$$

Applying a straightforward partial summation to the second sum inside the curly brackets, we can now easily prove that this expression tends to a limit as $N \to \infty$.

COROLLARY. *If ρ is a positive integer, and if*

$$\sum^\infty \frac{s_n}{n^2} \tag{16}$$

converges, then

$$\sum^\infty \frac{S_n^{(\rho)}}{n^{2+\rho}}$$

converges.

3. We can now prove Theorem 1. Suppose first that s is summable $(C^*, 1)^r$; there is no loss of generality in supposing that it is summable to 0, so that, with the notation of Lemma 4, $t_n^{(r)} = o(1)$. It will be enough to prove that s is summable (C, r) to 0; in other words, that

$$S_n^{(r)} = o(n^r) . \tag{17}$$

For the applicability of $(C^*, 1)^r$, and thus, a fortiori, the $(C^*, 1)^r$ summability of s requires, in particular, that $t_n^{(1)}$ should be defined; and this is equivalent to the convergence of (16). But it follows from [3, Theorem 3] or [5, Theorem 4] that, if s is summable (C, r), and if (16) converges, then s is summable (C^*, r).

Now, by Lemma 4, and with the notation used there,

$$\begin{aligned} S_n^{(r)} &= \sum_{\nu=0}^{n} \binom{n-\nu+r-1}{n-\nu} s_\nu \\ &= \sum_{\nu=0}^{n} \binom{n-\nu+r-1}{n-\nu} \sum_{\rho=0}^{r} (-1)^\rho P_\rho^{(r)}(\nu) t_{\nu+\rho}^{(r)} \\ &= \sum_{\rho=0}^{r} (-1)^\rho \sum_{\nu=0}^{n} \binom{n-\nu+r-1}{n-\nu} P_\rho^{(r)}(\nu) t_{\nu+\rho}^{(r)} \\ &= \sum_{\rho=0}^{r} (-1)^\rho \sum_{k=\rho}^{n+\rho} \binom{n-k+\rho+r-1}{n-k+\rho} P_\rho^{(r)}(k-\rho) t_k^{(r)} . \end{aligned} \tag{18}$$

We may replace the lower limit of summation in the inner sum in (18) by $k = 0$, since, by Lemma 4(i) $P_\rho^{(r)}(k-\rho)$ vanishes for the extra terms. Similarly, since the polynomial

$$\binom{n-k+\rho+r-1}{n-k+\rho}$$

vanishes for $k = n+\rho+1, \cdots, n+r-1$, we may, except in the case $\rho = r$, replace the upper limit of summation in the inner sum by $n+r-1$. If we then invert the order of summation, we obtain

$$\begin{aligned} S_n^{(r)} = &\sum_{k=0}^{n+r-1} t_k^{(r)} \sum_{\rho=0}^{r} (-1)^\rho \binom{n-k+\rho+r-1}{n-k+\rho} P_\rho^{(r)}(k-\rho) \\ &+ (-1)^r P_r^{(r)}(n) t_{n+r}^{(r)} = \sum_{k=0}^{n+r} \alpha_{nk}^{(r)} t_k^{(r)} , \end{aligned}$$

say. But since $\binom{n-k+r-1}{n-k}$ is a polynomial in n, k of degree $r-1$, it follows from Lemmas 3, 4(ii) that, for $0 \leqq k \leqq n+r-1$, $\alpha_{nk}^{(r)}$ is a polynomial in n, k of degree not exceeding $r-1$. Further, $\alpha_{n,n+r}^{(r)}$ is a polynomial in n of degree r; and, since $t_k^{(r)} = o(1)$, (17) now follows, as required.

We now consider the converse implication. Suppose, then, that s is summable (C^*, r); we may again suppose that it is summable to 0. It follows that (16) converges; also, by [3, Theorem 4] or [5, Theorem 5], s is summable (C, r), so that (17) holds. Now let $R^{(\nu)}(n)$ denote a rational function of n (possibly different at each occurrence), the degree of the denominator exceeding that of the numerator by ν, and the denominator being a product of factors of the form $(n+p)$, with p a positive integer (repetitions being allowed). With this notation, we will prove that, for $\rho = 1, 2, \cdots r$, $t_n^{(\rho)}$ exists, and that

$$t_n^{(\rho)} = \sum_{\nu=\rho}^{r-1} S_{n-\rho}^{(\nu)} R^{(\nu)}(n) + o(1) . \tag{19}$$

When $\rho = r$, the sum in (19) is empty, so that (19) reduces to $t_n^{(r)} = o(1)$. Thus, once (19) has been proved, the proof of the theorem will be completed. We prove (19) by an induction argument. Consider first the case $\rho = 1$. It follows by partial summation from the convergence of (16) that

$$S_n^{(1)} = o(n^2) .$$

Hence, for $\nu \geqq 1$,

$$S_n^{(\nu)} = o(n^{\nu+1}) . \tag{20}$$

Using (20), we deduce from (13), by repeated partial summations, that

$$\begin{aligned} t_n^{(1)} &= (n+1)\left\{-\frac{S_{n-1}^{(1)}}{(n+1)(n+2)} + 2\sum_{k=n}^{\infty} \frac{S_k^{(1)}}{(k+1)(k+2)(k+3)}\right\} \\ &= (n+1)\left\{-\sum_{\nu=1}^{r} \frac{\nu!\, S_{n-1}^{(\nu)}}{(n+1)(n+2)\cdots(n+\nu+1)}\right. \\ &\quad \left. + (r+1)! \sum_{k=r}^{\infty} \frac{S_k^{(r)}}{(k+1)(k+2)\cdots(k+r+2)}\right\} \\ &= -\sum_{\nu=1}^{r-1} \frac{\nu!\, S_{n-1}^{(\nu)}}{(n+2)\cdots(n+\nu+1)} + o(1) , \end{aligned}$$

since, when $\nu = r$, we can replace (20) by the stronger result (17). Hence (19) holds when $\rho = 1$.

We now assume that (19) holds for ρ, where $1 \leqq \rho < r$, and prove that it holds for $\rho + 1$. By definition, $\{t_n^{(\rho+1)}\}$ is the $(C^*, 1)$ transform of $\{t_n^{(\rho)}\}$. The $(C^*, 1)$ transform of the term $o(1)$ in (19) exists and is $o(1)$, by the regularity of $(C^*, 1)$. It is therefore enough to consider the $(C^*, 1)$ transform of a typical term in the sum (19); that is to say, to consider

$$(n+1)\sum_{k=n}^{\infty} \frac{S_{k-\rho}^{(\nu)} R^{(\nu)}(k)}{(k+1)(k+2)} , \tag{21}$$

where $\rho \leqq \nu < r$. This series converges, by Lemma 5, Corollary. Also, by repeated partial summation, again using (20), the expression (21) is equal to

$$(n+1)\Big\{-\sum_{\mu=\nu+1}^{r} S_{n-\rho-1}^{(\mu)}\Delta^{\mu-\nu-1}\Big(\frac{R^{(\nu)}(n)}{(n+1)(n+2)}\Big) + \sum_{k=n}^{\infty} S_{k-\rho}^{(r)}\Delta^{r-\nu}\Big(\frac{R^{(\nu)}(k)}{(k+1)(k+2)}\Big)\Big\}$$
$$= \sum_{\mu=\nu+1}^{r-1} S_{n-\rho-1}^{(\mu)}R^{(\mu)}(n) + o(1) .$$

Here, again, we use (17) to deal with the second sum, and the term $\mu = r$ of the first sum, inside the curly brackets. Thus (19), if true for ρ, is true for $\rho+1$, and the proof of the theorem is completed.

4. In order to prove the remaining theorems, we require some further lemmas.

LEMMA 6. *Let r be a positive integer. Then*
(i) *For $k \geqq n$,*

$$\Delta^{k-n}\Big(\frac{1}{(n+2)^r}\Big) = \frac{(k-n)!\,(n+1)!}{(k+2)!}K_r(n, k) , \tag{22}$$

where $K_r(n, k)$ is defined by induction (on r) by

$$K_1(n, k) = 1 ;$$

$$K_r(n, k) = \sum_{\nu=n}^{k} \frac{K_{r-1}(\nu, k)}{\nu+2} \qquad (r \geqq 2) . \tag{23}$$

Alternatively, (23) *may be replaced by*

$$K_r(n, k) = \sum_{\nu=n}^{k} \frac{K_{r-1}(n, \nu)}{\nu+2} \qquad (r \geqq 2) . \tag{24}$$

(ii) *For fixed n,*

$$K_r(n, k) = \frac{(\log k)^{r-1}}{(r-1)!} + O((\log k)^{r-2}) \tag{25}$$

as $k \to \infty$. Further,

$$(\log k)^{-(r-1)}K_r(n, k)$$

is of bounded variation in $k \geqq n$.

The result that (22) holds is familiar, and easily verified, when $r = 1$. Assume the result true for $r-1$, where $r \geqq 2$. Applying the familiar formula

$$\Delta^q(a_n b_n) = \sum_{\nu=0}^{q} \binom{q}{\nu} \Delta^\nu a_n \Delta^{q-\nu} b_{n+\nu} \tag{26}$$

with

$$a_n = \frac{1}{(n+2)}, \qquad b_n = \frac{1}{(n+2)^{r-1}}, \qquad q = k - n,$$

we obtain

$$(27) \quad \Delta^{k-n}\left(\frac{1}{(n+2)^r}\right) = \sum_{\nu=0}^{k-n} \binom{k-n}{\nu} \frac{\nu!}{(n+2)(n+3)\cdots(n+\nu+2)}$$
$$\times \frac{(k-n-\nu)!}{(n+\nu+2)\cdots(k+2)} K_{r-1}(n+\nu, k)$$
$$= \frac{(k-n)!\,(n+1)!}{(k+2)!} \sum_{\nu=0}^{k-n} \frac{K_{r-1}(n+\nu, k)}{n+\nu+2}.$$

On changing the notation by replacing $(n+\nu)$ by ν in the sum in (27), we see that (22) holds for r, with $K_r(n, k)$ given by (23).

If we had applied (26) with

$$a_n = \frac{1}{(n+2)^{r-1}}, \qquad b_n = \frac{1}{n+2},$$

a similar argument would have yielded (24). We remark that it may be verified directly that the two induction definitions are equivalent; for either gives, for $r \geqq 2$,

$$K_r(n, k) = \sum \frac{1}{(\nu_1+2)(\nu_2+2)\cdots(\nu_{r-1}+2)},$$

the sum being taken over all $\nu_1, \nu_2 \cdots, \nu_{r-1}$ for which

$$n \leqq \nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_{r-1} \leqq k.$$

Once (i) has been proved, (25) follows at once by induction on r (using (24)). Further, again using (24), we have, for $r \geqq 2$

$$\Delta\{(\log k)^{-(r-1)} K_r(n, k)\}$$
$$= (\log(k+1))^{-(r-1)} \Delta_k K_r(n, k) + K_r(n, k) \Delta((\log k)^{-(r-1)})$$
$$= -(\log(k+1))^{-(r-1)} \frac{K_{r-1}(n, k+1)}{k+3} + \frac{(r-1)}{k} K_r(n, k)(\log k)^{-r}\left(1 + O\left(\frac{1}{k}\right)\right)$$
$$= O\left(\frac{1}{k \log^2 k}\right),$$

by (25). The result follows.

LEMMA 7. *For fixed $n > 0$,*

$$\frac{K_r(n, k)}{K_r(0, k)}$$

is a non-decreasing function of k for $k \geqq n$.

The proof is by induction. The result is trivial when $r = 1$. Assume the result true for $r - 1$, where $r \geqq 2$. Then, by (24),

$$\frac{K_r(n, k)}{K_r(0, k)} - \frac{K_r(n, k+1)}{K_r(0, k+1)} = \frac{L_r(n, k)}{K_r(0, k)K_r(0, k+1)},$$

where

$$\begin{aligned} &L_r(n, k) \\ &= \sum_{\nu=0}^{k+1} \frac{K_{r-1}(0, \nu)}{\nu + 2} \sum_{\nu=n}^{k} \frac{K_{r-1}(n, \nu)}{\nu + 2} - \sum_{\nu=0}^{k} \frac{K_{r-1}(0, \nu)}{\nu + 2} \sum_{\nu=n}^{k+1} \frac{K_{r-1}(n, \nu)}{\nu + 2} \\ &= \frac{1}{k+3}\left\{K_{r-1}(0, k+1) \sum_{\nu=n}^{k} \frac{K_{r-1}(n, \nu)}{\nu + 2} - K_{r-1}(n, k+1) \sum_{\nu=0}^{k} \frac{K_{r-1}(0, \nu)}{\nu + 2}\right\}. \end{aligned}$$

But, by the induction hypothesis, we have

$$K_{r-1}(0, k+1)K_{r-1}(n, \nu) \leqq K_{r-1}(n, k+1)K_{r-1}(0, \nu)$$

for $n \leqq \nu \leqq k$. Hence

$$\begin{aligned} K_{r-1}(0, k+1) \sum_{\nu=n}^{k} \frac{K_{r-1}(n, \nu)}{\nu + 2} &\leqq K_{r-1}(n, k+1) \sum_{\nu=n}^{k} \frac{K_{r-1}(0, \nu)}{\nu + 2} \\ &< K_{r-1}(n, k+1) \sum_{\nu=0}^{k} \frac{K_{r-1}(0, \nu)}{\nu + 2}. \end{aligned}$$

Thus $L_r(n, k) < 0$, which gives the conclusion.

We now note that, if the (H^*, r) transform of s is denoted by $\{h_n^{(r)}\}$, then it follows from (22) that $h_n^{(r)}$ is defined by

$$h_n^{(r)} = (n + 1) \sum_{k=n}^{\infty} \frac{K_r(n, k)}{(k+1)(k+2)} s_k \tag{28}$$

whenever this series converges. Further, it follows from Lemma 6 (ii) that, if (28) converges for one value of n, then it converges for all n, and that a necessary and sufficient condition for this to happen is that

$$\sum^{\infty} \frac{(\log k)^{r-1}}{k^2} s_k \tag{29}$$

should converge.

LEMMA 8. *If the (H^*, r) transformation is applicable to s, then the $(C^*, 1)^r$ transformation is also applicable to s, and the $(C^*, 1)^r$ transform is equal to the (H^*, r) transform.*

We again prove the result by induction. The result is trivial when $r = 1$, since, in this case, the definitions of (H^*, r), $(C^*, 1)^r$ are the same.

Suppose, then, the result true for $r-1$, where $r \geqq 2$. Suppose the (H^*, r) transformation is applicable. Then (29) converges; and hence the corresponding series with r replaced by $r-1$ also converges, so that $(H^*, r-1)$ is also applicable. By (23) and (28),

$$(30) \qquad h_n^{(r-1)} = (n+1)\sum_{k=n}^{\infty} \frac{K_{r-1}(n,k)}{(k+1)(k+2)} s_k$$

$$= (n+1)(n+2)\sum_{k=n}^{\infty} \frac{[K_r(n,k) - K_r(n+1,k)]}{(k+1)(k+2)} s_k$$

$$= (n+2)h_n^{(r)} - (n+1)h_{n+1}^{(r)} .$$

But, in view of Lemma 7, it follows easily from the convergence of (28) with $n = 0$ that

$$h_n^{(r)} = o(n) .$$

We therefore deduce from (30) that

$$(31) \qquad h_n^{(r)} = (n+1)\sum_{k=n}^{\infty} \frac{h_k^{(r-1)}}{(k+1)(k+2)} .$$

By the induction hypothesis, and with the notation used in the proof of Theorem 1, $t_k^{(r-1)}$ exists and equals $h_k^{(r-1)}$. Hence, by (31) and the definition of $t_k^{(r)}$, $t_n^{(r)}$ exists and equals $h_n^{(r)}$.

5. The positive part of Theorem 2 follows at once from Theorem 1 and Lemma 8. In order to prove the negative part of Theorem 2, and also of Theorem 3, we consider the example

$$s_k = \begin{cases} t^{-\lambda}2^{2t} & (k = 2^t,\ t = 1, 2, \cdots) ; \\ -t^{-\lambda}2^{2t} & (k = 2^t + 1,\ t = 1, 2, \cdots) ; \\ 0 & (\text{otherwise}) . \end{cases}$$

where $\lambda > 0$. Then

$$S_k^{(1)} = \begin{cases} t^{-\lambda}2^{2t} & (k = 2^t,\ t = 1, 2, \cdots) ; \\ 0 & (\text{otherwise}) . \end{cases}$$

Since

$$\sum_{t=1}^{T} t^{-\lambda}2^{2t} = O(T^{-\lambda}2^{2T}) ,$$

we see that $S_k^{(2)} = o(k^2)$, so that s is summable $(C, 2)$ to 0. The series (29) diverges if $r \geqq \lambda + 1$, since the general term does not tend to 0; and it is easily proved that it converges if $r < \lambda + 1$. In particular, (29) converges when $r = 1$; in other words, (16) converges, so that (C^*, r) is applicable (for any r). Thus, by [3, Theorem 3] or [5, Theorem 4], s is summable (C^*, r) for $r \geqq 2$. But, if $r \geqq 2$ and we choose $\lambda \leqq r-1$, (H^*, r) is not applicable. Further, if $2 \leqq r < r_1$, we may choose λ so that $r-1 < \lambda \leqq r_1 - 1$. Then (H^*, r) is applicable, so that, since s is summable (C^*, r), it is summable also (H^*, r); but (H^*, r_1) is not applicable.

It remains only to consider the case $r = 1$ of Theorem 3. Summability $(H^*, 1)$ is the same as $(C^*, 1)$, and this is known to be equivalent to $(C, 1)$. It follows, a fortiori that if s is summable $(H^*, 1)$ then the $(C, 1)$ means are bounded; that is to say

$$S_k^{(1)} = O(k) . \tag{32}$$

The convergence of (29) (with r replaced by r_1) follows at once by partial summation; indeed, a weaker result that (32) would suffice for this. This gives the conclusion.

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