

## THE INVERSION FORMULAE FOR SOME HYPERGEOMETRIC TRANSFORMS

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**Abstract.** By means of Laplace transform and their inverses we have obtained three inversion formulae for some hypergeometric transforms whose kernels are Whittaker's function  $W_{k,m}$ , confluent hypergeometric function  ${}_1F_1$  and Gauss hypergeometric function  ${}_2F_1$  respectively. Corresponding inversion formulae derived from Erdélyi's work are also mentioned.

**1. Introduction.** The Laplace transform of  $\phi(x)$  is denoted by  $\mathcal{L}\{\phi(x)\}$  and defined by

$$(1) \quad \mathcal{L}\{\phi(x)\} = \int_0^{\infty} e^{-xt} \phi(x) dx = \psi(t),$$

with  $\phi(x)$  and  $\psi(t)$  related as in (1) the inverse Laplace transform of  $\psi(t)$  is written

$$\mathcal{L}^{-1}\{\psi(t)\} = \phi(x).$$

The Mellin transform of  $f(u)$  is denoted either by  $M\{f(u)\}$  or by  $F(s)$ . Usually  $u$  is real and positive while  $s$  is a complex variable of the form  $s = \sigma + i\tau$ ,  $\sigma$  and  $\tau$  are both real.

We denote the Mellin transform [9, p. 46] of a function  $f(u)$ ,  $M\{f(u)\} = F(s)$  as:

$$(2) \quad M\{f(u)\} = F(s) = \int_0^{\infty} f(u) u^{s-1} du,$$

where  $f(u)u^{s-1} \in L_2(0, \infty)$ .

If  $f(u)$  and  $F(s)$  are related as in (2) then the inverse Mellin transform of  $F(s)$  can be denoted by  $M^{-1}\{F(s)\} = f(u)$  or by equation [9, p. 60],

$$(3) \quad f(u) = \frac{1}{2\pi i} \int_C F(s) u^{-s} ds,$$

where  $F(s) \in L_2(\sigma - i\tau, \sigma + i\tau)$ ,  $-\infty < \tau < \infty$ , and  $C$  is a suitable contour in the complex  $s$ -plane.

If  $M\{h(u)\} = H(s)$ , and  $M\{f(u)\} = F(s)$ , then the Mellin-Parseval theorem [9, p. 94] states that

$$(4) \quad \int_0^\infty h(u)f(u)du = \frac{1}{2\pi i} \int_C H(s)F(1-s)ds,$$

where  $C$  is a suitable contour in the  $s$ -plane.

An attempt has been made in the present paper to obtain three inversion formulae by a systematic use of  $\mathcal{L}$  and  $\mathcal{L}^{-1}$  operators. As the tables of Laplace transform [3, 4] are available this methods seems to be more simple and convenient. Some particular cases are mentioned and the solutions obtained are illustrated by simple examples.

The operators of fractional integration play an important role in the inversion of hypergeometric transforms, in this connection the work of Erdélyi [2] is worth mentioning here, who developed a general theory. Here we have mentioned corresponding inversion formulae deduced from Erdélyi's results [2].

We shall make use of following results [3, p. 337, (14), (9), (19)] in our investigations

$$(5) \quad M\{(tu)^{\lambda-1/2}e^{-(1/2)tu}W_{k,m}(tu)\} = t^{-s} \frac{\Gamma(\lambda+m+s)\Gamma(\lambda-m+s)}{\Gamma(\lambda-k+1/2+s)},$$

$$R(\lambda \pm m + s) > 0, t > 0,$$

where we considered that  $M$  acts upon a function of  $u$ .

$$(6) \quad M\{e^{-ux}{}_1F_1(\gamma-\alpha; \gamma; ux)\} = u^{-s} \frac{\Gamma(s)\Gamma(\gamma)\Gamma(\alpha-s)}{\Gamma(\alpha)\Gamma(\gamma-s)},$$

$$R(u) > 0, R(\alpha) > R(s) > 0, \gamma \neq 0, -1, -2, \dots$$

$$(7) \quad M\left\{H_{22}^{12} \left[ \frac{1}{u} x \begin{matrix} (1-\alpha, 1), (1-\beta, 1) \\ (0, 1), (1-\gamma, 1) \end{matrix} \right] \right\} = u^s \frac{\Gamma(s)\Gamma(\alpha-s)\Gamma(\beta-s)}{\Gamma(\gamma-s)},$$

$$R\{s+1+\min(0, 1-\gamma)\} > 0$$

$$R\{\max(-\alpha, -\beta)+s+1\} < 0.$$

Our analysis of the present paper is based on the following interesting theorem due to C. Fox [5, p. 300], which describes annihilating power of  $\mathcal{L}^{-1}$ .

If: (i)  $\alpha > 0, (1/2)\alpha + \beta > 0, t > 0$ ;

(ii)  $s = \sigma + i\mu, \sigma$  and  $\mu$  both real

$$F(s) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$$

then

$$(8) \quad \mathcal{L}^{-1}\left\{\frac{1}{2\pi i} \int_C \Gamma(\alpha s + \beta)F(s)t^{-\alpha s - \beta} ds\right\} = \frac{1}{2\pi i} \int_C F(s)x^{\alpha s + \beta - 1} ds,$$

where, for both integrals, the contour  $C$  may be the line  $\sigma = 1/2$  a line parallel to the imaginary axis in the complex  $s$ -plane. And

$$(9) \quad \mathcal{L} \left\{ \frac{1}{2\pi i} \int_C \frac{H(s)}{\Gamma(\alpha s + \beta)} x^{\alpha s + \beta - 1} ds \right\} = \frac{1}{2\pi i} \int_C H(s) \cdot t^{-\alpha s - \beta} ds .$$

**2. Inversion formula for the transform whose kernel is the Whittaker's function.** Mainra [7] gives a generalization of the Laplace transform (1) as

$$(10) \quad g(t) = \int_0^\infty [(tu)^{\lambda - 1/2} e^{-(1/2)tu} W_{k,m}(tu)] f(u) du ,$$

where  $W_{k,m}(tu)$  is Whittaker's confluent hypergeometric function [11, p. 334].

In this section we shall obtain an inversion formula for the transform (10).

**THEOREM 1.** *If*

- (i)  $(\lambda \pm m) > 0, (\lambda - k + 1) > 0,$
- (ii)  $f(x) \in L_2(0, \infty),$  then  $g(t)$  is defined and also belongs to  $L_2(0, \infty),$
- (iii)  $s^{\lambda + k - 1} F(1 - s) \in L_2(1/2 - i\infty, 1/2 + i\infty),$   
 $s^{m + k - 1} F(1 - s) \in L_2(1/2 - i\infty, 1/2 + i\infty),$
- (iv)  $F(1 - s) \in L_2(1/2 - i\infty, 1/2 + i\infty),$
- (v)  $y^{-1/2} f(y) \in L_2(0, \infty),$

where  $f(y)$  is of bounded variation near the point  $y = x,$  then the inversion formula for the transform (10) is

$$(11) \quad f(x) = x^{-(\lambda + m)} \mathcal{L}^{-1} \{ t^{-m - k + 1/2} \mathcal{L} [ x^{\lambda - k + 1/2 - (\lambda - m)} \mathcal{L}^{-1} \{ t^{-(\lambda - m)} g(t) \} ] \} .$$

**PROOF.** We first apply (4) to the right hand of (10). For large positive  $u$  and  $t > 0$  the kernel of our integral equation behaves as [11, p. 336],

$$(12) \quad (tu)^{\lambda - 1/2} e^{-(1/2)tu} W_{k,m}(tu) = (tu)^{\lambda + k - 1/2} e^{-tu} \{ 1 + O(u^{-1}) \} .$$

For small positive  $u$  we express the  $W_{k,m}(u)$  in terms of the  $M_{k,m}(u)$  and  $M_{k,-m}(u)$  series [11, p. 340], where the two  $M$  series are defined in [10, p. 332]. We then have for  $t > 0, u$  positive and small

$$(13) \quad (tu)^{\lambda - 1/2} e^{-(1/2)tu} W_{k,m}(tu) = (tu)^{\lambda \pm m} \{ 1 + O(u) \} ,$$

since condition  $(\lambda \pm m) > 0$  it follows from (12) and (13), that the expression inside the square braces in (10) belongs to  $L_2(0, \infty)$  when considered as a function of  $u.$  Further we have that  $f(u) \in L_2(0, \infty)$  and hence we can apply (4) to the righthand side of (10) by virtue of the result [9, p. 95, Theorem 72]. Using (5) we have

$$(14) \quad g(t) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(m+\lambda+s)\Gamma(\lambda-m+s)}{\Gamma(\lambda-k+1/2+s)} t^{-s} F(1-s) ds, \quad t > 0$$

where the contour in the  $s$ -plane is the straight line  $s = 1/2 + i\tau$  and  $\tau$  varies from  $-\infty$  to  $\infty$ .

Following the work of Fox [5], we shall now try to eliminate the three gamma function occurring in the integrand.

From the asymptotic expansion of the gamma function [11, p. 273] along the line  $s = 1/2 + i\tau$ , for large  $|\tau|$ , we have

$$(15) \quad \frac{\Gamma(\lambda+m+s)\Gamma(\lambda-m+s)}{\Gamma(\lambda-k+1/2+s)} F(1-s) = s^{\lambda+k-1} F(1-s) \{1 + O(s^{-1})\}$$

and

$$(16) \quad \frac{\Gamma(\lambda+m+s)}{\Gamma(\lambda-k+1/2+s)} F(1-s) = s^{m+k-1} F(1-s) \{1 + O(s^{-1})\}.$$

Multiplying both sides of (14) by  $t^{-\lambda+m}$  and then operating  $\mathcal{L}^{-1}$ , we have

$$(17) \quad \begin{aligned} & \mathcal{L}^{-1}\{t^{-\lambda+m}g(t)\} \\ &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s) \frac{\Gamma(m+\lambda+s)}{\Gamma(\lambda-k+1/2+s)} x^{s+(\lambda-m)-1} ds. \end{aligned}$$

Now to eliminate  $\Gamma(\lambda-k+1/2+s)$ , we multiply (17) by  $x^{\lambda-k+1/2-(\lambda-m)}$ , then by means of the  $\mathcal{L}$  operator we have

$$(18) \quad \begin{aligned} & \mathcal{L}[x^{\lambda-k+1/2-(\lambda-m)} \mathcal{L}^{-1}\{t^{-(\lambda-m)}g(t)\}] \\ &= \int_0^\infty e^{-xt} x^{\lambda-k+1/2-(\lambda-m)} \left\{ \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(\lambda+m+s)}{\Gamma(\lambda-k+1/2+s)} x^{s+(\lambda-m)-1} F(1-s) ds \right\} dx. \end{aligned}$$

The integral involved in (18) are absolutely convergent, as  $s = 1/2 + i\tau$ , the real part of the power of  $x$  in (18) is  $(\lambda - k + 1) > 0$  and also by (16)  $s^{m+k-1/2} F(1-s) \in L_2(1/2 - i\infty, 1/2 + i\infty)$ .

Now we integrate the  $x$ -integral, which results to the elimination of the factor  $\Gamma(\lambda - k + 1/2 + s)$  from the denominator, to give

$$(19) \quad \begin{aligned} & \mathcal{L}[x^{\lambda-k+1/2-(\lambda-m)} \mathcal{L}^{-1}\{t^{-(\lambda-m)}g(t)\}] \\ &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s) \Gamma(m+\lambda+s) t^{-s-(\lambda-k+1/2)} ds. \end{aligned}$$

Applying inverse Laplace operator (8) to (19), we obtain

$$(20) \quad \begin{aligned} & \mathcal{L}^{-1}\{t^{-m-k+1/2} \mathcal{L}[x^{\lambda-k+1/2-(\lambda-m)} \mathcal{L}^{-1}\{t^{-(\lambda-m)}g(t)\}]\} \\ &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s) x^{(s+\lambda+m)-1} ds, \end{aligned}$$

on replacing  $s$  by  $1 - s$  in (20) we have

$$\begin{aligned} & \mathcal{L}^{-1}\{t^{-m-k+1/2} \mathcal{L}[x^{\lambda-k+1/2-(\lambda-m)} \mathcal{L}^{-1}\{t^{-(\lambda-m)}g(t)\}]\} \\ &= x^{\lambda+m} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)x^{-s}ds = x^{\lambda+m}f(x), \end{aligned}$$

by an appeal to the theorem [9, p. 46, Theorem 28], the application of which is justified by the condition (v) stated above.

If we simplify the triple integrals in (11), we are lead to

$$\begin{aligned} f(u) &= \frac{\Gamma(m - k + 3/2)}{\Gamma(2m + 1)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (tu)^{m-\lambda} \\ &\times {}_1F_1(m - k + 3/2; 2m + 1, tu)g(t)dt, \quad (c > 0) \end{aligned}$$

a result obtained by Erdélyi [2].

Particular Cases: If we set  $\lambda = m$  in (10) then the Whittaker function transform reduces to the Varma transform [10] and consequently our theorem reduces to the one, considered recently by Fox [6, p. 195].

EXAMPLE. Now we consider a simple example to illustrate (11) as a solution of the integral equation (10).

Let

$$f(u) = u^{k-\lambda-1-1/2}e^{-1/u}.$$

Evaluating the integral equation (10) by [3, p. 217], we obtain  $g(t) = 2t^\lambda K_{2m}(2t^{1/2})$ , with this value of  $g(t)$  we apply (11). From [3, p. 283; (40)], we obtained  $f(x)$ .

**3. Inversion formula for the transform whose kernel is the Gauss hypergeometric function.** Rajendra Swaroop [8] utilized Gauss hypergeometric function  ${}_2F_1$  [1] to define the transform

$$(21) \quad g(u) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} \int_0^\infty {}_2F_1\left(\alpha, \beta; \gamma; -\frac{1}{u}x\right)f(x)dx,$$

which is a generalization of Laplace and Stieltjes transform. He calls it as Gauss-hypergeometric transform.

In this section we shall obtain an inversion formula for the transform (21).

**THEOREM 2.** *If*

- (1)  $\alpha, \beta > 1, (\gamma - 1/2) > 0,$
- (2)  $f(x) \in L_2(0, \infty),$  then  $g(u)$  is defined and also belongs to  $L_2(0, \infty),$
- (3)  $s^{\alpha+\beta-\gamma-1/2}F(1 - s) \in L_2(1/2 - i\infty, 1/2 + i\infty),$
- (4)  $F(1 - s) \in L_2(1/2 - i\infty, 1/2 + i\infty),$

(5)  $y^{1/2}f(y) \in L_2(0, \infty)$

where  $f(y)$  is bounded variation near the point  $y = x$  then the inversion formula for the transform (21) is

(22)  $\mathcal{L}^{-1}\{t^{\alpha-1-\beta}[\mathcal{L}^{-1}\{t^{1-\alpha}[\mathcal{L}^{-1}\{\tau^{-\gamma}[\mathcal{L}\{u^{\gamma-1}g(u^{-1})\}]_{t=\tau^{-1}}\}]_{x=i}\}]_{x=t^{-1}}\} = x^{\beta-2}f(x^{-1}) .$

PROOF. We first apply (4) to the right-hand side of (21) and from (7) we have

(23)  $g(u) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s)u^s \frac{\Gamma(s)\Gamma(\alpha-s)\Gamma(\beta-s)}{\Gamma(\gamma-s)} ds .$

Now following the same procedure as we have done in previous case, we eliminate the four gamma function factor from the integrand, then we have

(24)  $\mathcal{L}^{-1}\{t^{\alpha-1-\beta}[\mathcal{L}^{-1}\{t^{1-\alpha}[\mathcal{L}^{-1}\{\tau^{-\gamma}[\mathcal{L}\{u^{\gamma-1}g(u^{-1})\}]_{t=\tau^{-1}}\}]_{x=i}\}]_{x=t^{-1}}\} = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s)x^{-s+\beta-1}ds ,$

on replacing  $s$  by  $1-s$  in (24)

$$\begin{aligned} &= x^{\beta-2} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)(x^{-1})^{-s} ds \\ &= x^{\beta-2}f(x^{-1}) , \end{aligned}$$

which completes the proof of our theorem.

As we have [2]

$$\frac{\Gamma(\beta+1)}{\Gamma(\gamma)} {}_2F_1(\alpha, \beta; \gamma; -ux) = I_u[(1+ux)^{-\alpha}; 1, \gamma-\beta-1, \beta] ,$$

therefore, if we use systematically the operators  $I$  and  $K$  of Erdélyi [2], we may obtain an inversion formula for the transform (21) as

$$\begin{aligned} f(x) &= \frac{\beta\Gamma(\gamma-\alpha+1)}{\Gamma(\alpha-1)\Gamma(\beta-\alpha+2)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (xy)^{\alpha-2} \\ &\times {}_2F_1(2-\alpha, \gamma-\alpha+1; \beta-\alpha+2; 1/(xy))g(1/y)dy, (c > 0) . \end{aligned}$$

EXAMPLE. To illustrate (22), we set

$$f(x) = {}_2F_1(a, b; c; -x) ,$$

then evaluating the integral (21) by [4, p. 422, (14)] we obtain

$$g(u) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)} uG_{44}^{33}[u \mid \begin{smallmatrix} 0, 1-a, 1-b, \gamma-1 \\ \alpha-1, \beta-1, 0, 1-c \end{smallmatrix}] ,$$

replacing this value of  $g(u)$  in the left-hand side of (22) and using [3, p. 222, (34)] we obtained  $x^{\beta-2}f(x^{-1})$ .

4. **Inversion formula for the transform whose kernel is the confluent hypergeometric function.** By using the confluent hypergeometric function  ${}_1F_1$  [1] as kernel, Erdélyi [2] gave a generalization of the Laplace transform as

$$(25) \quad g(u) = \int_0^\infty {}_1F_1(\alpha; \gamma; -ux)f(x)dx .$$

In this section we shall obtain an inversion formula for the transform (25).

**THEOREM 3.** *If*

- (1)  $\alpha > 1, (\gamma - 1/2) > 0,$
- (2)  $f(x) \in L_2(0, \infty),$  then  $g(u)$  is defined and also belongs to  $L_2(0, \infty),$
- (3)  $s^{\alpha-\gamma-1}F(1-s) \in L_2(1/2-i\infty, 1/2+i\infty),$
- (4)  $F(1-s) \in L_2(1/2-i\infty, 1/2+i\infty),$
- (5)  $y^{1/2}f(y) \in L_2(0, \infty),$

where  $f(y)$  is of bounded variation near the point  $y = x,$  then the inversion formula for the transform (25) is

$$(26) \quad \begin{aligned} &\mathcal{L}^{-1}\{t^{-\alpha+1}[\mathcal{L}^{-1}\{\tau^{-\gamma}[\mathcal{L}\{u^{\gamma-1}g(u)\}]_{t=\tau^{-1}}\}]_{x=t}\} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} x^{\alpha-2}f(x^{-1}) . \end{aligned}$$

**PROOF.** From Kummer's relations [1] we have

$${}_1F_1(\alpha; \gamma; -ux) = e^{-ux}{}_1F_1(\gamma - \alpha; \gamma; ux) .$$

Then the equation (25) is

$$(27) \quad g(u) = \int_0^\infty [e^{-ux}{}_1F_1(\gamma - \alpha; \gamma; ux)]f(x)dx ,$$

we first apply (4) to the right-hand side of (27) and from (6) we have

$$(28) \quad g(u) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s) \frac{\Gamma(s)\Gamma(\alpha-s)}{\Gamma(\gamma-s)} u^{-s} ds .$$

Using the same procedure as we have done previously, on applying  $\mathcal{L}$  and  $\mathcal{L}^{-1}$  we obtain

$$(29) \quad \begin{aligned} &\mathcal{L}^{-1}\{t^{-\alpha+1}[\mathcal{L}^{-1}\{\tau^{-\gamma}[\mathcal{L}\{u^{\gamma-1}g(u)\}]_{t=\tau^{-1}}\}]_{x=t}\} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(1-s)x^{-s+\alpha-1}ds , \end{aligned}$$

on replacing  $s$  by  $1-s$  in (29), we have

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} x^{\alpha-2} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)(x^{-1})^{-s} ds$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\gamma)} x^{\alpha-2} f(x^{-1}).$$

By an appeal to the Erdélyi's operators  $\mathfrak{S}$  [2], we can obtain an another inversion formula for the transform (25) as

$$f(x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (-xu)^{(\gamma/2)-1} \\ \times e^{xu/2} W_{(\gamma/2)-\alpha+1, (\gamma-1)/2}(-xu) g(u) du, \quad (c > 0).$$

As Erdélyi pointed out, given a hypergeometric transform, the inversion formulae are not necessarily unique.

EXAMPLE. To illustrate (26) we set

$$F(x) = {}_1F_1(\alpha; \gamma; -ux),$$

then evaluating the integral (27) by [4, p. 422, (14)] we obtain

$$g(u) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \frac{\Gamma(c)}{\Gamma(a)} u^{-1} G_{33}^{20} \left[ \frac{1}{u} \begin{matrix} 0, 1-a, -1+\gamma \\ -1+\alpha, 0, 1-c \end{matrix} \right]$$

replacing this value of  $g(u)$  to the left-hand side of (26) and using [3, p. 222, (34)], we obtained  $(\Gamma(\gamma)/\Gamma(\alpha)) x^{\alpha-2} f(x^{-1})$ .

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