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APPROXIMATION OPERATORS ON BANACH SPACES OF DISTRIBUTIONS

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Abstract. An approximation process $\{\Gamma_n\}_{n\in P}$ on a Banach subspace X of $\mathscr{A}'$ [Zemanian A. H. [36]], satisfying either a Jackson type inequality or a Bernstein type inequality of order $\rho(n)$ on X with respect to Y of X, is being related to a class of Banach subspaces $\{X_\lambda\}_{\lambda\in J}$ of $\mathscr{A}'$, on each of which, $\{\Gamma_n\}_{n\in P}$ defines a sequence of multiplier type operators, satisfying the same inequality with same order. Sufficient conditions for $X_\lambda\subset\mathscr{A}'$, $\lambda\in J$ are given. Results are illustrated by examples.

1. **Introduction.** For a Banach space X, a sequence $\{\Gamma_n\}_{n\in P}$ of bounded linear operators $\Gamma_n: X\to X$, with $P=\{1,2,3,\cdots\}$ is called an approximation process on X, if $\Gamma_n f\to f$ in X $\forall f\in X$. For suitable subspaces Y, Λ of X (Λ being fixed, $\dim(\Lambda)<\infty$) and function $\rho(n)\geqq 0$, $\rho(n)\searrow 0$ on P, an approximation process $\{\Gamma_n\}$ on X is said to,

(I) satisfy a Jackson-type inequality of order $\rho(n)$ on X with respect to Y, if $\forall f\in Y$, $\|\Gamma_n f - f\|_X \leqq C\rho(n)\|f\|_Y$;

(II) satisfy a Bernstein type inequality of order $\rho(n)$ on X with respect to Y, if $\bigcup_{n\in P}\Gamma_n(X)\subset Y$ and $\forall f\in X$, $\|\Gamma_n f\|_Y \leqq C_1(\rho(n))^{-1}\|f\|_X$. ($C$, C_1 constants independent of n);

(III) be saturated with order $\rho(n)$ on X with saturation class Y, if for $f\in X$, $\|\Gamma_n f-f\|_X=\begin{cases} o(\rho(n))\Leftrightarrow f\in\Lambda \\ O(\rho(n))\Leftrightarrow f\in Y, Y-\Lambda\neq\varnothing.\end{cases}$

For such $\{\Gamma_n\}$ as in (III) above, the inverse problem is the characterization of elements of the sets

$$\{f\in X \mid \|\Gamma_n f-f\|_X=O(\eta(n))\} \text{ with some } \eta(n)\geqq 0,\ \eta(n)\searrow 0,\ \frac{\rho(n)}{\eta(n)}\to 0 \text{ as } n\to\infty.$$

Given a Banach subspace X of a certain space $\mathscr{A}'$ of generalized functions, each $f\in\mathscr{A}'$ having Fourier expansion with respect to an orthonormal system $\{\psi_n\}_{n\in N}$ ($N=0,1,2,3,\cdots$) and given an approximation process $\{\Gamma_n\}_{n\in P}$ related to $\{\psi_n\}_{n\in N}$ on X, satisfying (J) Jackson-type inequality or (B) Bernstein-type inequality or for X, having (S) saturation and inverse theorems, the aim of this paper is to determine a family of related

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Banach subspaces of $\mathscr{A}'$, on each of which $\{\Gamma_n\}_{n\in P}$ satisfy the above (i.e (J) or (B) or (S)).

Let I be an open interval of $\boldsymbol{R}$. Let $\mathscr{U}$ be a self-adjoint differential operator of the form $\mathscr{U} = \theta_0 D^{n_1}\theta_1 D^{n_2}\cdots D^{n_\nu}\theta_\nu = \bar{\theta}_\nu(-D)^{n_\nu}\cdots\bar{\theta}_1(-D)^{n_1}\bar{\theta}_0$ $\theta_i\in C^\infty(I)$, $n_i\in P$, $1\leqq i\leqq\nu$, with discrete spectrum, with $\{\psi_n\}_{n\in N}$ a sequence of orthonormal C^∞-functions on I, as eigenfunctions, corresponding to eigenvalues $\{\lambda_n\}_{n=0}^\infty$. Let $|\lambda_n|\uparrow\infty$ as $n\to\infty$. Let $\mathscr{A}$ be the space of test functions, $\mathscr{A}'$ = dual of $\mathscr{A}$, be as constructed by Zemanian [[36], [37], Chap. IX]. $\forall f\in\mathscr{A}'$, f has Fourier expansion $f\sim\sum_{k=0}^\infty\langle f,\psi_k\rangle\psi_k$ such that $\sum_{k=0}^n\langle f,\psi_k\rangle\psi_k\to f$ in $\mathscr{A}'$ as $n\to\infty$. There exists only finite number of $i_k\in N$, $0\leqq k\leqq l$ such that $\lambda_{i_k}=0$, $0\leqq k\leqq l$. Let Λ = span of $\{\psi_{i_k}\,|\,0\leqq k\leqq l\}$. Let us call Λ the trivial class. Let $[\{\psi_n\}]$ = span of $\{\psi_n\}_{n\in N}$.

The main results are presented as follows: Given a Banach subspace X of $\mathscr{A}'$, with X^* denoting the dual of X, a family of related Banach subspaces $\{X_\lambda\}_{\lambda\in J}$, J being a parameter set, is constructed so that (i) every multiplier type operator related to $\{\psi_n\}$ on X, defines a similar operator on each X_λ, $\lambda\in J$; (ii) every approximation process on X satisfying Jackson-type inequality or Bernstein type inequality with certain order on X with respect to a subspace Y of X, also satisfies the same inequalities with the same order on each X_λ, with respect to suitable subspace Y_λ of X_λ, $\lambda\in J$. Sufficient conditions for each X_λ to be subspace of $\mathscr{A}'$, $\lambda\in J$ are given in terms of estimates of ψ_n, $\left(\frac{d}{dx}\right)^k\psi_n$, $n,k\in N$ in the norm of $X\cap X^*$. Using these results and those of Butzer-Scherer [17, 18], Trebels [30] both saturation and inverse problems are studied for various approximation processes related to $\{\psi_n\}_{n\in N}$ on each X_λ, $\lambda\in J$. Finally, these results are illustrated by means of classical orthonormal systems, like Hermite, Laguerre or Jacobi functions.

As an illustration we cite the following example. Let $I=(-\infty,\infty)$. Let $\mathscr{U} = -e^{x^2/2}De^{-x^2}De^{x^2/2}$, $D=\frac{d}{dx}$, $\psi_n(x)=\frac{e^{-x^2/2}H_n(x)}{[2^n n!\,\pi^{1/2}]^{1/2}}$, where $H_n(x)$ are Hermite polynomials. Let $X=L^p(-\infty,\infty)$ for some $p\in(1,\infty)$. Here, $\lambda_n=2n$, $\lambda_0=0$, $\Lambda=\{de^{-x^2/2}\,|\,d\in\boldsymbol{R}\}$. $\mathscr{A}=\mathscr{S}$, $\mathscr{A}'=\mathscr{S}'$ [37]. Let $\forall n\in P$, $\{\gamma_{n,k}\}_{n\in P}$ be real sequence with $\gamma_{n,k}=O(k^{q_n})$ for some $q_n\in P$. For $f\in\mathscr{S}'(R)$, let $\Gamma_n f=\sum_{k=0}^\infty\gamma_{n,k}\langle f,\psi_k\rangle\psi_k$ $(n\in P)$. Then $\Gamma_n f\in\mathscr{S}'$ for all $n\in P$. For $\beta>0$:

(1) *If* $\{\Gamma_n\}_{n\in P}\subset[L^p]$, *then* $\{\Gamma_n\}_{n\in P}\subset[Z]$.

(2) *If* $\{\Gamma_n\}_{n\in P}\subset[L^p]$ *and* $\forall f\in L^p_\beta=\left\{f\in L^p\,|\,g\sim\sum_{k=0}^\infty k^\beta\langle f,\psi_k\rangle\psi_k\in L^p\right\}$, $\|\Gamma_n f-f\|_{L^p}=O(n^{-\beta})$, *then* $\forall f\in Z_\beta=\left\{f\in Z\,|\,g\sim\sum_{k=0}^\infty k^\beta\langle f,\psi_k\rangle\psi_k\in Z\right\}$, $\|\Gamma_n f-f\|_Z=O(n^{-\beta})$;

(3) *If* $\{\Gamma_n\}\subset[L^p]$, $\bigcup_{n\in P}\Gamma_n(L^p)\subset L^p_\beta$ *and* $\|\Gamma_n f\|_{L^p_\beta}\leqq cn^\beta\|f\|_{L^p}$ $\forall f\in L^p$, *then* $\bigcup_{n\in P}\Gamma_n(Z)\subset Z_\beta$ *and* $\|\Gamma_n f\|_{Z_\beta}\leqq C_1 n^\beta\|f\|_Z$ $\forall f\in Z$, *where* Z denotes any one of the following spaces: $H^{q,m}(\boldsymbol{R})$, $H^{q,-m}(\boldsymbol{R})$, $(H^{q,-m}(\boldsymbol{R}), H^{q,m}(\boldsymbol{R}))_{\theta,q_1}$, $0<\theta<1$, $1\leqq q\leqq\infty$, $p\leqq q\leqq p'$, $m\in P$. For definition of these spaces, the reader is referred to [31, Chapter 31] [13, p. 167]. The intermediate spaces constructed by the K-method of J. Peetre [13, p. 167] are defined as follows: Let X, Y be Banach subspaces of $\mathscr{D}'(I)$-the space of Schwartz distributions on I. Let $X+Y=\{f_1+f_2\,|\,f_1\in X, f_2\in Y\}$ with norm $\|f\|_{X+Y}=\inf\{\|f_1\|_X+\|f_2\|_Y\,|\,f_1\in X, f_2\in Y, f=f_1+f_2\}$ $(f\in X+Y)$. For $f\in X+Y$, $0<t<\infty$, let $K(t,f,X,Y)=\inf\{\|f_1\|_X+t\|f_2\|_Y\,|\,f=f_1+f_2,\ f_1\in X,\ f_2\in Y\}$,

$$(X,Y)_{\theta,q}=\begin{cases}\left\{f\in X+Y\,\middle|\,\|f\|_{\theta,q}=\left[\int_0^\infty[t^{-\theta}K(t,f,X,Y)]^q\dfrac{dt}{t}\right]^{1/q}<\infty\right\} & \text{if } 1\leqq q<\infty,\ 0<\theta<1\,,\\ \{f\in X+Y\,|\,\|f\|_{\theta,\infty}=[\sup_{0<t<\infty}t^{-\theta}K(t,f,X,Y)]<\infty\} & \text{if } q=\infty,\ 0\leqq\theta\leqq1\,.\end{cases}$$

The spaces of Bessel potentials $H^{p,m}$, and its dual $H^{p',-m}(\boldsymbol{R})$ are special cases of the following spaces defined as follows. For $m\in P$, and for a Banach subspace X of $\mathscr{D}'(I)$, let

$$W^{-m}(X)=\left\{f\in\mathscr{S}'(I)\,\middle|\,f=\sum_{\alpha=0}^m D^\alpha f_\alpha;\ f_0,f_1,\cdots,f_m\in X\right\}$$

with

$$\|f\|_{W^{-m}(X)}=\inf\left\{\sum_{\alpha=0}^m\|f_\alpha\|_X\,\middle|\,f=\sum_\alpha D^\alpha f_\alpha,\ f_\alpha\in X,\ 0\leqq\alpha\leqq m\right\}\ (f\in W^{-m}(X))\,.$$

Here $D^\alpha f$ denotes the distributional derivative of f of order α, $\alpha\in P$. Let $W^m(X)=\{f\in X\,|\,D^\alpha f\in X, 0\leqq\alpha\leqq m\}$. For $f\in W^m(X)$, $\|f\|_{W^m(X)}=\sum_{\alpha=0}^m\|D^\alpha f\|_X$. $W^{m,0}(X)=$ closure of $\mathscr{D}(I)$ in $W^m(X)$, where $\mathscr{D}(I)=\{f\in C^\infty(I)\,|\,\operatorname{supp}f$ is compact$\}$. $[W^{m,0}(L^p(\boldsymbol{R}^n))\cong H^{p,m}(\boldsymbol{R}^n)\cong W^m(L^p(\boldsymbol{R}^n))$; $W^{-m}(L^{p'}(\boldsymbol{R}^n))\cong H^{p',-m}\cong$ dual of $H^{p,m}(\boldsymbol{R}^n)]$.

In a series of papers by Favard [[19], [20]], Sunouchi and Watari [28], Aljancic [[1], [2], [3]], and Buchwalter [10], saturation behaviour of various approximation processes related to Trigonometric polynomials on $C(-\pi,\pi)$, $L^p(-\pi,\pi)$ $1\leqq p<\infty$ had been studied. Buchwalter [9] studied the same problem on a normed linear space for various approximation processes related to a biorthogonal system. Bavinck [6] studied both saturation and inverse problems of various approximation processes on $L^p(\mu)$ $1\leqq p<\infty$, $C(-1,1)$, where $d\mu(x)=(1-x)^\alpha(1+x)^\beta dx$, $x\in(-1,1)$, $\alpha>-1$, $\beta>-1$, related to Jacobi polynomials using the convolution structre for Jacobi series, introduced by Askey and Waigner [5]. Recently in a series of papers by P. L. Butzer and his colleagues [[16], [21]], both saturation and inverse problems related to classical orthogonal polynomials were investigated on $L^p(\mu)$ $1 \leqq p < \infty$ where $d\mu(x) = w(x)dx$, $w(x) \geqq 0$, $x \in (a, b)$, $-\infty \leqq a < b \leqq \infty$.

2. Definitions and Notations. In order to present the main results of this paper, we need to define certain spaces as follows. For Banach subspaces X, Y of $\mathscr{A}'$, let $[X, Y]$ = the space of bounded linear operators from X to Y. For $X \subset Y$, let $\mathrm{Cl}(X, Y)$ denote the closure of X in the topology of Y. Let $M(X, Y)$ denote the space of all real sequences $\{\gamma_k\}$ such that for some $\Gamma \in [X, Y]$, $\Gamma f \sim \sum_{k=0}^{\infty} \gamma_k \langle f, \psi_k \rangle \psi_k$, $(f \in X)$ with a norm $\|\{\gamma_k\}\|_{M(X,Y)} = \|\Gamma\|_{[X,Y]}$.

$$UM(X, Y) = \left\{ \{\gamma_{\tau,k}\}_{k \in N, \tau \in \Omega} \left| \begin{array}{l} \Omega \text{ a parameter set and} \\ \forall \tau \in \Omega,\ \{\gamma_{\tau,k}\}_{k \in N} \in M(X, Y) \text{ defining} \\ \Gamma_\tau \in [X, Y] \text{ with } \sup_{\tau \in \Omega} \|\Gamma_\tau\| < \infty \end{array} \right. \right\}.$$

For $f \in \mathscr{A}'$, $\forall \delta > 0$, an element $\mathscr{U}^\delta f$ of $\mathscr{A}'$ can be defined as follows: $\langle \mathscr{U}^\delta f, \psi_k \rangle = \lambda_k^\delta \langle f, \psi_k \rangle$ $(k \in N)$. $\mathscr{U}^\delta f$ is well defined by completeness of $\{\psi_n\}$ on $\mathscr{A}'$ and by Theorems 9.5.2, 9.6.1 of [[37], p. 260–261]. For a Banach subspace X of $\mathscr{A}'$ and for $\delta > 0$, $X_\delta = \{f \in X \mid \mathscr{U}^\delta f \in X\}$ with norm $\|f\|_{X_\delta} = \|f\|_X + \|\mathscr{U}^\delta f\|_X$ $(f \in X_\delta)$; $X_{-\delta} = \{f \in \mathscr{A}' \mid f = f_0 + \mathscr{U}^\delta f_1;\ f_0, f_1 \in X\}$. For $f \in X_{-\delta}$, $\|f\|_{X_{-\delta}} = \inf\{\|f_0\|_X + \|f_1\|_X \mid f = f_0 + \mathscr{U}^\delta f_1;\ f_0, f_1 \in X\}$. For $\delta > 0$ let $\nu_{k,\delta} = \begin{cases} \lambda_k^{-\delta} & \text{if } \lambda_k \neq 0,\ k \in N \\ 0 & \text{if } \lambda_k = 0,\ k \in N \end{cases}$. For each $f \in \mathscr{A}'$, an element $G_\delta f$ of $\mathscr{A}'$ can be defined as $\langle G_\delta f, \psi_k \rangle = \nu_{k,\delta} \langle f, \psi_k \rangle$ $(k \in N)$. $\forall \phi \in \mathscr{A}$, $\forall \delta > 0$, $\mathscr{U}^\delta \phi \in \mathscr{A}$, $G_\delta \phi \in \mathscr{A}$ [Ref. Lemma 9.3.3, Theorem 9.6.1, [37]].

3. Main Results. First, we need to choose suitably, Banach subspace X of $\mathscr{A}'$, from which, we like to extend Jackson or Bernstein type inequalities satisfied by approximation processes, to various other related Banach subspaces of $\mathscr{A}'$. For this we need the notion of families $\mathscr{F}(m)$, $\mathscr{F}(m, \delta)$ of Banach spaces. Let $m \in P$, m be *fixed* throughout the rest of the paper.

DEFINITION 3.1. A Banach space $Z \in \mathscr{F}(m)$ if (1) $\mathrm{Cl}(\mathscr{D}(I), Z) = Z \subset \mathscr{A}'$, (2) $\mathscr{A} \subset W^{m,0}(Z) \cap Z^*$, (3) $W^{-m}(Z + Z^*) \subset \mathscr{A}'$, (4) $\forall \delta > 0$ $\{\nu_{k,\delta}\}_{k \in N} \in M(Z)$ defining $G_\delta \in [Z]$.

DEFINITION 3.2. For $\delta > 0$, a space $Z \in \mathscr{F}(m, \delta)$ if (1) $Z \in \mathscr{F}(m)$, (2) $\mathrm{Cl}(\mathscr{D}(I), Z^*) = Z^*$, $\forall f \in Z^*_{-\delta} + Z_{-\delta}$, $D^k f \in \mathscr{A}'$ $0 \leqq k \leqq m$.

The families of related Banach subspaces of $\mathscr{A}'$ can be given as follows:

DEFINITION 3.3. Let $\delta > 0$, $X \in \mathscr{F}(m, \delta)$ be reflexive. Then $Y(m, \delta, X)$ be the family consisting of the following spaces:

$$Y = (\text{any one of } X,\ X^*,\ (X, X^*)_{\theta_1, q_1}\ 0 < \theta_1 < 1,\ 1 < q_1 < \infty),\ Y_{-\delta}\ ,$$
$$\left.\begin{array}{l} W^{-m}(Y),\ W^{m,0}(Y),\ (W^{-m}(Y), W^{m,0}(Y))_{\theta,q} \\ W^{-m}(Y_{-\delta}),\ W^{m,0}(Y_{-\delta}),\ (W^{-m}(Y_{-\delta}), W^{m,0}(Y_{-\delta}))_{\theta,q} \end{array}\right\} \begin{array}{l} 0 < \theta < 1 \\ 1 \leqq q \leqq \infty\ . \end{array}$$

DEFINITION 3.4. A space $X \in Q(m)$, if $X \in \mathscr{F}(m)$ and there exists $X' \in \mathscr{F}(m)$ with $X \subset (X')^*$, $X' \subset X^*$, on X $\|\ \|_X = \|\ \|_{(X')^*}$; on X' $\|\ \|_{X'} = \|\ \|_{X^*}$. For $\delta > 0$, $X \in Q(m)$, let $Q(m, \delta, X)$ be the family consisting of the following spaces: $E_1(=$ any one of X, X', $(X, X')_{\theta,q}$, $0 < \theta < 1, 1 \leqq q < \infty)$, $W^{-m}(E_1)$, $(W^{-m}(E_1), E_1)_{\theta,q}$, $0 < \theta < 1, 1 \leqq q \leqq \infty$; $E_2(=$ any one of $X^*_{-\delta}$, $(X')^*_{-\delta}$, $(X^*_{-\delta}, (X')^*_{-\delta})_{\theta,q}$, $0 < \theta < 1, 1 \leqq q \leqq \infty)$; $E_3(=$ any one of X^*, $(X')^*$, $(X^*, (X')^*)_{\theta,q}$, $0 < \theta < 1, 1 \leqq q \leqq \infty)$; $E_4(=$ any one of $(X, X^*)_{\theta,q}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty)$.

THEOREM 3.1. (1) *Let $\beta > 0$ and $X \in \mathscr{F}(m, \beta)$ be reflexive. Then $M(X) \subset M(Z)$, $UM(X) \subset UM(Z), \forall Z \in Y(m, \beta, X)$.*

(2) *Let $\beta > 0$, $X \in Q(m)$. Then $M(X) \subset M(Z)$, $UM(X) \subset UM(Z)$, $\forall Z \in Q(m, \beta, X)$.*

Assertion (1) implies that every multiplier type operator on X defines a multiplier type operator on members of $Y(m, \beta, X)$ or $Q(m, \beta, X)$. Assertion (2) and Banach Steinhaus Theorem imply that every approximation process related to $\{\psi_n\}_{n \in N}$ on X, defines an approximation process related to $\{\psi_n\}$ on every $Z \in Y(m, \beta, X)$ or $Q(m, \beta, X)$ with $\mathrm{Cl}(\mathscr{A}, Z) = Z$.

Given a Banach subspace Z of $\mathscr{A}'$, $\delta > 0$, we need the notion of the space $\tilde{Z}_\delta =$ relative completion of Z_δ in Z, for describing the saturation classes in the theorem given below. For origin of definition of such spaces and for their properties see [14, p. 373], [16], [8]].
$\tilde{Z}_\delta = \{f \in Z \mid$ There exists $\{f_n\} \subset Z_\delta$, $\sup_{n \in P} \|f_n\|_{Z_\delta} \leqq \rho$, $f_n \to f$ in $Z\}$.
For $f \in \tilde{Z}_\delta$, $\|f\|_{\tilde{Z}_\delta} = \inf\{\rho > 0 \mid \{f_n\} \subset Z_\delta, \sup_{n \in P} \|f_n\|_{Z_\delta} \leqq \rho, f_n \to f \text{ in } Z\}$.

REMARK. $Z_\delta \subset \tilde{Z}_\delta$, on Z_δ $\|\ \|_{Z_\delta} \geqq \|\ \|_{\tilde{Z}_\delta}$ and $\tilde{Z}_\delta = Z_\delta$ if Z is reflexive.

THEOREM 3.2. *Suppose $\rho(\tau) \searrow 0$, $\tau \to \tau_0$ and $\delta > 0$, $\beta > 0$. Suppose $X \in \mathscr{F}(m, \beta)$ be reflexive (resp. $X \in Q(m)$) and $\forall \tau$, $\{\gamma_{\tau,k}\}_{k \in N} \in M(X)$ defining $\Gamma_\tau \in [X]$. Then we have the following:*

(a) *If $\forall f \in X_\delta$, $\|\Gamma_\tau f - f\|_X \leqq C_1 \rho(\tau) \|f\|_{X_\delta}$, then $\forall Z \in Y(m, \beta, X)$ (resp. $\forall Z \in Q(m, \beta, X)$) we have: $\forall f \in \tilde{Z}_\delta$, $\|\Gamma_\tau f - f\|_Z \leqq C_1 \rho(\tau) \|f\|_{\tilde{Z}_\delta}$.*

(b) *If $\forall f \in X$, $\Gamma_\tau f \in X_\delta$ and $\|\Gamma_\tau f\|_{X_\delta} \leqq C_2(\rho(\tau))^{-1} \|f\|_X$, then $\forall Z \in Y(m, \beta, X)$ (resp. $Q(m, \beta, X)$), we have: $\forall f \in Z$, $\Gamma_\tau f \in Z_\delta$, and $\|\Gamma_\tau f\|_{Z_\delta} \leqq C_2(\rho(\tau))^{-1} \|f\|_Z$.*

(c) *If, $\sup_n \|\sum_{k=0}^{n} (1 - k/(n+1)) \langle f, \psi_k \rangle \psi_k\|_X < \infty$ $\forall f \in X$; $\forall f \in X_\delta$,*

$\|\Gamma_\tau f - f\|_X \leqq C_1\rho(\tau)\|f\|_{X_\delta}$ *and for some* $c \neq 0$, $\dfrac{1-\gamma_{\tau,k}}{\rho(\tau)} \to c\lambda_k^\beta$ *as* $\tau \to \tau_0$, $\forall$ *fixed* $k \in N$, *then* $\forall Z \in Y(m, \beta, X)$ (*resp.* $Q(m, \beta, X)$), *we have, for* $f \in Z$, $\|\Gamma_\tau f - f\|_Z = \begin{cases} o(\rho(\tau)) \Leftrightarrow f \in A \\ O(\rho(\tau)) \Leftrightarrow f \in \tilde{Z}_\delta \end{cases}$.

In the following theorem, some sufficient conditions for members of $Y(m, \delta, X)$, $Q(m, \delta, X)$ to be subspaces of $\mathscr{A}'$, are given.

THEOREM 3.3. *Let* X, Y *be Banach subspaces of Lebesgue measurable, real or complex valued functions on* I *such that* $X \subset Y^*$, $Y \subset X^*$, $\mathrm{Cl}(\mathscr{D}(I), X) = X$, $\mathrm{Cl}(\mathscr{D}(I), Y) = Y$. *Let* $D = \dfrac{d}{dx}$.

(a) *Suppose* $\|\mathscr{U}^k D\psi_n\|_{L^2(I)} = O(|\lambda_n|^{s+k})$ ($n, k \in N$, $s \in P$ *independent of* n, k). *Then* $D: \mathscr{A}' \to \mathscr{A}'$ *is continuous linear operator of* $\mathscr{A}'$ *into* $\mathscr{A}'$ *and hence the spaces under consideration are subspaces of* $\mathscr{A}'$.

(b) *Suppose* $\forall k \in N$, $0 \leqq k \leqq m$, $\|D^k\psi_n\|_{X\cap X^*} = O(|\lambda_n|^{s_k})$ ($s_k \in P$, *depending only on* k). *Then* $\forall k \in N$, $0 \leqq k \leqq m$, $D^k: X + X^* \to \mathscr{A}'$ *is continuous,* $\langle D^k f, \psi\rangle = (-1)^k\langle f, D^k\psi\rangle$, ($f \in X + X^*$, $\psi \in \mathscr{A}$).

(c) *Suppose* $\|\psi_n\|_{X\cap Y} = O(|\lambda_n|^s)$ ($s \in P$, *independent of* $n \in N$) *and* $\forall n \in N$, *there exists* $n_1 \in P$, $\{n_q\}_{q=0}^{n_1}$ *in* N, *a finite sequence* $\{C_q^n\}_{q=0}^{n_1}$ *of constants with* $D\psi_n = \sum_{q=0}^{n_1} C_q^n \psi_{n_q}$, *and* $\sum_{q=0}^{n_1} |C_q^n| \leqq C_1|\lambda_n|^{q_1}$, $\sup_{0\leqq q\leqq n_1} |\lambda_{n_q}| \leqq C_2|\lambda_n|^{q_2}$ ($q_1, q_2 \in P$, $C_1 > 0$, $C_2 > 0$; q_1, q_2, C_1, C_2 *all independent of* $n \in N$}. *Then we have* (i) $\mathscr{A} \subset X \cap X^*$; $X, X^*, W^{-m}(X + X^*)$, $W^{-m}(X_{-\beta} + X^*_{-\beta})$ $\beta > 0$, *are all subspaces of* $\mathscr{A}'$. (ii) $\mathrm{Cl}(\{\psi_n\}, W^{m,0}(X)) = W^{m,0}(X)$ *and hence* $\mathrm{Cl}(\mathscr{A}, W^{m,0}(X)) = W^{m,0}(X)$.

(d) *Let* $k_0 \in P$ (k_0 *fixed*). *Suppose* $\forall k \in P$, $0 \leqq k \leqq m$ $\|\mathscr{U}^{k_0}D^k\psi_n\|_{X\cap X^*} = O(|\lambda_n|^{s_{k,k_0}})$ ($s_{k,k_0} \in P$, *depending only on* k, k_0). *Then* $\forall k \in P$, $0 \leqq k \leqq m$, $D^k\mathscr{U}^{k_0}: X + X^* \to \mathscr{A}'$ *is continuous. Hence* $W^{-m}(X_{-k_0} + X^*_{-k_0}) \subset \mathscr{A}'$.

4. In this section, we state and prove certain lemmas needed in the proof of main results of §3.

LEMMA 4.1. *Let* X *be a Banach subspace of* $\mathscr{D}'(I)$ *and* $\mathrm{Cl}(\mathscr{D}(I), X) = X$. *Then* (a) $(W^{m,0}(X))^* = W^{-m}(X^*)$ *with equivalent norms,* (b) *If* X *is reflexive then* $W^{m,0}(X)$ *is reflexive.*

PROOF. (a) Proof of (a) is analogous to that of Prop. 31.3, p. 325 Treves [31];
(b) Let X be reflexive. $W^{m,0}(X)$ is reflexive since $W^{m,0}(X)$ can be embedded as a closed linear subspace of the reflexive space $E = \underbrace{X \times X \cdots \times X}_{m+1 \text{ times}}$ under the norm $\|f\|_E = \sum_{i=0}^{m} \|f_i\|_X$ with $f = (f_0, f_1, \cdots, f_m) \in E$.

LEMMA 4.2. *Let* X, Y *be Banach subspaces of* $\mathscr{D}'(I)$. *Then there exists an extension of* $T \in [X, Y]$, $\bar{T}$, $\bar{T} \in [W^{-m}(X), W^{-m}(Y)]$ *such that* $\|\bar{T}\| \leqq \|T\|$ *and when* $\mathrm{Cl}(\mathscr{D}(I), X) = X$; $\bar{T}$ *is uniquely determined.*

PROOF. On $f \in W^{-m}(X)$, define $\bar{T}$ by $\bar{T}f = T\left(\sum_{j=0}^{m} D^j f_j\right) = \sum_{j=0}^{m} D^j T f_j$. This definition is independent of the representation of f, since $f = \sum_{j=0}^{m} D^j f_j = \sum_{j=0}^{m} D^j g_j$ implies $0 = \bar{T}0 = \bar{T}\left(\sum_{j=0}^{m} D^j (f_j - g_j)\right) = \sum_{j=0}^{m} D^j T f_j - \sum_{j=0}^{m} D^j T g_j$. Also, for $f = \sum_{j=0}^{m} D^j f_j$, $f_j \in X$, $0 \leqq j \leqq m$, $\|\bar{T}f\|_{W^{-m}(Y)} \leqq \sum_{j=0}^{m} \|Tf_j\|_Y \leqq \|T\| \sum_{j=0}^{m} \|f_j\|_X$. Hence $\|\bar{T}\| \leqq \|T\|$. Uniqueness of $\bar{T}$ follows from $\mathrm{Cl}(\mathscr{D}(I), X) = X$.

LEMMA 4.3. *Let* $Z \in \mathscr{F}(m)$ *and* $\delta > 0$ *then* (a) $\mathrm{Cl}(\mathscr{A}, Z_\delta) = Z_\delta \subset Z$, Z_δ *is Banach space.* (b) $(Z_\delta)^* = (Z^*)_{-\delta}$ $Z^* \subset (Z^*)_{-\delta}$. (c) $\mathrm{Cl}(\mathscr{A}, Z^*) = Z^*$ *implies* $\mathrm{Cl}(\mathscr{A}, (Z^*)_{-\delta}) = (Z^*)_{-\delta}$. (d) *For* $0 < \alpha < \delta$, *we have* $Z_\delta \subset Z_\alpha$, $Z^*_{-\alpha} \subset Z^*_{-\delta}$. (e) *If* Z *is reflexive, so is* Z_δ. (f) $M(Z) \subset M(W^{-m}(Z))$. (g) $W^{-m}(Z_\delta) = (W^{-m}(Z))_\delta$. (h) $Z^* \in \mathscr{F}(m)$ *and* Z *reflexive imply* $(W^{-m}(Z_\delta))^* = (W^{m,0}(Z^*))_{-\delta}$ (i) $(Z^*_{-\delta})_\delta = Z^*$; $(Z_\delta)_{-\delta} = Z$. *Here* $Z^*_{-\delta}$ *denotes* $(Z^*)_{-\delta}$.

PROOF. For $\mathscr{k} \in Z$, let $\phi_{\mathscr{k}} = \sum_{k=0}^{l} \langle \mathscr{k}, \psi_{i_k} \rangle \psi_{i_k}$. Then $\phi_{\mathscr{k}} \in \Lambda$ and $\|\phi_{\mathscr{k}}\|_Z \leqq C\|\mathscr{k}\|_Z$, with $C = \sum_{k=0}^{l} \|\psi_{i_k}\|_{Z^*} \|\psi_{i_k}\|_Z$. (a) Clearly $\mathscr{A} \subset Z_\delta \subset Z$. For $f \in Z_\delta$, $\mathscr{U}^\delta f \in Z$ and since $\mathrm{Cl}(\mathscr{A}, Z) = Z$, for $\rho > 0$ there exists $\phi \in \mathscr{A}$ for which $\|\mathscr{U}^\delta f - \phi\|_Z \leqq \rho$. If $g = \mathscr{U}^\delta f - \phi$, then $\|g - \phi_g\| \leqq (1 + C)\rho$, $f - \phi_f - G_\delta \phi = G_\delta g$. $\|f - (\phi_f + G_\delta \phi)\|_{Z_\delta} \leqq (\|G_\delta\| + 1 + C)\rho$, $G_\delta \phi \in \mathscr{A}$. Hence $\mathrm{Cl}(\mathscr{A}, Z_\delta) = Z_\delta$. Since $\mathscr{U}^\delta$ is closed on Z_δ and Z is complete, Z_δ is Banach.

(b) The map $T: Z_\delta \to Z \times Z$, given by $Tf = (f, \mathscr{U}^\delta f)$, $f \in Z_\delta$ is an isometry. $T^*: Z^* \times Z^* \to (Z_\delta)^*$ is *onto* by Hahn-Banach Theorem. For $f \in Z^*_{-\delta}$ with $f = f_0 + \mathscr{U}^\delta f_1$, $f_0, f_1 \in Z^*$, define $\bar{f}$ on Z_δ given by $\bar{f}(\phi) = \langle f_0, \phi \rangle + \langle f_1, \mathscr{U}^\delta \phi \rangle$, $(\phi \in Z_\delta)$. $\bar{f}$ is well defined and $\bar{f} \in (Z_\delta)^*$. The map $I: Z^*_{-\delta} \to (Z_\delta)^*$ given by $I(f) = \bar{f}$, $f \in Z^*_{-\delta}$, is one to one. We prove that I is *onto*: Let $f \in (Z_\delta)^*$. Since T^* is onto, there exists $\mathscr{k}_0, \mathscr{k}_1 \in Z^*$ such that $T^*(\mathscr{k}_0, \mathscr{k}_1) = f$. Define $v \in Z^*_{-\delta}$ as $v = \mathscr{k}_0 + \mathscr{U}^\delta \mathscr{k}_1$. Then $Iv = f$. Hence $Z^*_{-\delta} = (Z_\delta)^*$. It is easy to prove (c) and the fact $Z^* \subset Z^*_{-\delta} \subset \mathscr{A}'$.

(d) Let $0 < \alpha < \delta$. For $f \in Z_\delta$, $\mathscr{U}^\alpha f = G_{\delta-\alpha} \mathscr{U}^\delta f \in Z$. Hence $Z_\delta \subset Z_\alpha$. $\mathscr{A} \subset Z_\delta \subset Z_\alpha$ and $\mathrm{Cl}(\mathscr{A}, Z_\alpha) = Z_\alpha$ imply $\mathrm{Cl}(Z_\delta, Z_\alpha) = Z_\alpha$. Hence $Z^*_{-\alpha} \subset Z^*_{-\delta} \subset \mathscr{A}'$.

(e) If Z is reflexive, so is Z_δ, as Z_δ can be embedded as a strongly closed subspace of $Z \times Z$.

(f)

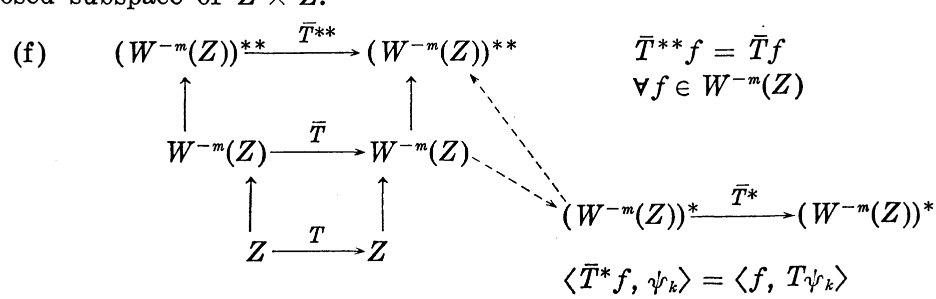


FIGURE 1.

In this diagram, $\rightharpoonup$ (resp. $\longrightarrow$) denotes the direction to which proof proceeds, taking transpose (resp. extension) of the operator under consideration:

$\mathrm{Cl}(\mathscr{A}, W^{-m}(Z)) = W^{-m}(Z) \subset \mathscr{A}'$. Hence $\mathscr{A} \subset (W^{-m}(Z))^* \subset \mathscr{A}'$. Let $\{\gamma_k\} \in M(Z)$ defining $T \in [Z]$. By Lemma 4.2 there exists $\bar{T} \in [W^{-m}(Z)]$. $\bar{T}^* \in [(W^{-m}(Z))^*]$ such that $\langle \bar{T}^* f, \psi_k \rangle = \langle f, T\psi_k \rangle = \gamma_k \langle f, \psi_k \rangle$, $(k \in N, f \in (W^{-m}(Z))^*)$. Hence $\{\gamma_k\}_{k \in N} \in M((W^{-m}(Z))^{**})$ defining $\bar{T}^{**} \in [(W^{-m}(Z))^{**}]$. Since $\bar{T}^{**} f = \bar{T} f$ $\forall f \in W^{-m}(Z)$, $\{\gamma_k\}_{k \in N} \in M(W^{-m}(Z))$ defining $\bar{T} \in [W^{-m}(Z)]$.

(g) Let $f \in W^{-m}(Z_\delta)$. $f = \phi_f + \sum_{j=0}^{m} D^j G_\delta g_j$ with $\phi \in \Lambda$, $g_j \in Z$, $0 \leqq j \leqq m$. By (f) of this lemma, $f = \phi_f + G_\delta\left[\sum_{j=0}^{m} D^j g_j\right]$. This implies $\mathscr{U}^\delta f = \sum_{j=0}^{m} D^j g_j \in W^{-m}(Z)$ thus $W^{-m}(Z_\delta) \subset (W^{-m}(Z))_\delta$. Conversely, let $f \in (W^{-m}(Z))_\delta$. Then $\mathscr{U}^\delta f = \sum_{j=0}^{m} D^j g_j \in W^{-m}(Z)$; $g_j \in Z$, $0 \leqq j \leqq m$. This implies $f = \phi_f + G_\delta\left(\sum_{j=0}^{m} D^j g_j\right) = \phi_f + \sum_{j=0}^{m} D^j G_\delta g_j \in W^{-m}(Z_\delta)$, with $\phi_f \in \Lambda$. Hence $(W^{-m}(Z))_\delta \subset W^{-m}(Z_\delta)$.

(h) Since $Z^* \in \mathscr{F}(m)$ and Z reflexive $W^{m,0}(Z^*)$ is reflexive. The rest follows by steps similar to those of (b) of this lemma.

(i) $\forall f \in Z^*$, f, $\mathscr{U}^\delta f \in Z^*_{-\delta}$. Hence $f \in (Z^*_{-\delta})_\delta$ and $\|f\|_{(Z^*_{-\delta})_\delta} \leqq 2\|f\|_{Z^*}$. This gives $Z^* \subset (Z^*_{-\delta})_\delta$. $\forall f \in (Z^*_{-\delta})_\delta$, $\mathscr{U}^\delta f \in Z^*_{-\delta}$. Hence $\mathscr{U}^\delta f = f_0 + \mathscr{U}^\delta f_1$, $f_0, f_1 \in Z^*$ or $f = \phi_{f+f_1} + G_\delta f_0 + f_1$, $\phi_{f+f_1} \in \Lambda$. Hence $f \in Z^*$, $\|f\|_{Z^*} \leqq C_1(1 + \|G_\delta\|)\|f\|_{(Z^*_{-\delta})_\delta}$. This gives $(Z^*_{-\delta})_\delta \subset Z^*$. Hence $(Z^*_{-\delta})_\delta = Z^*$. The identity $(Z_\delta)_{-\delta} = Z$ is easy to prove.

LEMMA 4.4. *Suppose X, Y be Banach subspaces of $\mathscr{A}'$ each containing $\mathscr{A}$ as a dense subspace and for $\delta > 0$, $\{\nu_{k\,\delta}\}_{k \in N} \in M(X) \cap M(Y)$. Then for*

$$\left.\begin{array}{l} 0 < \theta < 1,\ 1 \leqq q < \infty \\ 0 \leqq \theta \leqq 1,\ q = \infty \end{array}\right\} ((X, Y)_{\theta,q})_\delta = (X_\delta, Y_\delta)_{\theta,q};\ ((X^*, Y^*)_{\theta,q})_{-\delta} = (X^*_{-\delta}, Y^*_{-\delta})_{\theta,q}.$$

PROOF. For $f \in ((X, Y)_{\theta,q})_\delta$, taking $\mathscr{U}^\delta f = f_1 + f_2$, with $f_1 \in X$, $f_2 \in Y$, we can prove for $0 < t < \infty$, $K(t, f, X_\delta, Y_\delta) \leqq (1 + \|G_\delta\|_{[X]} + \|G_\delta\|_{[Y]}) K(t, \mathscr{U}^\delta f, X, Y)$. This implies $((X, Y)_{\theta,q})_\delta \subset (X_\delta, Y_\delta)_{\theta,q}$. Conversely for $f \in (X_\delta, Y_\delta)_{\theta,q}$ with $f = f_1 + f_2$, $f_1 \in X_\delta$, $f_2 \in Y_\delta$, we can prove, for $0 < t < \infty$, $K(t, f, X, Y) \leqq K(t, f, X_\delta, Y_\delta)$; $K(t, \mathscr{U}^\delta f, X, Y) \leqq K(t, f, X_\delta, Y_\delta)$. This gives $(X_\delta, Y_\delta)_{\theta,q} \subset ((X, Y)_{\theta,q})_\delta$. Hence the first identity.

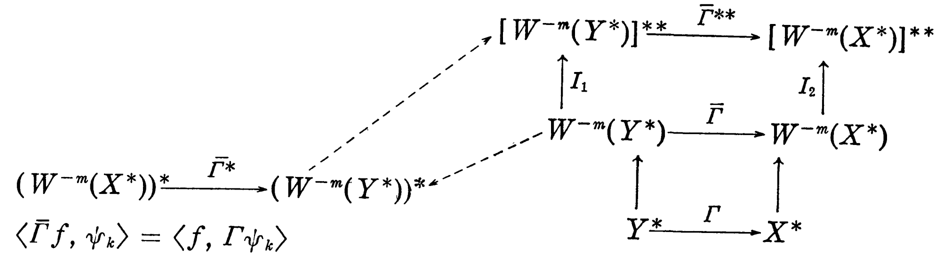
$$((X^*, Y^*)_{\theta,q})_{-\delta} = (((X^*_{-\delta})_\delta, (Y^*_{-\delta})_\delta)_{\theta,q})_{-\delta} = (((X^*_{-\delta}, Y^*_{-\delta})_{\theta,q})_\delta)_{-\delta} = (X^*_{-\delta}, Y^*_{-\delta})_{\theta,q}\,.$$

LEMMA 4.5. *Let X, Y be Banach subspaces of $\mathscr{A}'$ such that $\mathscr{A}$ is dense in both X and Y^*, $\mathscr{A} \subset Y$, $W^{-m}(Y^*) \subset \mathscr{A}'$. Then*

(a) $M(X, Y) \subset M(W^{-m}(Y^*), W^{-m}(X^*)) \subset M((W^{-m}(X^*))^*, (W^{-m}(Y^*))^*)$

(b) $UM(X, Y) \subset UM(W^{-m}(Y^*), W^{-m}(X^*)) \subset UM((W^{-m}(X^*))^*, (W^{-m}(Y^*))^*)$.

PROOF. $\mathrm{Cl}(\mathscr{A}, W^{-m}(Y^*)) = W^{-m}(Y^*)$ and $\mathrm{Cl}(W^{-m}(Y^*), \mathscr{A}') = \mathscr{A}'$.



$\bar{\Gamma}^{**}I_1 = I_2\bar{\Gamma}$, I_1, I_2 are identity maps.

FIGURE 2.

Hence $\mathscr{A} \subset (W^{-m}(Y^*))^* \subset \mathscr{A}'$. It is enough to prove (a). $M(X, Y) \subset M(Y^*, X^*)$. Let $\{\delta_k\}_{k\in P} \in M(Y^*, X^*)$ defining $\Gamma \in [Y^*, X^*]$. By Lemma 4.2, there exists $\bar{\Gamma} \in [W^{-m}(Y^*), W^{-m}(X^*)]$. It is easy to check that $\{\delta_k\}_{k\in P} \in M((W^{-m}(X^*))^*, (W^{-m}(Y^*))^*)$ defining $\bar{\Gamma}^* \in [(W^{-m}(X^*))^*, (W^{-m}(Y^*))^*]$. $\bar{\Gamma}^{**} \in [(W^m(Y^*))^{**}, (W^{-m}(X^*))^{**}]$ and $\bar{\Gamma}^{**}f = \bar{\Gamma}f$ $\forall f \in W^{-m}(Y^*)$. Hence $\{\delta_k\}_{k\in N} \in M(W^{-m}(Y^*), W^{-m}(X^*))$ defining $\bar{\Gamma} \in [W^{-m}(Y^*), W^{-m}(X^*)]$ and $\|\bar{\Gamma}\| \leqq \|\Gamma\|$. (Refer Lemma 4.3.f for symbols $\rightarrow$, $(\rightarrow)$).

COROLLARY 4.1. *Let* $X \in \mathscr{F}(m)$ *and* $\mathrm{Cl}(\mathscr{A}, X^*) = X^*$. *Then for* $0 < \theta < 1$, $1 \leqq q \leqq \infty$

(i) $M(X) \subset M(W^{m,0}(X)) \cap M(W^{-m}(X)) \subset M((W^{-m}(X), W^{m,0}(X))_{\theta,q})$

(ii) $UM(X) \subset UM(W^{m,0}(X)) \cap UM(W^{-m}(X)) \subset UM((W^m(X), W^{m,0}(X))_{\theta,q})$.

PROOF. Apply Lemma 4.5 and Theorem 3.2.23, [13, p. 180].

LEMMA 4.6. (a) *Suppose* $Z \in \mathscr{F}(m, \delta)$, *for some* $\delta > 0$. *Then*

(1) $\{\nu_{k,\delta}\} \in M(Z, Z_\delta) \subset M(Z^*_{-\delta}, Z^*) \subset M(W^{-m}(Z^*_{-\delta}), W^{-m}(Z^*))$,
$\{\nu_{k,\delta}\} \in M(W^{m,0}(Z^*_{-\delta}), W^{m,0}(Z^*))$

(2) $(W^{-m}(Z^*_{-\delta}))_\delta = W^{-m}(Z^*)$

(3) $W^{-m}(Z^*_{-\delta}) = (W^{-m}(Z^*))_{-\delta}$

(b) *If, in addition Z is reflexive then*

(1) $(W^{m,0}(Z))_\delta = (W^{-m}(Z^*_{-\delta}))^*$

(2) $\{\nu_{k,\delta}\} \in M(E, E_\delta)$ $\forall E \in Y(m, \delta, Z)$

(3) $W^{m,0}(Z^*_{-\delta}) = (W^{m,0}(Z^*))_{-\delta}$

(4) $UM(Z) \subset UM(E_{-\delta})$

where $E =$ *any one of* Z^*, $W^{-m}(Z^*)$, $W^{m,0}(Z^*)$, $(W^{-m}(Z^*), W^{m,0}(Z^*))_{\theta,q}$ $0 < \theta < 1$, $1 \leqq q \leqq \infty$.

PROOF. (a) (1) Follows from Lemma 4.5 and by similar steps as in the proof of Lemma 4.3 (f).

(2) $W^{-m}(Z^*) \subset W^m(Z^*_{-\delta})$. For $f \in W^{-m}(Z^*)$ with $f = \sum_{j=0}^{m} D^j f_j$, $f_j \in Z^*$, $0 \leqq j \leqq m$, let $g_\delta(f) = \sum_{j=0}^{m} D^j \mathscr{U}^\delta f_j \in W^{-m}(Z^*_{-\delta})$; $f = \phi + G_\delta(g_\delta(f))$ with $\phi \in \Lambda$,

$\mathscr{U}^\delta f = g_\delta(f) \in W^{-m}(Z^*_{-\delta})$. Thus, $W^{-m}(Z^*) \subset (W^{-m}(Z^*_{-\delta}))_\delta$. For $f \in (W^{-m}(Z^*_{-\delta}))_\delta$, $\mathscr{U}^\delta f \in W^{-m}(Z^*_{-\delta})$. By (1), $f = \phi_f + G_\delta(\mathscr{U}^\delta f) \in W^{-m}(Z^*)$ with $\phi_f \in \Lambda$. Thus $(W^{-m}(Z^*_{-\delta}))_\delta \subset W^{-m}(Z^*)$.

(3) $(W^{-m}(Z^*))_{-\delta} = (((W^{-m}(Z^*_{-\delta}))_\delta)_{-\delta} = W^{-m}(Z^*_{-\delta})$.

(b) (1) If Z is reflexive, so are $W^{m,0}(Z)$ and $(W^{m,0}(Z))_\delta$. Hence $(W^{m,0}(Z))_\delta = [(W^{m,0}(Z))_\delta]^{**} = ((W^{m,0}(Z))_\delta)^*)^* = ((W^{-m}(Z^*))_{-\delta})^* = (W^{-m}(Z^*_{-\delta}))^*$.

(2) Follows from Lemma 4.5 by letting $X = Z$, $Y = Z_\delta$, and $X = Z^*_{-\delta}$, $Y = Z^*$, and by Theorem 3.2.23, in [13] and by Lemma 4.4.

(3) $W^{m,0}(Z^*_{-\delta}) = (W^{-m}(Z_\delta))^* = ((W^{-m}(Z))_\delta)^* = (W^{-m}(Z))^*_{-\delta} = (W^{m,0}(Z^*))_{-\delta}$.

(4) Let $\{\delta_k\}_{k\in N} \in M(Z^*)$ defining $\Gamma \in [Z^*]$. For $f \in Z^*_{-\delta}$ with $f = f_0 + \mathscr{U}^\delta f_1$, $f_0, f_1 \in Z^*$. Define $\bar{\Gamma} f = \Gamma f_0 + \mathscr{U}^\delta \Gamma f_1$. It is easy to check that $\{\delta_k\}_{k\in N} \in M(Z^*_{-\delta})$ defining $\bar{\Gamma} \in [Z^*_{-\delta}]$, $M(Z) \subset M(Z^*_{-\delta})$, $UM(Z) \subset UM(Z^*_{-\delta})$. The rest follows from Lemma 4.5, Lemma 4.4 and from [13, Theorem 3.3.23].

Using the definition of $M(X, Y)$ we like to give a simple characterization of elements of $M(X_\delta, X)$ for a Banach subspace X of $\mathscr{A}'$ and $\delta > 0$.

Indeed, for $\{\gamma_k\} \in M(X_\delta, X)$ defining $\Gamma \in [X_\delta, X]$. We have, for every $f \in X$, $G_\delta f \in X_\delta$ and hence $\Gamma(G_\delta f) \in X$. Thus $\{\gamma_k \nu_{k,\delta}\}_{k\in N} \in M(X)$ defining $\Gamma G_\delta \in [X]$ with $\|\Gamma G_\delta\|_{[X]} \leqq \|\Gamma\|_{[X_\delta, X]}(C + \|G_\delta\|_{[X]})$ (C an independent constant). This gives $\gamma_k = \delta_k \lambda_k^\delta$ ($k \in N$, $k \neq i_0, \cdots, i_l$) for some $\{\delta_k\} \in M(X)$ with $\|\{\delta_k\}\|_{M(X)} \leqq C_1 \|\{\gamma_k\}\|_{M(X_\delta, X)}$. Conversely, for $\{\eta_k\} \in M(X)$, $\{\eta_k \lambda_k^\delta\} \in M(X_\delta, X)$ with $\|\{\eta_k \lambda_k^\delta\}\|_{M(X_\delta, X)} \leqq \|\{\eta_k\}\|_{M(X)}$.

Thus we have proved the following:

LEMMA 4.7. *Let X be a Banach subspace of $\mathscr{A}'$ and $\delta > 0$. Then $\{\gamma_k\} \in M(X_\delta, X)$ if and only if there exists $\{\eta_k\} \in M(X_\delta, X)$ satisfying*

$$\gamma_k = \delta_k \lambda_k^\delta \qquad (k \in N,\ k \neq i_0, \cdots, i_l)\ .$$

In this case

$$\|\{\gamma_k\}\|_{M(X_\delta, X)} \leqq \|\{\eta_k\}\|_{M(X)} \leqq e_1 \|\{\gamma_k\}\|_{M(X_\delta, X)}\ .$$

5. In this section we present the proofs of our main results, utilizing the techniques developed and results obtained in § 4.

PROOF OF THEOREM 3.1. (1) Let $\delta > 0$, $X \in \mathscr{F}(m, \delta)$ be reflexive. Then Y (= any one of $X, X^*, (X, X^*)_{\theta,q}$, $0 < \theta < 1$, $1 < q < \infty$), and $Y_{-\delta} \in \mathscr{F}(m)$ and are reflexive. Hence (1) follows from Corollary 4.1 and Lemma 4.6(b).

(2) For $Z \in \mathscr{F}(m)$, $UM(Z) \subset UM(Z_\delta)$ since, for a multiplier type $\Gamma \in [Z]$ and $f \in Z, \Gamma(\mathscr{U}^\delta f) = \mathscr{U}^\delta(\Gamma f)$ in $\mathscr{A}'$. Hence, $\|\Gamma f\|_{Z_\delta} \leqq \|\Gamma\| \|f\|_{Z_\delta}$, ($f \in Z_\delta$). For $X \in Q(m)$ $UM(X) \subset UM(X^*)$. Since $\mathrm{Cl}([\{\psi_n\}], X') = X' \subset X^*, UM(X^*) \subset UM(X')$. If $E=$ either X or X', we have $UM(X) \subset UM(E) \subset UM(E^*)$ and $UM(X) \subset$

$UM(E) \subset UM(E_\delta) \subset UM(E^*_{-\delta})$. The rest of the theorem follows from Lemma 4.3 (f) and by [13, Theorem 3.2.23].

PROOF OF THEOREM 3.2. Let $\rho(\tau) \searrow 0$ as $\tau \to \tau_0$. Let $\beta > 0$, $\delta > 0$, X, $\{\gamma_{\tau,k}\}$, Γ_τ be as given in Theorem 3.2.

(a) The inequality $\|\Gamma_\tau f - f\|_X \leqq C_1\rho(\tau)\|f\|_{X_\delta}$ for every $f \in X_\delta$ implies $\left\{\frac{\gamma_{\tau,k}-1}{\rho(\tau)}\right\} \in UM(X_\delta, X)$, with

$$\sup_\tau \left\|\left\{\frac{\gamma_{\tau,k}-1}{\rho(\tau)}\right\}\right\|_{M(X_\delta,X)} \leqq \sup_\tau \left\|\frac{\Gamma_\tau - I}{\rho(\tau)}\right\|_{[X_\delta,X]} < d_1 < \infty .$$

By Lemma 4.7, for each τ, there exists $\{\eta_{\tau,k}\} \in M(X)$ satisfying

$$\frac{\gamma_{\tau,k}-1}{\rho(\tau)} = \eta_{\tau,k}\lambda_k^\delta \qquad (k \in N,\ k \neq i_0, \cdots, i_l)$$

with

$$\sup_\tau \|\{\eta_{\tau,k}\}\|_{M(X)} \leqq e_1 \sup_\tau \left\|\left\{\frac{\gamma_{\tau,k}-1}{\rho(\tau)}\right\}\right\|_{M(X_\delta,X)} < e_1 d_1 < \infty .$$

By Theorem 3.1 we have, $\{\eta_{\tau,k}\} \in UM(Z)$. By Lemma 4.7 we have, for $Z \in Y(m, \beta, X)$ (resp. $Q(m, \beta, X)$), $\left\{\frac{\gamma_{\tau,k}-1}{\rho(\tau)}\right\} \in UM(Z_\delta, Z)$, i.e. $\forall f \in Z_\delta$ $\|\Gamma_\tau f - f\|_Z \leqq C_{11}\rho(\tau)\|f\|_{Z_\delta}$. For $Z \in Y(m, \beta, X)$, Z is reflexive and hence $\widetilde{Z}_\delta = Z_\delta$. We have proved (a) for $Z \in Y(m, \beta, X)$. In order to prove that $\{\Gamma_\tau\}$ satisfies Jackson-type inequality of order $\rho(\tau)$ on Z with respect to $\widetilde{Z}_\delta$ for $Z \in Q(m, \beta, X)$, we have to prove that, $\left\{\frac{\gamma_{\tau,k}-1}{\rho(\tau)}\right\} \in UM(\widetilde{Z}_\delta, Z)$ $\forall Z \in Q(m, \beta, X)$. Let $Z \in Q(m, \beta, X)$. $\forall \tau$, $\left\{\frac{\gamma_{\tau,k}-1}{\rho(\tau)}\right\} \in M(X)$. Hence, by Theorem 3.1 $\left\{\frac{\gamma_{\tau,k}-1}{\rho(\tau)}\right\} \in M(Z)$, defining $\left\{\frac{\Gamma_\tau - I}{\rho(\tau)}\right\} \in [Z]$, $\forall \tau$. For $f \in \widetilde{Z}_\delta$, there exists a sequence $\{f_n\}$ in Z_δ such that $\sup_{n \in P}\|f_n\|_{Z_\delta} \leqq 2\|f\|_{\widetilde{Z}_\delta}$ and $f_n \to f$ in Z. This implies $\forall \tau$, $\frac{\Gamma_\tau f_n - f_n}{\rho(\tau)} \to \frac{\Gamma_\tau f - f}{\rho(\tau)}$ in Z and $\left\|\frac{\Gamma_\tau f - f}{\rho(\tau)}\right\|_Z \leqq$ $\limsup_{n \in P} \left\|\frac{\Gamma_\tau f_n - f_n}{\rho(\tau)}\right\|_Z \leqq C_{11} \sup_n \|f_n\|_{Z_\delta} \leqq 2C_{11}\|f\|_{\widetilde{Z}_\delta}$.

(b) Let $Z \in Y(m, \beta, X)$ (resp. $Q(m, \beta, X)$). By hypothesis (b), we have: $\forall f \in X$, $\Gamma_\tau f \in X_\delta$ and $\|\Gamma_\tau f\|_{X_\delta} \leqq C_2(\rho(\tau))^{-1}\|f\|_X$; i.e. $\|\rho(\tau)\mathscr{U}^\delta\Gamma_\tau f\|_X \leqq \rho(\tau)\|\Gamma_\tau f\|_{X_\delta} \leqq C_2\|f\|_X$; i.e. $\{\rho(\tau)\lambda_k^\delta\gamma_{\tau,k}\} \in UM(X)$. By Theorem 3.1, $\{\rho(\tau)\lambda_k^\delta\gamma_{\tau,k}\} \in UM(Z)$ i.e. $\rho(\tau)\|\mathscr{U}^\delta\Gamma_\tau f\|_Z \leqq C_2\|f\|_Z$ for every $f \in Z$. By Theorem 3.1, $\{\nu_{k,\delta}\} \in M(Z)$ defining $G_\delta \in [Z]$. $\forall f \in Z$, $\rho(\tau)\Gamma_\tau f = G_\delta[\rho(\tau)\mathscr{U}^\delta\Gamma_\tau f] +$

$\rho(\tau)\Gamma_\tau\phi_f$, $\phi_f = \sum_{k=0}^{l} \langle f, \psi_{i_k}\rangle\psi_{i_k} \in \Lambda$. Hence $\rho(\tau)\,\|\Gamma_\tau f\|_{Z_\delta} \leqq (A + \|G_\beta\|)\,\|f\|_Z$, $f \in Z$, $A \equiv A(\psi_{i_k}\cdots\psi_{i_l}) > 0$. Hence $\forall f \in Z, \Gamma_\tau f \in Z_\delta$, $\|\Gamma_\tau f\|_{Z_\delta} \leqq C_{22}(\rho(\tau))^{-1}\|f\|_Z$.

(c) Let $Z \in Y(m,\beta,X)$ (resp. $Q(m,\beta,X)$). By (a), we have $\|\Gamma_\tau f - f\|_Z \leqq C_1\rho(\tau)\,\|f\|_{Z_\delta}$, $\forall f \in Z_\delta$.

Case 1: Suppose $\mathrm{Cl}([\{\psi_n\}],Z) = Z$. $-c\mathscr{U}^\beta$ is a closed operator with dense domain Z_δ and range in Z. We will show that (i) $\forall f \in Z_\delta$, $\dfrac{\Gamma_\tau f - f}{\rho(\tau)} \to -c\mathscr{U}^\delta f$ in Z, (ii) there exists $\{J_n\}_{n\in P} \subset [Z]$, $\bigcup_{n\in P} J_n(Z) \subset Z_\delta$; $J_n f \to f$ in Z, $\forall f \in Z$; and J_n and Γ_τ commute $\forall n \in P$, $\forall \tau$. Then (c) follows by Theorem 13.4.1, Butzer-Nessel [14, p. 502] [Ref. Berens [8]]. For $f \in Z_\delta$, let $T_\tau f = \dfrac{\Gamma_\tau f - f}{\rho(\tau)} + c\mathscr{U}^\delta f$. By uniform boundedness principle $\sup_\tau \|T_\tau\|_{[Z_\delta, Z]} < \infty$. $\forall k \in P$, $T_\tau\psi_k = \left[\dfrac{\gamma_{\tau,k}-1}{\rho(\tau)} + c\lambda_k^\delta\right]\psi_k \to 0$ as $\tau \to \tau_0$. Since $\mathrm{Cl}([\{\psi_n\}],Z) = Z$, Banach Steinhaus theorem implies that $\forall f \in Z_\delta$, $\dfrac{\Gamma_\tau f - f}{\rho(\tau)} \to -c\mathscr{U}^\delta f$ in Z as $\tau \to \tau_0$.

For $f \in \mathscr{A}'$, let $R_n f = \sum_{k=0}^{n}\left(1 - \dfrac{k}{n+1}\right)\langle f, \psi_k\rangle\psi_k$. $R_n \in [X]$, $\sup_n \|R_n\|_{[X]} < d_1 < \infty$, $R_n f \to f$ in X, $\forall f \in X$. (see Corollary 3.6, [16, I]). Theorem 3.1 implies that $\{R_n\} \in [Z]$, R_n and Γ_τ commute, $\|R_n\|_{[Z]} \leqq d_1$, $R_n f \to f$ in Z, $\forall f \in Z$.

Case 2: Suppose Z is the dual of a Banach space F with $F = \mathrm{Cl}([\{\psi_n\}],F)$, we only have to prove, for $f \in Z$, $\|\Gamma_\tau f - f\|_Z = \begin{cases} o(\rho(\tau)) \Rightarrow f \in \Lambda \\ O(\rho(\tau)) \Rightarrow f \in \tilde{Z}_\delta \end{cases}$. For $f \in Z$ let $\|\Gamma_\tau f - f\|_Z = O(\rho(\tau))$. Since bounded sets in Z are weakly* compact there exists $f^0 \in Z$ and $\{\tau_l\}_{l\in P}$ such that $\tau_l \to \tau_0$ as $l \to \infty$, $\dfrac{\Gamma_{\tau_l} f - f}{\rho(\tau_l)} \to f^0$ as $l \to \infty$, in the weak* topology of Z. $\forall k \in N$, $\left\langle \dfrac{\Gamma_{\tau_l} f - f}{\rho(\tau_l)}, \psi_k \right\rangle = \left(\dfrac{\gamma_{\tau_l,k}-1}{\rho(\tau_l)}\right)\langle f, \psi_k\rangle \to \langle f^0, \psi_k\rangle = \langle -c\mathscr{U}^\delta f, \psi_k\rangle$. Hence $\mathscr{U}^\delta f = -\dfrac{1}{c} f^0 \in Z$; i.e. $f \in Z_\delta \subset \tilde{Z}_\delta$. If big O is replaced by small o, then $\mathscr{U}^\delta f = 0$, i.e. $f \in \Lambda$.

PROOF OF THEOREM 3.3. Let $s_0 \in P$ such that $\sum_{\substack{k=0\\ \lambda_k \neq 0}}^{\infty} |\lambda_k|^{-2s_0} < M_0 < \infty$,

(a) Suppose, $\forall k, n \in N$, $\|\mathscr{U}^k D\psi_n\|_{L^2(I)} \leqq M_1(|\lambda_n|^{s+k})$, ($s \in P$, independet of $n, k \in N$). Let $\phi \in \mathscr{A}$. $D\phi = \sum_{k=0}^{\infty} \langle \phi, \psi_k\rangle D\psi_k \in C^\infty(I)$. $\forall k \in N$, $\|\mathscr{U}^k D\phi\|_{L^2} \leqq \sum_{n=0}^{\infty} |\langle \phi, \psi_n\rangle|\, \|\mathscr{U}^k D\psi_n\|_{L^2} \leqq M_1 \sum_{n=0}^{\infty} |\langle \phi, \psi_n\rangle|\,|\lambda_n|^{s+k} \leqq M_0 M_1 \left\{\sum_{k=0}^{\infty} |\langle \phi, \psi_n\rangle|^2 |\lambda_n|^{2(s+k+s_0)}\right\}^{1/2} < \infty$. Hence $\mathscr{U}^k D\phi \in L^2(I)$, $\forall k \in N$. Since $D\phi$, $\psi_n \in$ domain of $\mathscr{U}^k$ in $L^2(I)$,

$\forall n, k \in N$, we have $\langle \mathscr{U}^k D\phi, \psi_n\rangle = \langle D\phi, \mathscr{U}^k\psi_n\rangle$, $(k, n \in N)$. Hence $D\phi \in \mathscr{A}$, by definition of $\mathscr{A}$ [see [37], p. 252]. Let $\{\phi_n\}_{n\in N}$ be a sequence in $\mathscr{A}$ such that $\phi_n \to \phi$ in $\mathscr{A}$. Let $\phi_n = \sum_{k=0}^{\infty} a_{n,k}\psi_k$, $\phi = \sum_{k=0}^{\infty} a_k\psi_k$. Since $\phi_n \to \phi$ in $\mathscr{A}$ $\forall l \in N$, $\sum_{k=0}^{\infty} |a_{n,k} - a_k|^2 |\lambda_k|^{2l} \to 0$ as $n \to \infty$. $\forall l \in N$, $\|\mathscr{U}^l(D\phi_n - D\phi)\|_{L^2} \leqq \sum_{k=0}^{\infty} |a_{n,k} - a_k| \, \|\mathscr{U}^l D\psi_k\|_{L^2(I)} \leqq M_1M_0\left\{\sum_{k=0}^{\infty} |a_{n,k} - a_k|^2 |\lambda_k|^{2(s+l+s_0)}\right\}^{1/2} \to 0$ as $n \to \infty$. Hence $D\phi_n \to D\phi$ in $\mathscr{A}$ as $n \to \infty$. This proves that the mappings $D: \mathscr{A} \to \mathscr{A}$, $D: \mathscr{A}' \to \mathscr{A}'$ are continuous.

(b) Let $\forall k \in N$, $0 \leqq k \leqq m$, $\|D^k\psi_n\|_{X\cap X^*} \leqq M_1 |\lambda_n|^{s_k}$, ($s_k \in P$ depending only on k; M_1, M_2 constants >0). For $\phi \in \mathscr{A}$, $\|D^k\phi\|_{X\cap X^*} \leqq \sum_{n=0}^{\infty} |\langle \phi, \psi_n\rangle| \times \|D^k\psi_n\|_{X\cap X^*} \leqq M_1M_0 \|\mathscr{U}^{s_k+s_0}\phi\|_{L^2} < \infty$, $0 \leqq k \leqq m$. Thus $(-1)^kD^k: \mathscr{A} \to X$, $(-1)^kD^k: \mathscr{A} \to X^*$ are continuous. Hence (b) follows.

(d) By steps similar to those in the proof of (b), we can show, $\|\mathscr{U}^{k_0}D^k\phi\|_{X\cap X^*} \leqq \text{Const}\, \|\mathscr{U}^{s_{k,k_0}+s_0}\phi\|_{L^2(I)}$, $0 \leqq k \leqq m$. Thus, $(-1)^k\mathscr{U}^{k_0}D^k: \mathscr{A} \to X$, $(-1)^k\mathscr{U}^{k_0}D^k: \mathscr{A} \to X^*$ are continuous. Hence $D^k\mathscr{U}^{k_0}: X + X^* \to \mathscr{A}'$, $0 \leqq k \leqq m$ is continuous.

(c) (i) $\forall \phi \in \mathscr{A}$, $\|\phi\|_{X\cap X^*} \leqq \text{Const}\, \|\phi\|_{X\cap Y} \leqq \text{Const}\, \|\mathscr{U}^{s+s_0}\phi\|_{L^2(I)} < \infty$. This gives $\mathscr{A} \subset X \cap Y$. Since $\text{Cl}(\mathscr{D}(I), X) = X$, $\text{Cl}(\mathscr{D}(I), Y) = Y$, we get $\text{Cl}(\mathscr{A}, X) = X$, $\text{Cl}(\mathscr{A}, Y) = Y \Rightarrow X + X^* \subset Y^* + X^* \subset \mathscr{A}'$. Let $\|\psi_n\|_{X\cap X^*} \leqq B_1 |\lambda_n|^{\beta}$, $B_1 > 0$. Then $\|D\psi_n\|_{X\cap X^*} \leqq \sum_{q=0}^{n_1} |C_q^n| \, \|\psi_{n_q}\|_{X\cap X^*} \leqq B_1C_1C_2^s |\lambda_n|^{q_1+q_2s}$, $D^2\psi_n = \sum_{q=0}^{n_1} C_q^n D\psi_{n_q}$. This gives $\|D^2\psi_n\|_{X\cap X^*} = O(|\lambda_n|^{q_1+q_1q_2+q_2^2s})$. By similar arguments $\|D^k\psi_n\|_{X\cap X^*} = O(|\lambda_n|^{s_k})$, $s_k \in P$, depending only on $k \in N$. Hence $W^{-l}(X + X^*) \subset \mathscr{A}'$ $\forall l \in P$ by (b). $\forall k, n \in N$, we can write $D^k\psi_n = \sum_{q=0}^{N_k} C_{k,q}^n \psi_{n,k,q}$ where $N_k \in P$, depending only on k, $C_{k,q}^n$ constants, with $\sum_{q=0}^{N_k} |C_{k,q}^n| = O(|\lambda_n|^{d_k})$, $\sup_{0\leqq q \leqq N_k} |\lambda_q| = O(|\lambda_n|^{e_k})$; $d_k, e_k \in P$ depending only on k. This implies, for $\beta > 0$, $k \in P$, $\mathscr{U}^{\beta}D^k\psi_n = \sum_{k=0}^{N_k} C_{k,q}^n \lambda_{n_q}^{\beta} \psi_{n,k,q}$, $\|\mathscr{U}^{\beta}D^k\psi_n\|_{X\cap X^*} = O(|\lambda_n|^{s_{k,\beta}})$ with $s_{k,\beta} = d_k + e_k(\beta + s)$. Hence by (d), $W^{-m}(X^*_{-\beta} + X_{-\beta}) \subset \mathscr{A}'$.

(ii) The map $T: W^{+m}(X) \to \underbrace{X \times X \times \cdots \times X}_{(m+1) \text{ times}} = E$ given by $Tf = (f, Df, D^2f, \cdots, D^mf) \in E$ for $f \in W^m(X)$, is an isometry. $T^*: \underbrace{X^* \times X^* \times \cdots \times X^*}_{(m+1) \text{ times}} = E^* \to (W^m(X))^*$ is *onto* by Hahn Banach theorem. Suppose, for some $n_0 \in N$, $\psi_{n_0} \notin W^{m,0}(X)$. Since $\mathscr{A} \subset W^m(X)$, there exists $\ell' \in (W^m(X))^*$ with $\langle \ell', \psi_{n_0}\rangle \neq 0$, $\langle \ell', \phi\rangle = 0$, $\forall \phi \in W^{m,0}(X)$. Since T^* is onto, $\ell' = T^*(\ell_0, \ell_1, \cdots, \ell_m)$ with $\ell_i \in X^*$, $0 \leqq i \leqq m$. Define $v = \sum_{j=0}^{m} (-1)^jD^j\ell_j$. Now $v \in W^{-m}(X^*)$, $\langle v, \phi\rangle = \left\langle \sum_{j=0}^{m} (-1)^jD^j\ell_j, \phi\right\rangle = \sum_{j=0}^{m} \langle \ell_j, D^j\phi\rangle = \langle (\ell_0, \ell_1, \cdots, \ell_m), T\phi\rangle = \langle \ell', \phi\rangle = 0$ $\forall \phi \in \mathscr{D}(I)$, $v = 0$ in $W^{-m}(X^*) \subset \mathscr{A}'$. Hence $\langle v, \psi_k\rangle = 0$ $k \in N$.

But $\langle v, \psi_{n_0}\rangle = \left\langle \sum_{j=0}^{m} (-1)^j D^j \ell_j, \psi_{n_0}\right\rangle = \sum_{j=0}^{m} \langle \ell_j, D^j \psi_{n_0}\rangle = \langle (\ell_0, \ell_1, \cdots, \ell_m), T\psi_{n_0}\rangle = \langle \ell', \psi_{n_0}\rangle \neq 0$. This leads to contradiction. Hence $\psi_n \in W^{m,0}(X)$ $\forall n \in \boldsymbol{N}$. $\mathscr{A} \subset W^{m,0}(X)$ since, for $\phi \in \mathscr{A}$, $\phi_n = \sum_{k=0}^{n} \langle \phi, \psi_k\rangle \psi_k \in W^{m,0}(X)$, $\phi_n \to \phi$ as $n \to \infty$ in $W^m(X)$-norm and $W^{m,0}(X)$ is norm closed subset of $W^m(X)$. Since $\mathscr{D}(I) \subset \mathscr{A} \subset W^{m,0}(X)$, $\mathrm{Cl}(\mathscr{A}, W^{m,0}(X)) = W^{m,0}(X)$. This implies $\mathrm{Cl}([\{\psi_n\}], W^{m,0}(X)) = W^{m,0}(X)$.

6. Applications. In this section we illustrate our main results of this paper by means of classical summability methods and classical orthonormal functions.

6.1. First of all we give examples of spaces $X \in \mathscr{F}(m, \delta)$ or $Q(m)$ $m \in P$, $\delta > 0$. Suppose $\forall f \in L^1(I) + L^\infty(I)$, $D^k f \in \mathscr{A}'$, $0 \leqq k \leqq m$, and $\mathscr{A} \subset L^1(I) \cap L^\infty(I)$. Then $\mathrm{Cl}([\{\psi_n\}], X) = X$ where $X =$ any one of $L^p(I)$, $1 \leqq p < \infty$ or $C_0(I)$. For $\delta > 0$, $m \in P$, let $P_{m,\delta}$ denote the set $\{p \mid 1 < p < \infty, \{\nu_{k,\delta}\}_{k\in N} \in M(L^p), \forall f \in (L^p_{-\delta} + L^{p'}_{-\delta}), D^k f \in \mathscr{A}', 0 \leqq k \leqq m\}$. Then $\forall p \in P_{m,\delta}$ $L^p \in \mathscr{F}(m, \delta)$ and L^p is reflexive. $L^1(I)$, $C_0(I) \in Q(m)$ and $Q(m, \delta, L^1) \supset \bigcup_{p \in P_{m,\delta}} Y(m, \delta, L^p)$; $Y(m, \delta, L^p) \supset \{L^q(I) \mid p \leqq q \leqq p'\}$ $(p \in P_{m,\delta})$. Here $C_0(I) = C(I)$ if I is finite interval.

For a Banach subspace X of $\mathscr{S}'(\boldsymbol{R})$ let $X^\wedge =$ the set of $f \in \mathscr{S}'$, such that, $f =$ distributional Fourier transform of some $g_f \in X$. $X^\wedge$ is a Banach space under the norm $\|f\|_{X^\wedge} = \|g_f\|_X$; $(X^\wedge)^* = (X^*)^\wedge$ if $\mathrm{Cl}(\mathscr{S}(\boldsymbol{R}), X) = X$. For $I = R$, $m \in P$, $\delta > 0$, $1 < p < \infty$, $L^{p,\wedge} \in F(m, \delta)$ and $L^{p,\wedge}$ is reflexive. $(L^1(\boldsymbol{R}))^\wedge$, $(C_0(\boldsymbol{R}))^\wedge \in Q(m)$ $\forall m \in P$. For more details about $L^{p,\wedge}$ spaces see Katznelson [22]. $L^{p,q}(\boldsymbol{R}) \in \mathscr{F}(m, \delta)$ $m \in P$, $\delta > 0$, $1 < p < \infty$, $1 < q < \infty$.

6.2. Examples of Multiplier Operators. Here we like to give examples of multiplier type approximation processes satisfying Jackson and Bernstein tpye inequalities on a Banach subspace of $\mathscr{A}'$. Let $g_\delta(v) =$ any one of the functions $r_{\delta,\mu}(v)$ $\mu \geqq 1$, $w_\delta(v)$, $C_\delta(v)$, $\delta > 0$, $v \geqq 0$, *where* $r_{\delta,\mu}(v) = \begin{cases} (1 - v^\delta)^\mu & \text{if } 0 \leqq v \leqq 1 \\ 0 & \text{if } v > 1 \end{cases}$, $w_\delta(v) = e^{-v^\delta}$, $C_\delta(v) = \dfrac{1}{1 + v^\delta}$. Then $g_\delta(v)$, $v^\delta g_\delta(v)$, $\dfrac{1 - g_\delta(v)}{v^\delta}$ are quasi convex $C_0(0, \infty)$ functions. [see [14]]. Let $Z \in \mathscr{F}(m, \delta)$ be reflexive space (resp. $Z \in Q(m)$) $m \in P$. Let $\left\| \sum_{k=0}^{n} \left(1 - \dfrac{k}{n+1}\right) \langle f, \psi_k\rangle \psi_k \right\|_Z \leqq C \|f\|_Z$ ($f \in Z$, C independent of n). Let $\lambda_k = (k + b)^s$, $s > 0$, $b \geqq 0$. $\rho_\delta(n) = \lambda_{n+1}^{-\delta} = (n + 1 + b)^{-\delta s}$. Let $\gamma_{n,\delta,k} = g_\delta\left(\dfrac{\lambda_k}{\lambda_{n+1}}\right)$. Then by a result of Trebels [30, Theorem 3.9, p. 30] [also ref. [16,I]] we obtain $\{\gamma_{n,\delta,k}\}$, $\left\{\dfrac{1 - \gamma_{n,\delta,k}}{\lambda_k^\delta \lambda_{n+1}^{-\delta}}\right\}$, $\{\rho_\delta(n) \lambda_k^{-\delta} \gamma_{n,\delta,k}\}_{k \in N, n \in P} \in UM(Z)$. This implies that if $\Gamma_n f \sim \sum_{k \in P} \gamma_{n,\delta,k} \langle f, \psi_k\rangle \psi_k$ $(f \in Z)$ *then* $\{\Gamma_n\}_{n \in P} \subset [Z]$ satisfies both Jackson and Bernstein-type inequalities on Z with respect to Z_δ of order $\rho_\delta(n)$. Further $\frac{1-\gamma_{n,\delta,k}}{\rho_\delta(n)} \to c\lambda_k^\delta$ $(n \to \infty)$ $\forall$ fixed $k \in N$ (c a constant $\neq 0$). Hence, using the results of [17, 18] and those of this paper, one can obtain saturation and inverse results for various $\{\Gamma_n\}$ as given above.

6.3. Finally, let us give examples of orthonormal functions $\{\psi_n\}$, corresponding spaces $\mathscr{A}$, $\mathscr{A}'$, in terms of classical orthonormal functions. Let $\sigma_n(f) = \sum_{k=0}^{n} (1 - k/(n+1))\langle f, \psi_k\rangle\psi_k$ $(f \in \mathscr{A}', n \in P)$.

I. *Hermite functions*: $I = (-\infty, \infty)$, $X =$ any one of $L^p(-\infty, \infty)$, $1 < p < \infty$ or $C_0(-\infty, \infty)$. $\mathscr{U} = -e^{x^2/2}\frac{d}{dx}e^{-x^2}\frac{d}{dx}e^{x^2/2} = -D^2 + x^2 - 1$. $\psi_n(x) = \frac{e^{-x^2/2}H_n(x)}{[2^n n!\, \pi^{1/2}]^{1/2}}$, $n \in N$, with $H_n(x) =$ Hermite polynomial of order n. $\lambda_n = 2n$, $n \in N$. $\lambda_0 = 0$. Hence $\Lambda = \{ce^{-x^2/2} \mid c \in \boldsymbol{R}\}$, $\mathscr{A} = \mathscr{S}$, $\mathscr{A}' = \mathscr{S}'$ [36, 37]. (i) $\forall f \in X$, $\sup_{n \in P} \|\sigma_n(f)\|_X < \infty$ [see [25]], (ii) $\frac{d}{dx}\psi_n(x) = -\sqrt{\frac{n}{2}}\psi_{n-1} + \sqrt{\frac{n+1}{2}}\psi_{n+1}$, (iii) $\|\psi_n\|_{X \cap X^*} = O(n^{1/4})$, (iv) $\|\mathscr{U}^k D\psi_n\|_{L^2} = O(\lambda_n^{k+1})$, $k \in P$, (v) $\forall \delta > 0$, $\{\nu_{k,\delta}\}_{k \in N} \in M(X)$.

II. *Laguerre functions* ($\alpha = 0$ *case*): $I = [0, \infty)$, $X =$ any one of $L^p[0, \infty)$, $1 \leqq p < \infty$, or $C_0[0, \infty)$. $\mathscr{U} = -e^{+x/2}\frac{d}{dx}e^{-x}\frac{d}{dx}e^{x/2} = -xD^2 + D + \frac{x}{4} - \frac{1}{2}$, $\psi_n(x) = e^{-x/2}\sum_{m=0}^{n}\binom{n}{m}\frac{(-x)^m}{m!}$, $\lambda_n = n$, $n \in N$. (i) $\lambda_0 = 0$, $\Lambda = \{ce^{-x/2} \mid c \in \boldsymbol{R}\}$, (ii) $\forall f \in X$, $\sup_{n \in P}\|\sigma_n(f)\|_X < \infty$ (see [25]), (iii) $\frac{d}{dx}\psi_n(x) = -\frac{1}{2}\psi_n - \sum_{k=0}^{n-1}\psi_k(x)$, $\|\psi_n\|_{X \cap X^*} = O(n)$, $\|\mathscr{U}^k D\psi_n\|_{L^2(0,\infty)} = O(n^{k+1})$, $\forall \delta > 0$, $\{\nu_{k,\delta}\}_{k \in N} \in M(X)$.

III. *Laguerre functions* ($\alpha \neq 0$ *case*): $I = [0, \infty)$, $X =$ any one of $L^p[0, \infty)$, $C_0[0, \infty)$, $1 \leqq p < \infty$. Let $m \in P$. Let $\alpha > 2m - 1$, α, m fixed. $\mathscr{U}_\alpha = -x^{-\alpha/2}e^{x/2}\frac{d}{dx}e^{-x}x^{\alpha+1}\frac{d}{dx}e^{x/2}x^{-\alpha/2} = -\left[xD^2 + D - \frac{x}{4} + \frac{\alpha^2}{4x} + \frac{\alpha+1}{2}\right]$; $\psi_n^{(\alpha)}(x) = \left[\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right]^{1/2} x^{\alpha/2}e^{-x/2}L_n^{(\alpha)}(x)$ with $\{L_n^{(\alpha)}(x)\}_{n \in N}$ are generalized Laguerre polynomials, $\lambda_n = n$. (i) $\lambda_0 = 0$, $\Lambda = \{cx^{\alpha/2}e^{-x/2} \mid c \in \boldsymbol{R}\}$, (ii) $\forall f \in X$, $\sup_{n \in P}\|\sigma_n(f)\|_X < \infty$ [see [25]], (iii) $\|\psi_n\|_{X \cap X^*} = O(n)$, (iv) $\frac{d}{dx}\psi_n^{(\alpha)} = \frac{\alpha}{2}\sum_{k=0}^{n}\sum_{l=0}^{k}\left[\frac{n!}{\Gamma(n+\alpha+1)}\frac{\Gamma(l+\alpha+1)}{l!}\right]^{1/2}\psi_k^{(\alpha-2)} - \frac{1}{2}\psi_n^{(\alpha)}(x) - \sum_{k=0}^{n-1}\left[\frac{n!}{k!}\frac{\Gamma(k+\alpha+1)}{\Gamma(n+\alpha+1)}\right]^{1/2}\psi_k^{(\alpha)}$, (v) $\|\mathscr{U}_\alpha^k D\psi_n^{(\alpha)}\|_{L^2[0,\infty)} = O(n^{k+2})$, $0 \leqq k \leqq m$; $\forall \delta > 0$, $\{\nu_{k,\delta}\}_{k \in N} \in M(X)$.

IV. *Legendre functions*: $I = (-1, 1)$, $X =$ any one of $L^p(-1, 1)$,

$1 \leqq p < \infty$ or $C(-1, 1)$. $\mathscr{U} = \frac{d}{dx}(x^2-1)\frac{1}{dx} - \frac{1}{4}$, $\psi_n(x) = \sqrt{n+\frac{1}{2}}P_n(X)$, $P_n(x) =$ Legendre polynomial of degree n. $\lambda_n = \left(n+\frac{1}{2}\right)^2$, $\Lambda = \{0\}$. (i) $\forall f \in X$, $\|\sigma_n(f)\|_X < \infty$ [see [4]]. (ii) $\psi_n'(x) = \sum_{k=1}^{[(n+1)/2]}\left[\frac{n+1-2k}{\sqrt{2n+7/2-4k}}\right]\psi_{2n-4k+3}(x)$. (iii) $\|\mathscr{U}^k D\psi_n\|_{L^2(-1,1)} = O(\lambda_n^{k+1})$, $k \in P$. (iv) $\forall \beta > 0$, $\left\{\left(k+\frac{1}{2}\right)^{-2\beta}\right\} \in M(X)$, $k \in N$.

V. *Jacobi functions*: $I = (-1,1)$, $m \in P$. Let $\kappa > 0$. Let $\kappa_0 = \kappa$ if $\kappa \in P$, $\kappa_0 = [\kappa] = 1$ otherwise. Let $\alpha > 2(m+\kappa_0)+1$, $\beta > 2(m+\kappa_0)+1$, m, κ, α, β all fixed. $W_{\alpha,\beta} = (1-x)^\alpha(1+x)^\beta$, $\mathscr{U}^{\alpha,\beta} = \frac{1}{\sqrt{W_{\alpha,\beta}}}\frac{d}{dx}(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}\frac{1}{\sqrt{W_{\alpha,\beta}}} + \frac{(\alpha+\beta+1)^2}{4}$, $P_n^{(\alpha,\beta)} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^{n+m}$ are Jacobi polynomials [30]. $\psi_n^{(\alpha,\beta)} = \sqrt{W_{\alpha,\beta}(x)}\frac{P_n^{(\alpha,\beta)}}{\sqrt{h_n^{(\alpha,\beta)}}}$ where $h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(\beta+n+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$; $\lambda_{n,\alpha,\beta} = \left[n+\left(\frac{\alpha+\beta+1}{2}\right)\right]^2$. Let $X =$ any one of $L^p(-1,1)$, $1 \leqq p < \infty$ or $C(-1,1)$. Then, by direct computation, it can be shown that (i) $\|D^k\psi_n^{(\alpha,\beta)}\|_{X\cap X^*} = O(|\lambda_{n,\alpha,\beta}|^{s_k})$, $\|\mathscr{U}^{k_0}D^k\psi_n^{(\alpha,\beta)}\|_{X\cap X^*} = O(\lambda_{n,\alpha,\beta}^{l_k})$, $0 \leqq k \leqq m$. s_k, $l_k \in P$ depending only on k. (ii) $\Lambda = \{0\}$, (iii) If $P_{\sigma,\alpha,\beta} = \{p \mid 1 < p < \infty$, $\forall f \in L^p(-1,1)$, $\sup_{n\in P}\|\sigma_n(f)\|_{L^p} < \infty\}$ then $\left(\frac{4}{3}, 4\right) \subset P_{\sigma,\alpha,\beta}$ [see [31]] and $\forall \delta > 0$, $\left\{\left(k+\left(\frac{\alpha+\beta+1}{2}\right)\right)^{-2\delta}\right\}_{k\in N} \in M(L^p(-1,1))$, $\forall p \in P_{\sigma,\alpha,\beta}$.

VI. *Trigonometric functions (first form)*: $X =$ any one of $L^p(-\pi,\pi)$, $1 \leqq p < \infty$ or $C(-\pi,\pi)$. $\mathscr{U} = i^{-1/2}\frac{d}{dx}i^{1/2} = -iD$, $\psi_n(X) = \frac{e^{inx}}{\sqrt{2\pi}}$, $\lambda_n = n$, $(n \in \boldsymbol{Z})$. $\Lambda = \{0\}$, $\|\mathscr{U}^k D\psi_n\|_{L^2(I)} = O(n^{k+1})$, $k \in P$. $\forall \beta > 0$, $\{\nu_{k,\beta}\}_{k\in N} \in M(X)$.

Second form: $I = (0,\pi)$, $\mathscr{U} = -D^2$, $\psi_n(x) = \sqrt{\frac{2}{\pi}}\cos nx$, $\lambda_n = n^2$, $\lambda_0 = 0$, $\Lambda =$ constants. $\|\mathscr{U}^k\psi_n'(x)\|_{L^2(0,\pi)} = O(n^{2k+1})$, $k \in P$. $\beta > 0$, $\{\nu_{k,\beta}\}_{k\in N} \in M(X)$.

Third form: $I = (0,\pi)$, $\mathscr{U} = -D^2$, $\psi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$, $\lambda_n = n^2$ $(n \in N)$.

The results of this paper hold true if, instead of taking $\delta > 0$ in the Definitions 3.1–3.4 and $\beta > 0$ in the Theorems 3.1, 3.2, we take $\delta > \delta_0 > 0$, $\beta > \delta_0 > 0$ there, for some fixed constant $\delta_0 > 0$ depending only on $\{\psi_n\}$. In this case we can cite orthonormal functions constructed through Bessel functions as examples.

VII. *Bessel functions* (*First form*)

$$I = (0, 1),\ \mathscr{U} = -S_\mu = -x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2},\ \mu \geqq -1$$

$$\psi_n(x) = \frac{\sqrt{2x}J_\mu(y_{\mu,n}x)}{J_{\mu+1}(y_{\mu,n})} \qquad n = 1, 2, 3, \cdots$$

where $J_\mu(x)$ is the μ-th order Bessel function of first kind and the $y_{\mu,n}$ denote all the positive roots of $J_\mu(y) = 0$ with

$$0 < y_{\mu,1} < y_{\mu,2} < y_{\mu,3} \cdots;\ \lambda_n = y^2_{\mu,n} \qquad n = 1, 2, 3, \cdots .$$

Using the inequality $J^2_{\mu+1}(y_{\mu,n}) > B_2(y_{\mu,n})^{-1}$, ($B_2 > 0$ a constant) [see Wing [33, Relation 6.2]] we can prove $\left\|\left(\frac{d}{dx}\right)^k \psi_n\right\|_{L^1\cap L^\infty} = O(\lambda_n^{s_k})$ ($k \in P$, $s_k \in P$ independent of $n \in P$).

Wing [33] has shown that $\{\psi_n\}$ forms a Schauder basis in $L^p(0, 1)$ $1 < p < \infty$ for $\mu \geqq -1/2$ and Benedek and Panzone [7] have extended this result to $-1 < \mu < -1/2$ provided $\frac{1}{\mu + 3/2} < p < \frac{1}{(-\mu - 1/2)}$. Further $\sum_{n=1}^\infty \frac{1}{y^{2\delta}_{\mu,n}} < \infty$ ($\delta \in P$) [see Watson [32, p. 502]]. By these results we have, for $\delta \geqq 1$ $\{\lambda_k^{-\delta}\} \in M(X)$ $X = L^p(0, 1)$ with $1 < p < \infty$ if $\mu \geqq -1/2$ and $\frac{1}{\mu + 3/2} < p < \frac{1}{(-\mu - 1/2)}$ if $-1 < \mu < -1/2$.

Bessel functions (*Second form*)

$I = (0, 1)$. Let $\mu \geqq -1/2$. Let a be a real number $a > |\mu|$.

$$\mathscr{U} = S_\mu = -x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2} + a^2 - \mu^2$$

$$\psi_n(x) = \sqrt{\frac{2x}{h_n}}J_\mu(z_{\mu,n}x) \qquad n = 1, 2, 3, \cdots$$

where the $z_{\mu,n}$ denote all the positive roots of

$$zJ^{(1)}_\mu(z) + aJ_\mu(z) = 0$$

with $0 < z_{\mu,1} < z_{\mu,2} < z_{\mu,3} \cdots$. Here $J^{(1)}_\mu(z) = \frac{d}{dz}(J_\mu(z))$. Also $h_n = [J^{(1)}_\mu(z_{\mu,n})]^2 + \left[1 - \frac{\mu^2}{z^2_{\mu,n}}\right][J_\mu(z_{\mu,n})]^2$. We have $\left\|\left(\frac{d}{dx}\right)^k \psi_n\right\|_{L^1\cap L^\infty} = O(\lambda_n^{s_k})$ ($k \in P$, $s_k \in P$ independent of n). $\sum_{n=1}^\infty \frac{1}{z^2_{\mu,n} + a^2 - \mu^2} \leqq \frac{1}{2(a + \mu)} < \infty$ [see Lamb [23, p. 273]]. Further $\{\psi_n\}$ forms a Schauder basis in $L^p(0, 1)$, $1 < p < \infty$. See Wing [33]. These results imply that $\{\lambda_n^{-\delta}\} \in M(L^p)$, $1 < p < \infty$, $\delta \geqq 1$.

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