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ON DEFORMATIONS OF AUTOMORPHISM GROUPS OF COMPACT COMPLEX MANIFOLDS

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Introduction. By *an analytic space*, we mean a reduced, Hausdorff, complex analytic space. By *a complex fiber space*, we mean a triple (X, π, S) of analytic spaces X and S and a holomorphic map π of X onto S. By *a family of complex manifolds*, we mean a complex fiber space (X, π, S) such that there are an open covering $\{X_\alpha\}_{\alpha \in A}$ of X, open sets $\{\Omega_\alpha\}_{\alpha \in A}$ of C^n, an open covering $\{S_\alpha\}_{\alpha \in A}$ of S and holomorphic isomorphisms

$$\eta_\alpha : X_\alpha \to \Omega_\alpha \times S_\alpha$$

such that the diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\quad \eta_\alpha \quad} & \Omega_\alpha \times S_\alpha \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}} & \\ & S_\alpha & \end{array}$$

is commutative for each $\alpha \in A$. By the definition, each fiber $\pi^{-1}(s)$, $s \in S$, is a complex manifold. S is called *the parameter space of the family*. If, moreover, π is a proper map, we say that (X, π, S) is *a family of compact complex manifolds*. In this case, each fiber is a compact complex manifold.

Let V be a compact complex manifold. We denote by $\mathrm{Aut}(V)$ the group of automorphisms (holomorphic isomorphisms onto itself) of V. It is well known that $\mathrm{Aut}(V)$ is a complex Lie group (Bochner-Montgomery [1]).

The purpose of this paper is to prove the following theorem.

MAIN THEOREM. *Let (X, π, S) be a family of compact complex manifolds. We assume that S satisfies the second axiom of countability. Then the disjoint union*

$$A = \coprod_{s \in S} \mathrm{Aut}(\pi^{-1}(s))$$

admits an analytic space structure such that (1) *(A, λ, S) is a complex fiber space where $\lambda : A \to S$ is the canonical projection,* (2) *the map*

$$X \times_S A \to X$$

defined by

$$(P, f) \to f(P)$$

is holomorphic, where

$$X \times_S A = \{(P, f) \in X \times A \mid \pi(P) = \lambda(f)\},$$

the fiber product of X and A over S, (3) *the map*

$$S \to A$$

defined by

$$s \to I_s$$

is holomorphic, where I_s is the identity map of $\pi^{-1}(s)$, and (4) *the map*

$$A \times_S A \to A$$

defined by

$$(f, g) \to g^{-1}f$$

is holomorphic, where

$$A \times_S A = \{(f, g) \in A \times A \mid \lambda(f) = \lambda(g)\},$$

the fiber product of A and A over S.

The method of the proof of Main Theorem is based on those of [8] and [9], ideas of which are essentially due to Kuranishi's [6].

If we put $S =$ one point, our proof of Main Theorem gives a new proof of the above theorem of Bochner-Montgomery. In this case, $A = \mathrm{Aut}(V)$ has no singular point, for it is homogeneous. In general cases, A may admit singular points, even if S has no singular point. This is naturally expected, because dimensions of automorphism groups vary upper semicontinuously on parameters [5]. In the case of the family of Hopf surfaces, we have shown Main Theorem by direct calculations [10]. In this case, A admits singular points.

Main Theorem was conjectured by Professor Heisuke Hironaka. I express my thanks to him for his proposal of the problem, his comments and his encouragement.

1. Maximal families of holomorphic maps—Theorem 1. Let (X, π, S) be a family of complex manifolds. Let T be an analytic space. Let b be a holomorphic map of T into S. We put

$$b^*X = X \times_S T = \{(P, t) \in X \times T \mid \pi(P) = b(t)\}$$

and $b^*\pi =$ the restriction of the projection

$$X \times T \to T \quad \text{to} \quad b^*X .$$

Then it is easy to see that $(b^*X, b^*\pi, T)$ is a family of complex manifolds. Each fiber $(b^*\pi)^{-1}(t)$ is written as $\pi^{-1}(b(t)) \times t$. We sometimes identify $(b^*\pi)^{-1}(t)$ with $\pi^{-1}(b(t))$.

DEFINITION 1.1. Let (X, π, S) be a family of compact complex manifolds. Let (Y, μ, S) be a family of complex manifolds with the same parameter space S. Let T be an analytic space. A triple (E, T, b) is called *a family of holomorphic maps of* (X, π, S) *into* (Y, μ, S) if and only if (1) b is a holomorphic map of T into S and (2) E is a holomorphic map of b^*X into b^*Y such that the diagram

$$\begin{array}{ccc} b^*X & \xrightarrow{\quad E \quad} & b^*Y \\ & {\scriptstyle b^*\pi}\searrow \quad \swarrow{\scriptstyle b^*\mu} & \\ & T & \end{array}$$

is commutative.

T is called *the parameter space of* (E, T, g).

REMARK. For each $t \in T$, $(b^*\pi)^{-1}(t)$ and $(b^*\mu)^{-1}(t)$ are identified with $\pi^{-1}(b(t))$ and $\mu^{-1}(b(t))$ respectively. Thus we may consider (E, T, b) to be a collection $\{E_t\}_{t \in T}$ of holomorphic maps

$$E_t \colon \pi^{-1}(b(t)) \to \mu^{-1}(b(t)) .$$

DEFINITION 1.2. Let (X, π, S) and (Y, μ, S) be as above. A family (E, T, b) of holomorphic maps of (X, π, S) into (Y, μ, S) is said to be *maximal at a point* $t \in T$ if and only if, for any family (G, R, h) of holomorphic maps of (X, π, S) into (Y, μ, S) with a point $r \in R$ such that $b(t) = h(r)$ and

$$E_t = G_r \colon \pi^{-1}(b(t)) \to \mu^{-1}(b(t)) ,$$

there are an open neighborhood U of r in R and a holomorphic map

$$k \colon U \to T$$

such that

(1) $k(r) = t$,

(2) $bk = h$ and

(3) $G_q = E_{k(q)} \colon \pi^{-1}(h(q)) \to \mu^{-1}(h(q))$ for all $q \in U$.

A maximal family is a family which is maximal at every point of its parameter space.

THEOREM 1. *Let* (X, π, S) *be a family of compact complex manifolds. Let* (Y, μ, S) *be a family of complex manifolds with the same parameter* *space S. Let o be a point of S. Let f be a holomorphic map of $\pi^{-1}(o)$ into $\mu^{-1}(o)$. Then there exists a maximal family (E, T, b) of holomorphic maps of (X, π, S) into (Y, μ, S) with a point $t_o \in T$ such that*

(1) $b(t_o) = o$ *and*

(2) $E_{t_o} = f: \pi^{-1}(o) \to \mu^{-1}(o)$.

REMARK. Theorem 1 corresponds to Theorem of [9]. In fact, Theorem 1 is essentially reduced to Theorem of [9], if we consider the graph Γ_f of f. However, in order to prove Main Theorem, we need the concrete construction of the analytic space T. So we prove Theorem 1 in the sequel. The method is thus similar to that of [9].

2. **Banach spaces $C^p(F, |\ |)$.** In this section, we refer some results of §2 of [8], which will be used in the sequel. Let V be a compact complex manifold. Let F be a holomorphic vector bundle on V. Let $\{U_i\}_{i \in I}$ be a finite open covering of V such that (1) the closure $\bar{U}_i$ is contained in an open set $\tilde{U}_i$ having a local coordinate system

$$(z_i) = (z_i^1, \cdots, z_i^d),$$

(2) $U_i = \{z_i \in \tilde{U}_i \mid |z_i| < 1\}$, where

$$|z_i| = \max\{|z_i^1|, \cdots, |z_i^d|\} \quad \text{and}$$

(3) F is trivial on U_i.

Let e, $0 < e < 1$, be a small positive number such that the open sets U_i^e of V defined by

$$U_i^e = \{z_i \in U_i \mid |z_i| < 1 - e\}$$

again cover V.

We define additive groups $C^p(F)$, $p = 0, 1, \cdots$, as follows. An element $\xi = \{\xi_{i_0 \cdots i_p}\} \in C^p(F)$ is a function which associates to each $(p+1)$-ple $(i_0, \cdots, i_p)$ of indices in I a holomorphic section $\xi_{i_0 \cdots i_p}$ of F on $U_{i_0}^e \cap \cdots \cap U_{i_{p-1}}^e \cap U_{i_p}$. In particular, an element $\xi = \{\xi_i\} \in C^0(F)$ is a function which associates to each index $i \in I$ a holomorphic section ξ_i of F on U_i. We define the coboundary map

$$\delta: C^p(F) \to C^{p+1}(F)$$

by

$$(\delta\xi)_{i_0 \cdots i_{p+1}}(z) = \sum_\nu (-1)^\nu \xi_{i_0 \cdots i_{\nu-1} i_{\nu+1} \cdots i_{p+1}}(z)$$

for $z \in U_{i_0}^e \cap \cdots \cap U_{i_p}^e \cap U_{i_{p+1}}$. Then it is easy to see that $\delta^2 = 0$.

We introduce a norm $|\ |$ in $C^p(F)$. For each $\xi = \{\xi_{i_0 \cdots i_p}\} \in C^p(F)$, we define $|\xi|$ by

$$|\xi| = \sup\{|\xi^{\lambda}_{i_0\cdots i_p}(z)|\,|\lambda = 1, \cdots, r,\ z \in U^{\varepsilon}_{i_0} \cap \cdots \cap U^{\varepsilon}_{i_{p-1}} \cap U_{i_p}, (i_0, \cdots, i_p) \in I^{p+1}\},$$

where $\xi^{\lambda}_{i_0\cdots i_p}$ is the representation of the component $\xi_{i_0\cdots i_p}$ of ξ with respect to the local trivialization of F on U_{i_0}. In particular, we define $|\xi|$ for $\xi \in C^0(F)$ by

$$|\xi| = \sup\{|\xi^{\lambda}_{i}(z)|\,|\lambda = 1, \cdots, r, i \in I, z \in U_i\},$$

where ξ^{λ}_{i} is the representation of ξ_i with respect to the local trivialization of F on U_i. We note that we denoted $|\ |_e$ in [8] instead of $|\ |$.

We put

$$C^p(F, |\ |) = \{\xi \in C^p(F)\,|\,|\xi| < +\infty\}.$$

It is easy to see that $C^p(F, |\ |)$ is a Banach space and the coboundary map δ maps $C^p(F, |\ |)$ continuously into $C^{p+1}(F, |\ |)$. We put

$$Z^p(F, |\ |) = \{\xi \in C^p(F, |\ |)\,|\,\delta\xi = 0\},$$
$$B^p(F, |\ |) = (\delta C^{p-1}(F)) \cap C^p(F, |\ |) \quad \text{and}$$
$$H^p(F, |\ |) = Z^p(F, |\ |)/B^p(F, |\ |),$$

for $p = 0, 1, \cdots$. It is clear that $H^0(F, |\ |)$ is canonically isomorphic to the 0-th cohomology group $H^0(V, F)$ of F.

By Lemmas 2.3 and 2.4 of [8], there are continuous linear maps

$$E_1: B^2(F, |\ |) \to C^1(F, |\ |) \quad \text{and}$$
$$E_0: B^1(F, |\ |) \to C^0(F, |\ |)$$

such that

$$\delta E_1 = \text{the identity map on } B^2(F, |\ |) \quad \text{and}$$
$$\delta E_0 = \text{the identity map on } B^1(F, |\ |).$$

We put

$$\Lambda = 1 - E_1\delta.$$

Then Λ is a projection map of $C^1(F, |\ |)$ onto $Z^1(F, |\ |)$.

By Lemma 2.5 of [8], $B^1(F, |\ |) = \delta C^0(F, |\ |)$ and is closed in $Z^1(F, |\ |)$. Again, by Lemma 2.5 of [8], $H^1(F, |\ |)$ is canonically isomorphic to $H^1(V, F)$, the first cohomology group of F. Thus there is a subspace $H^1(F, |\ |)$, (we use the same notation for the convenience), of $Z^1(F, |\ |)$ isomorphic to $H^1(V, F)$ such that $Z^1(F, |\ |)$ splits into a direct sum of $B^1(F, |\ |)$ and $H^1(F, |\ |)$:

$$Z^1(F, |\ |) = B^1(F, |\ |) \oplus H^1(F, |\ |).$$

Let

$$B: Z^1(F, |\ |) \to B^1(F, |\ |) \quad \text{and}$$
$$H: Z^1(F, |\ |) \to H^1(F, |\ |)$$

be the projection maps corresponding to the splitting.

3. Some lemmas. Let (X, π, S) be a family of compact complex manifolds. Let (Y, μ, S) be a family of complex manifolds with the same parameter space S. Let o be a point of S. We put

$$V = \pi^{-1}(o)$$

and

$$W = \mu^{-1}(o) .$$

Let f be a holomorphic map of V into W. We show that there are families of open sets $\{X_i\}_{i\in I}$ and $\{\tilde{X}_i\}_{i\in I}$ of X and $\{Y_i\}_{i\in I}$ and $\{\tilde{Y}_i\}_{i\in I}$ of Y, with the same *finite* set I of indices, satisfying following conditions:

(1) $X_i \Subset \tilde{X}_i$ and $Y_i \Subset \tilde{Y}_i$ for each $i \in I$ where $A \Subset B$ means that the closure $\bar{A}$ is compact and is contained in B,

(2) $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ cover V and $f(V)$ respectively,

(3) there are an open neighborhood $\tilde{S}$ of o and holomorphic isomorphisms

$$\eta_i: \tilde{X}_i \to \tilde{U}_i \times \tilde{S} \quad \text{and}$$
$$\xi_i: \tilde{Y}_i \to \tilde{W}_i \times \tilde{S}$$

such that the diagrams

$$\begin{array}{ccc} \tilde{X}_i & \xrightarrow{\eta_i} & \tilde{U}_i \times \tilde{S} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\text{proj}} & \\ & \tilde{S} & \end{array}$$

and

$$\begin{array}{ccc} \tilde{Y}_i & \xrightarrow{\xi_i} & \tilde{W}_i \times \tilde{S} \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\text{proj}} & \\ & \tilde{S} & \end{array}$$

are commutative where $\tilde{U}_i$ and $\tilde{W}_i$ are open sets in $\boldsymbol{C}^d$ and $\boldsymbol{C}^r$ respectively ($d = \dim V$, $r = \dim W$),

(4) there are an open neighborhood S' of o with $S' \Subset \tilde{S}$ and open subsets U_i and W_i of $\tilde{U}_i$ and $\tilde{W}_i$ respectively with $U_i \Subset \tilde{U}_i$ and $W_i \Subset \tilde{W}_i$ such that

$$X_i = \eta_i^{-1}(U_i \times S') \quad \text{and}$$
$$Y_i = \xi_i^{-1}(W_i \times S')$$

for each $i \in I$,

(5) there are coordinate systems

$$(z_i) = (z_i^1, \cdots, z_i^d),$$
$$(w_i) = (w_i^1, \cdots, w_i^r) \quad \text{and}$$
$$(s) = (s^1, \cdots, s^k), (o = 0),$$

in $\tilde{U}_i$, $\tilde{W}_i$ and $\tilde{\Omega}$ respectively, where $\tilde{\Omega}$ is an ambient space of $\tilde{S}$, such that

$$U_i = \{z_i \in \tilde{U}_i \mid |z_i| < 1\},$$
$$W_i = \{w_i \in \tilde{W}_i \mid |w_i| < 1\}$$

for each $i \in I$ and

$$S' = \{s \in \tilde{S} \mid |s| < 1\}$$

($|z_i| = \max\{|z_i^1|, \cdots, |z_i^d|\}$ etc.),

(6) $f(\eta_i^{-1}(U_i \times o)) \subset \xi_i^{-1}(W_i \times o)$ and
$f(\eta_i^{-1}(\tilde{U}_i \times o)) \subset \xi_i^{-1}(\tilde{W}_i \times o)$

for each $i \in I$.

Let Γ_f be the grach of the map f. Then Γ_f is a compact subset of $V \times W$. On the other hand, $V \times W$ is naturally regarded as a subset of $X \times_S Y$. Hence we regard Γ_f as a compact subset of $X \times_S Y$. Then, for each $(P, f(P)) \in \Gamma_f$, there is a neighborhood $\tilde{X}_P \times_{\tilde{S}_p} \tilde{Y}_P$ of $(P, f(P))$ in $X \times_S Y$ such that there are holomorphic isomorphisms

$$\eta_P : \tilde{X}_P \to \tilde{U}_P \times \tilde{S}_P \quad \text{and}$$
$$\xi_P : \tilde{Y}_P \to \tilde{W}_P \times \tilde{S}_P$$

such that the diagrams

$$\begin{array}{ccc} \tilde{X}_P & \xrightarrow{\quad \eta_P \quad} & \tilde{U}_P \times \tilde{S}_P \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}} & \\ & \tilde{S}_P & \end{array}$$

and

$$\begin{array}{ccc} \tilde{Y}_P & \xrightarrow{\quad \xi_P \quad} & \tilde{W}_P \times \tilde{S}_P \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}} & \\ & \tilde{S}_P & \end{array}$$

are commutative, where $\tilde{U}_P$ and $\tilde{W}_P$ are open sets in C^d and C^r respectively. $\tilde{S}_P$ is an open neighborhood of o in S. Let S_P be an open neighborhood of o in S such that $S_P \Subset \tilde{S}_P$. Let U_P and W_P be open subsets of $\tilde{U}_P$ and $\tilde{W}_P$ respectively such that $U_P \Subset \tilde{U}_P$ and $W_P \Subset \tilde{W}_P$. We put

$$X_P = \eta_P^{-1}(U_P \times S_P) \quad \text{and}$$
$$Y_P = \xi_P^{-1}(W_P \times S_P) .$$

Taking U_P and $\tilde{U}_P$ sufficiently small, we may assume that

$$f(\eta_P^{-1}(U_P \times o)) \subset \xi_P^{-1}(W_P \times o) \quad \text{and}$$
$$f(\eta_P^{-1}(\tilde{U}_P \times o)) \subset \xi_P^{-1}(\tilde{W}_P \times o) .$$

We may assume that there are coordinate systems

$$(z_P) = (z_P^1, \cdots, z_P^d) \quad \text{and}$$
$$(w_P) = (w_P^1, \cdots, w_P^r)$$

in $\tilde{U}_P$ and $\tilde{W}_P$ respectively such that

$$U_P = \{z_P \in \tilde{U}_P \mid |z_P| < 1\} \quad \text{and}$$
$$W_P = \{w_P \in \tilde{W}_P \mid |w_P| < 1\} .$$

Now we cover Γ_f by $\{X_P \times_{S_P} Y_P\}_{P \in V}$. We choose a finite subcovering

$$\{X_{P_i} \underset{S_{P_i}}{\times} Y_{P_i}\}_{i \in I} .$$

We put

$$\eta_i = \eta_{P_i} ,$$
$$\xi_i = \xi_{P_i} ,$$
$$U_i = U_{P_i} ,$$
$$\tilde{U}_i = \tilde{U}_{P_i} ,$$
$$W_i = W_{P_i} \quad \text{and}$$
$$\tilde{W}_i = \tilde{W}_{P_i} .$$

We put

$$\tilde{S} = \bigcap_{i \in I} \tilde{S}_{P_i} .$$

Let $\tilde{\Omega}$ be an ambient space of $\tilde{S}$ with a coordinate system

$$(s) = (s^1, \cdots, s^k) .$$

Let Ω be an open subset of $\tilde{\Omega}$ such that $\Omega \subset \tilde{\Omega}$. We may assume that

$$\Omega = \{s \in \tilde{\Omega} \mid |s| < 1\} .$$

We assume that o is the origin of Ω. We put

$$S' = \tilde{S} \cap \Omega .$$

We may assume that

$$S' \subset \bigcap_{i \in I} S_{P_i} .$$

We put

$$\begin{aligned} X_i &= \eta_i^{-1}(U_i \times S') , \\ \tilde{X}_i &= \eta_i^{-1}(\tilde{U}_i \times \tilde{S}) , \\ Y_i &= \xi_i^{-1}(W_i \times S') \quad \text{and} \\ \tilde{Y}_i &= \xi_i^{-1}(\tilde{W}_i \times \tilde{S}) . \end{aligned}$$

Then it is clear that $\{X_i\}_{i\in I}$, $\{\tilde{X}_i\}_{i\in I}$, $\{Y_i\}_{i\in I}$ and $\{\tilde{Y}_i\}_{i\in I}$ satisfy above conditions (1)-(6).

Henceforth, we identify $\eta_i^{-1}(U_i\times o)$, $\eta_i^{-1}(\tilde{U}_i\times o)$, $\xi_i^{-1}(W_i\times o)$ and $\xi_i^{-1}(\tilde{W}_i\times o)$ with U_i, $\tilde{U}_i$, W_i and $\tilde{W}_i$ respectively.

Now, we consider maps

$$\begin{aligned} \eta_{ik} &= \eta_i\eta_k^{-1}\colon \eta_k(\tilde{X}_i \cap \tilde{X}_k) \to \eta_i(\tilde{X}_i \cap \tilde{X}_k) , \\ \xi_{ik} &= \xi_i\xi_k^{-1}\colon \xi_k(\tilde{Y}_i \cap \tilde{Y}_k) \to \xi_i(\tilde{Y}_i \cap \tilde{Y}_k) . \end{aligned}$$

η_{ik} and ξ_{ik} can be written as

$$\begin{aligned} \eta_{ik}(z_k, s) &= (g_{ik}(z_k, s), s) \quad \text{and} \\ \xi_{ik}(w_k, s) &= (h_{ik}(w_k, s), s) , \end{aligned}$$

where

$$\begin{aligned} &g_{ik}\colon \eta_k(\tilde{X}_i \cap \tilde{X}_k) \to \tilde{U}_i \quad \text{and} \\ &h_{ik}\colon \xi_k(\tilde{Y}_i \cap \tilde{Y}_k) \to \tilde{W}_i . \end{aligned}$$

We want to extend η_{ik} and ξ_{ik} to ambient spaces of $\eta_k(X_i \cap X_k)$ and $\xi_k(Y_i \cap Y_k)$ respectively.

Let P be a point of $\bar{U}_i \cap \bar{U}_k$. Then it is clear that there is an open neighborhood $U_P \times S_P$ of $\eta_k(P)$ in $\eta_k(\tilde{X}_i \cap \tilde{X}_k)$ such that

(1) $S_P = \Omega_P \cap S'$ where Ω_P is a polydisc in C^k contained in Ω with the center o and

(2) U_P is an open neighborhood of P in V contained in $\tilde{U}_i \cap \tilde{U}_k$.

We cover $\eta_k(\bar{U}_i \cap \bar{U}_k)$ by open sets $\{U_P \times S_P\}_P$ in $\eta_k(\tilde{X}_i \cap \tilde{X}_k)$ having above conditions (1) and (2). We choose a finite subcovering

$$\{U_\lambda \times S_\lambda\}_{\lambda=1,\cdots,q}$$

from $\{U_P \times S_P\}_P$, where $U_\lambda = U_{P_\lambda}$, $S_\lambda = S_{P_\lambda} = \Omega_\lambda \cap S'$ and $\Omega_\lambda = \Omega_{P_\lambda}$. Then $\{U_\lambda\}_{\lambda=1,\cdots,q}$ covers $\bar{U}_i \cap \bar{U}_k$. Let Ω_o be a polydisc in C^k with the center o, the origin, contained in $\bigcap_\lambda \Omega_\lambda$. We put $S_o = \Omega_o \cap S'$. We may assume that

$$\Omega_o = \{s \in \Omega \mid |s| < \varepsilon_o\}$$

for a positive number ε_o, $0 < \varepsilon_o < 1$.

The proofs of Lemmas 3.1 and 3.2 below are similar to those of Lemma 3.1 and 3.2 of [9] respectively, so we omit them.

LEMMA 3.1. *There is a Stein open set U_0 of $\tilde{U}_k$ such that*

$$\bar{U}_i \cap \bar{U}_k \subset U_0 \subset \bigcup_\lambda U_\lambda \subset \tilde{U}_i \cap \tilde{U}_k .$$

LEMMA 3.2. *Let U_o be the open set of $\tilde{U}_k$ in Lemma 3.1. Let S_o be sufficiently small. Then*

$$\eta_k(X_i \cap X_k) \cap (\tilde{U}_k \times S_o) \subset U_o \times S_o .$$

Now, it is clear that

$$U_o \times S_o \subset \eta_k(\tilde{X}_i \cap \tilde{X}_k) .$$

$U_o \times S_o$ is a closed subvariety of $U_o \times \Omega_o$, which is Stein. Thus the map

$$\eta_{ik}: U_o \times S_o \rightarrow \tilde{U}_i \times S_o$$

is extended to a holomorphic map

$$\eta_{ik}: U_o \times \Omega_o \rightarrow \tilde{U}_i \times \Omega_o .$$

The extended map η_{ik} is written as follows:

$$\eta_{ik}(z_k, s) = (g_{ik}(z_k, s), s) ,$$

where

$$g_{ik}: U_o \times \Omega_o \rightarrow \tilde{U}_i$$

is an extension of the map g_{ik} above.

In a similar way, we can find a Stein open set W_0 of $\tilde{W}_k$ such that

$$\bar{W}_i \cap \bar{W}_k \subset W_o \subset \tilde{W}_i \cap \tilde{W}_k ,$$
$$W_o \times S_o \subset \xi_k(\tilde{Y}_i \cap \tilde{Y}_k) \quad \text{and}$$
$$\xi_k(Y_i \cap Y_k) \cap (\tilde{W}_k \times S_o) \subset W_o \times S_o .$$

$W_o \times S_o$ is a closed subvariety of $W_0 \times \Omega_o$, which is Stein. Hence the map

$$\xi_{ik}: W_0 \times S_o \rightarrow \tilde{W}_i \times S_o$$

is extended to a holomorphic map

$$\xi_{ik}: W_o \times \Omega_o \rightarrow \tilde{W}_i \times \Omega_o .$$

The extended map ξ_{ik} is written as follows:

$$\xi_{ik}(w_k, s) = (h_{ik}(w_k, s), s)$$

where

$$h_{ik}: W_o \times \Omega_o \to \widetilde{W}_i$$

is an extension of the map h_{ik} above.

Let e, $0 < e < 1$, be a positive number. We put

$$U_i^e = \{z_i \in U_i \mid |z_i| < 1 - e\} \quad \text{and}$$
$$W_i^e = \{w_i \in W_i \mid |w_i| < 1 - e\} .$$

LEMMA 3.3. *If e is sufficiently small, then $\{U_i^e\}_{i\in I}$ and $\{W_i^e\}_{i\in I}$ cover V and $f(V)$ respectively.*

PROOF. We prove the first half. The second half is shown in a similar way. We assume the converse. Let

$$1 > e_1 > e_2 > \cdots > 0$$

be a sequence of positive numbers converging to 0. We put

$$A_n = V - \bigcup_{i\in I} U_i^{e_n}, \quad n = 1, 2, \cdots .$$

Then A_n, $n = 1, 2, \cdots$, are non-empty, compact and satisfy

$$A_1 \supset A_2 \supset \cdots .$$

Hence

$$\bigcap_n A_n \neq \varnothing .$$

On the other hand,

$$\bigcap_n A_n = \bigcap_n \Big(V - \bigcup_{i\in I} U_i^{e_n}\Big) = \bigcap_n \Big(\bigcap_i (V - U_i^{e_n})\Big)$$
$$= \bigcap_i \Big(\bigcap_n (V - U_i^{e_n})\Big) = \bigcap_i (V - U_i) = \varnothing ,$$

a contradiction. q.e.d.

LEMMA 3.4. *If e is sufficiently small, then*

$$f(\bar{U}_i) \subset W_i^e$$

for each $i \in I$.

PROOF. We assume the converse. Let

$$1 > e_1 > e_2 > \cdots > 0$$

be a sequence of positive numbers converging to 0. We put

$$A_n = (W_i - W_i^{e_n}) \cap f(\bar{U}_i)\,, \qquad n = 1, 2\cdots .$$

Then A_n, $n = 1, 2, \cdots$, are non-empty compact subsets of W_i. Since

$$A_1 \supset A_2 \supset \cdots ,$$

we have

$$\bigcap_n A_n \neq \varnothing .$$

On the other hand,

$$\bigcap_n A_n = \Big(\bigcap_n (W_i - W_i^{e_n})\Big) \bigcap f(\bar{U}_i) = \varnothing ,$$

a contradiction. q.e.d.

Let e and e', $0 < e < e' < 1$, be small positive numbers satisfying Lemmas 3.3 and 3.4.

For any positive number ε with $0 < \varepsilon < \varepsilon_0$, we put

$$\Omega_\varepsilon = \{s \in \Omega \mid |s| < \varepsilon\} \quad \text{and}$$
$$S_\varepsilon = \Omega_\varepsilon \cap S' .$$

The proofs of Lemmas 3.5, 3.6, and 3.7 below are similar to those of Lemma 3.3, 3.4, and 3.5 of [9] respectively, so we omit them.

LEMMA 3.5. *There is a small positive number ε (independent of indices in I) with $0 < \varepsilon < \varepsilon_0$ such that if $s \in \Omega_\varepsilon$, then $g_{ik}(z_k, s)$ (resp. $h_{ik}(w_k, s)$) is defined and is a point of U_i(resp. W_i) for all $z_k \in U_i^e \cap U_k$ (resp. for all $w_k \in W_i^e \cap W_k$).*

LEMMA 3.6. *There is a small positive number ε (independent of indices in I) with $0 < \varepsilon < \varepsilon_0$ such that if $s \in S_\varepsilon$, then*

$$\eta_k^{-1}(z_k, s) \in X_i \cap X_k$$

(resp. $\xi_k^{-1}(w_k, s) \in Y_i \cap Y_k$) for all $z_k \in U_i^e \cap U_k$ (resp. for all $w_k \in W_i^e \cap W_k$).

LEMMA 3.7. *There is a small positive number ε (independent of indices in I) with $0 < \varepsilon < \varepsilon_0$ such that if $s \in S_\varepsilon$ and if*

$$\eta_k^{-1}(z_k, s) \in X_i^{e'} \cap X_k ,$$

then $z_k \in U_i^e \cap U_k$.

The set U_0 in Lemma 3.1 and the set W_0 above depend on the indices i and k. On the other hand, we may assume that ε_0 is independent of indices, for the set I of indices is a finite set. Hence we may assume that Ω_0 and S_0 are independent of indices. We write

$$U_0 = U_{0(ik)} \quad \text{and}$$
$$W_0 = W_{0(ik)} ,$$

whenever we want to distinguish them. $\eta_{jk}^{-1}(U_{o(ij)} \times \Omega_o)$ and $\xi_{jk}^{-1}(W_{o(ij)} \times \Omega_o)$ are open sets of $U_{o(jk)} \times \Omega_o$ and $W_{o(ik)} \times \Omega_0$ respectively, and contain $\bar{U}_i \cap \bar{U}_j \cap \bar{U}_k$ and $\bar{W}_i \cap \bar{W}_j \cap \bar{W}_k$ respectively. The proof of the following Lemma is similar to that of Lemma 3.7 of [9], so we omit it.

LEMMA 3.8. *There is a small positive number* ε (*independent of indices in* I) *with* $0 < \varepsilon < \varepsilon_o$ *such that if* $s \in \Omega_\varepsilon$, *then*

(1) $(z_k, s) \in \eta_{jk}^{-1}(U_{o(ij)} \times \Omega_o)$
for all $z_k \in U_i \cap U_j \cap U_k$,

(1)′ $(w_k, s) \in \xi_{jk}^{-1}(W_{o(ij)} \times \Omega_o)$
for all $w_k \in W_i \cap W_j \cap W_k$,

(2) $g_{ik}(z_k, s) \in U_i^{e/2} \cap U_j^{e/2}$
for all $z_k \in U_i^e \cap U_j^e \cap U_k$, *where*

$$U_i^{e/2} = \{z_i \in U_i \mid |z_i| < 1 - e/2\} ,$$

(2)′ $h_{ik}(w_k, s) \in W_i^{e/2} \cap W_j^{e/2}$

for all $w_k \in W_i^e \cap W_j^e \cap W_k$, *where*

$$W_i^{e/2} = \{w_i \in W_i \mid |w_i| < 1 - e/2\} .$$

Let A be a compact subset of W_k. Let ε be a small positive number. We regard W_k as a polydisc

$$W_k = \{w_k \in \boldsymbol{C}^r \mid |w_k| < 1\}$$

in $\boldsymbol{C}^r$. We consider a subset

$$A_\varepsilon = \{w_k + x_k \mid w_k \in A \quad \text{and} \quad |x_k| \leqq \varepsilon\}$$

of $\boldsymbol{C}^r$, where the summation is taken in $\boldsymbol{C}^r$. A_ε is compact, for the summation is a continuous operation. Since the proof of the following lemma is straightforward, we omit it.

LEMMA 3.9. *There is a small positive number* ε *such that* $A_\varepsilon \subset W_k$.

Since $\overline{f(U_i)}$ is a compact subset of W_i^e, $\overline{f(U_i)} \cap \overline{f(U_k)}$ is a compact subset of $W_i^e \cap W_k$, which is open in W_k. By Lemma 3.9, there is a small positive number ε such that

$$(\overline{f(U_i)} \cap \overline{f(U_k)})_\varepsilon \subset W_k .$$

Since the proof of the following lemma is straightforward, we omit it.

LEMMA 3.10. *There is a small positive number* ε (*independent of indices in* I) *such that*

$$(\overline{f(U_i)} \cap \overline{f(U_k)})_\varepsilon \subset W_i^e \cap W_k \subset W_k .$$

4. **The linear map** σ. We use the same notations as §3. Henceforth, we assume that $\tilde{S} \subset \tilde{\Omega}$ is a neat imbedding of $\tilde{S}$ at o, [3]. Thus k is equal to the dimension of the Zariski tangent space T_oS at o. We assume that $\tilde{S}$ is defined in $\tilde{\Omega}$ as common zeros of holomorphic functions

$$e_1(s), \cdots, e_m(s) .$$

It is easy to see that

(1) $e_\alpha(o) = 0, \alpha = 1, \cdots, m,$

(2) $(\partial e_\alpha/\partial s^\beta)(o) = 0, \alpha = 1, \cdots, m, \beta = 1, \cdots, k.$

In §3, we extended the maps

$$\eta_{ik} = \eta_i\eta_k^{-1}: U_o \times S_o \to \tilde{U}_i \times S_o \quad and$$
$$\xi_{ik} = \xi_i\xi_k^{-1}: W_o \times S_o \to \tilde{W}_i \times S_o$$

to

$$\eta_{ik}: U_0 \times \Omega_o \to \tilde{U}_i \times \Omega_o \quad and$$
$$\xi_{ik}: W_o \times \Omega_o \to \tilde{W}_i \times \Omega_o .$$

The extended maps η_{ik} and ξ_{ik} were written as

$$\eta_{ik}(z_k, s) = (g_{ik}(z_k, s), s) \quad and$$
$$\xi_{ik}(w_k, s) = (h_{ik}(w_k, s), s) .$$

LEMMA 4.1. *Let z_k and w_k be points of $U_i \cap U_k$ and $W_i \cap W_k$ respectively. Then the matrices*

$$(\partial g_{ik}/\partial z_k)(z_k, o) ,$$
$$(\partial g_{ik}/\partial s)(z_k, o) ,$$
$$(\partial h_{ik}/\partial w_k)(w_k, o) \quad and$$
$$(\partial h_{ik}/\partial s)(w_k, o)$$

are independent how to extend maps η_{ik} and ξ_{ik}.

PROOF. We show that $(\partial h_{ik}/\partial s)(w_k, o)$ is independent how to extend the map ξ_{ik}. Others can be shown in similar ways. In a neighborhood of (w_k, o) in $W_0 \times \Omega_0$, another extension of ξ_{ik} is written as follows:

$$w_i = h'_{ik}(w_k, s) = h_{ik}(w_k, s) + \sum_{\alpha=1}^{m} a_{ik}^\alpha(w_k, s)e_\alpha(s)$$

where $\alpha_{ik}^\alpha, \alpha = 1, \cdots, m$, are vector valued holomorphic functions in the neighborhood. Hence

$$(\partial h'_{ik}/\partial s)(w_k, o) = (\partial h_{ik}/\partial s)(w_k, o)$$

$$+ \sum_{\alpha=1}^{m} (\partial a_{ik}^{\alpha}/\partial s)(w_k, o)e_\alpha(o)$$
$$+ \sum_{\alpha=1}^{m} a_{ik}^{\alpha}(w_k, o)(\partial e_\alpha/\partial s)(o)$$
$$= (\partial h_{ik}/\partial s)(w_k, o)$$

by (1) and (2) above. q.e.d.

Now, f maps $\tilde{U}_i$ into $\tilde{W}_i$. Using the local coordinates, it is expressed by the equations

$$w_i = f_i(z_i), \quad i \in I,$$

where f_i is a vector valued holomorphic function on $\tilde{U}_i$.

Let z_k^o be a point of $U_i \cap U_j \cap U_k$. Then there are neighborhoods A of (z_k^o, o) in $U_{o(jk)} \times \Omega_o$ and B of $(f_k(z_k^o), o)$ in $W_{0(jk)} \times \Omega_o$ and vector valued holomorphic functions

$$b^\alpha(z_k, s), \alpha = 1, \cdots, m \quad \text{and}$$
$$c^\alpha(w_k, s), \alpha = 1, \cdots, m$$

on A and B respectively such that $\eta_{ij}\eta_{jk}$ and $\xi_{ij}\xi_{jk}$ are defined on A and B respectively and such that

$$(3)\ g_{ik}(z_k, s) = g_{ij}(g_{jk}(z_k, s), s) + \sum_{\alpha=1}^{m} b^\alpha(z_k, s)e_\alpha(s)$$

for all $(z_k, s) \in A$ and

$$(4) \qquad h_{ik}(w_k, s) = h_{ij}(h_{jk}(w_k, s), s) + \sum_{\alpha=1}^{m} c^\alpha(w_k, s)e_\alpha(s)$$

for all $(w_k, s) \in B$.

LEMMA 4.2. *Let z_k^o be a point of $U_i \cap U_j \cap U_k$. Then*

$$(\partial h_{ik}/\partial w_k)(f_k(z_k^0), o)$$
$$= (\partial h_{ij}/\partial w_j)(f_j(z_j^0), o)(\partial h_{jk}/\partial w_k)(f_k(z_k^o), o)$$

where $z_j^o = g_{jk}(z_k^o, o)$.

PROOF. We differentiate (4) with respect to w_k at $(f_k(z_k^o), o)$. Since $h_{jk}(f_k(z_k^o), o) = f_j(z_j^o)$, we obtain the above equality by (1). q.e.d.

The holomorphic vector bundle on V defined by the transition matrices $\{(\partial h_{ik}/\partial w_k)(f_k(z_k), o)\}$ is nothing but the induced bundle f^*TW of the holomorphic tangent bundle TW over f.

LEMMA 4.3. *Let z_k^o be a point of $U_i \cap U_j \cap U_k$. Then*

$$(\partial h_{ik}/\partial s)(f_k(z_k^o), o) = (\partial h_{ij}/\partial s)(f_j(z_j^o), o)$$
$$+ (\partial h_{ij}/\partial w_j)(f_j(z_j^o), o)(\partial h_{jk}/\partial s)(f_k(z_k^o), o),$$

where $z_j^o = g_{jk}(z_k^o, o)$.

PROOF. We differentiate (4) with respect to s at $(f_k(z_k^o), o)$ and obtain the above equality by (1) and (2). q.e.d.

LEMMA 4.4. *Let* z_k^0 *be a point of* $U_i \cap U_j \cap U_k$. *Then*

$$(\partial g_{ik}/\partial s)(z_k^0, o) = (\partial g_{ij}/\partial s)(z_j^0, o) + (\partial g_{ij}/\partial z_j)(z_j^o, o)(\partial g_{jk}/\partial s)(z_k^o, o)$$

where $z_j^o = g_{jk}(z_k^o, o)$.

PROOF. We differentiate (3) with respect to s at (z_k^0, o) and obtain the above equality by (1) and (2). q.e.d.

LEMMA 4.5. *Let* z_k^o *be a point of* $U_i \cap U_j \cap U_k$. *Then*

$$\begin{aligned}(\partial f_i/\partial z_i)(z_i^o)(\partial g_{ik}/\partial s)(z_k^o, o) &= (\partial f_i/\partial z_i)(z_i^o)(\partial g_{ij}/\partial s)(z_j^o, o)\\ &\quad + (\partial h_{ij}/\partial w_j)(f_j(z_j^o), o)(\partial f_j/\partial z_j)(z_j^o)(\partial g_{jk}/\partial s)(z_k^o, o)\end{aligned}$$

where $z_i^o = g_{ik}(z_k^o, o)$ *and* $z_j^o = g_{jk}(z_k^o, o)$.

PROOF. f_i, $i \in I$, must satisfy the following compatibility conditions:

$$h_{ij}(f_j(z_j), o) = f_i(g_{ij}(z_j, o))$$

for all $z_j \in U_i \cap U_j$. Differentiating the equation with respect to z_j at z_j^o, we obtain

$$\begin{aligned}(\partial h_{ij}/\partial w_j)(f_j(z_j^o), o)(\partial f_j/\partial z_j)(z_j^o)\\ = (\partial f_i/\partial z_i)(z_i^o)(\partial g_{ij}/\partial z_j)(z_j^o, o)\ .\end{aligned}$$

Hence

$$\begin{aligned}&(\partial f_i/\partial z_i)(z_i^o)(\partial g_{ij}/\partial s)(z_j^o, o)\\ &\quad + (\partial h_{ij}/\partial w_j)(f_j(z_j^o), o)(\partial f_j/\partial z_j)(z_j^o)(\partial g_{jk}/\partial s)(z_k^o, o)\\ &= (\partial f_i/\partial z_i)(z_i^o)(\partial g_{ij}/\partial s)(z_j^o, o)\\ &\quad + (\partial f_i/\partial z_i)(z_i^o)(\partial g_{ij}/\partial z_j)(z_j^o, o)(\partial g_{jk}/\partial s)(z_k^o, o)\\ &= (\partial f_i/\partial z_i)(z_i^o)(\partial g_{ik}/\partial s)(z_k^o, o)\end{aligned}$$

by Lemma 4.4. q.e.d.

We put $F = f^*TW$. Then Lemma 4.3 and Lemma 4.5 show that

$$\{(\partial h_{ik}/\partial s)(f_k(z_k), o) - (\partial f_i/\partial z_i)(z_i)(\partial g_{ik}/\partial s)(z_k, o)\}$$

is an element of $Z^1(F, |\ |)$, (the space of 1-cocycles defined in §2), where $z_k \in U_i^e \cap U_k$ and $z_i = g_{ik}(z_k, o)$. This follows from the fact that

$$|(\partial f_i/\partial z_i)(z_i)|\,, \quad z_i \in U_i^e\,,$$

is estimated by $\sup\{|f_i(z_i)|\,|\,z_i \in U_i\}$, (<1). Hence we can define a continuous linear map

$$\sigma\colon T_oS \to Z^1(F, |\ |)$$

by

$$\sigma(a)_{ik}(z_i) = \sum_{\alpha=1}^{k} a^\alpha[(\partial h_{ik}/\partial s^\alpha)(f_k(z_k), o) - (\partial f_i/\partial z_i)(z_i)(\partial g_{ik}/\partial s^\alpha)(z_k, o)]$$

for $z_i \in U_i^e \cap U_k$, where $z_k = g_{ki}(z_i, o)$ and $a = \sum_{\alpha=1}^{k} a^\alpha(\partial/\partial s^\alpha)_o$.

REMARK. We write $\sigma(a)_{ik}(z_i)$ instead of writing $\sigma(a)_{ik}(z_k)$ following the definition of $|\ |$ in §2.

5. Proof of Theorem 1. We use the same notations as in §3 and §4. f maps $\bar{U}_i$ into W_i^e. Using the local coordinates, it is expressed by the equations

$$w_i = f_i(z_i)\,, \quad i \in I\,.$$

Then the vector valued holomorphic functions f_i, $i \in I$, must satisfy the following compatibility conditions:

$$h_{ik}(f_k(z_k), o) = f_i(g_{ik}(z_k, o))$$

for all $z_k \in U_i \cap U_k$. As in §4, we put $F = f^*TW$, the induced bundle over f of the holomorphic tangent bundle TW. Let T_oS be the Zariski tangent space to S at o. We consider the product

$$C^0(F, |\ |) \times T_0S\,,$$

where $C^0(F, |\ |)$ is the Banach space introduced in §2. We introduce a norm $|\ |$ in $C^0(F, |\ |) \times T_0S$ as follows:

$$|(\phi, s)| = \max\{|\phi|, |s|\}$$

for $(\phi, s) \in C^0(F, |\ |) \times T_0S$, where $|s| = \max_\alpha |a^\alpha|$, $s = \sum_{\alpha=1}^{k} a^\alpha(\partial/\partial s^\alpha)_0$. Then $C^0(F, |\ |) \times T_0S$ is a Banach space. We identify $\tilde{\Omega}$ with an open set of T_oS by

$$(a^1, \cdots, a^k) \in \tilde{\Omega} \to \sum_{\alpha=1}^{k} a^\alpha(\partial/\partial s^\alpha)_o \in T_oS\,.$$

Let f' be a holomorphic map of $\pi^{-1}(s)$ into $\mu^{-1}(s)$ for a point $s \in S'$ such that

$$f'(\pi^{-1}(s) \cap X_i) \Subset \mu^{-1}(s) \cap Y_i$$

for all $i \in I$. We express the map f' by the equations

$$w_i = f'_i(z_i)\,, \quad i \in I\,,$$

using the isomorphisms

$$\eta_i\colon X_i \to U_i \times S' \quad \text{and}$$
$$\xi_i\colon Y_i \to W_i \times S'\,.$$

Then the vector valued holomorphic functions f'_i satisfy $f'_i(U_i) \Subset W_i$. We write

$$f'_i = f_i + \phi_i$$

where ϕ_i is a vector valued holomorphic function on U_i. We regard $\phi = \{\phi_i\}_{i \in I}$ as an element of $C^0(F, |\ |)$. We associate to f' an element $(\phi, s) \in C^0(F, |\ |) \times T_o S$ where $s \in S' \subset \Omega \subset T_0 S$. Then it is clear that (ϕ, s) must satisfy the following compatibility conditions:

(1) $s \in S'$ and

(2) $h_{ik}(f_k(z_k) + \phi_k(z_k), s) = f_i(g_{ik}(z_k, s)) + \phi_i(g_{ik}(z_k, s))$ for

$(z_k, s) \in \eta_k(X_i \cap X_k) \cap \pi^{-1}(s)$ and $(f_k(z_k) + \phi_k(z_k), s) \in \xi_k(Y_i \cap Y_k) \cap \mu^{-1}(s)$.

Conversely, if an element $(\phi, s) \in C^0(F, |\ |) \times T_o S$ satisfies $|(\phi, s)| < \varepsilon$, (where ε satisfies Lemma 3.9 for $A = \overline{f_k(U_k)}$ for each $k \in I$), and satisfies the conditions (1) and (2) above, then the equations

$$w_i = f'_i(z_i) = f_i(z_i) + \phi_i(z_i)\,,$$

for $z_i \in U_i$ and $i \in I$, define a holomorphic map f' of $\pi^{-1}(s)$ into $\mu^{-1}(s)$. By Lemma 3.9, f' satisfies

$$f'(\pi^{-1}(s) \cap X_i) \Subset \mu^{-1}(s) \cap Y_i,\ i \in I\,.$$

Henceforth, let ε, $0 < \varepsilon < 1$, be a small positive number satisfying Lemma 3.5—Lemma 3.8, Lemma 3.9 for $A = \overline{f_k(U_k)}$ for each $k \in I$, and Lemma 3.10. Let B_ε be the open ε-ball of $C^0(F, |\ |)$ with the center 0. Let Ω_ε be the open ε-ball of $T_o S$ with the center o. We put $S_\varepsilon = S' \cap \Omega_\varepsilon$. We assume that S' is defined in Ω as common zeros of holomorphic functions

$$e_1(s), \cdots, e_m(s)\,.$$

We define a holomorphic map

$$e\colon \Omega \to \boldsymbol{C}^m$$

by

$$e(s) = (e_1(s), \cdots, e_m(s))\,.$$

Then

$$S_\varepsilon = \{s \in \Omega_\varepsilon \mid e(s) = 0\} .$$

Now we define a map

$$K: B_\varepsilon \times \Omega_\varepsilon \to C^1(F, |\ |)$$

by

$$K(\phi, s)_{ik}(z_i) = h_{ik}(f_k(z_k) + \phi_k(z_k), s) \\ - f_i(g_{ik}(z_k, s)) - \phi_i(g_{ik}(z_k, s))$$

for $z_i \in U_i^e \cap U_k$, where $z_k = g_{ki}(z_i, o)$. Then $K(0, o) = 0$. If $z_k \in U_i^e \cap U_k$ and $s \in \Omega_\varepsilon$, then $g_{ik}(z_k, s)$ is defined and is a point of U_i by Lemma 3.5. Hence $f_i(g_{ik}(z_k, s))$ and $\phi_i(g_{ik}(z_k, s))$ are defined. On the other hand, $f_k(z_k) + \phi_k(z_k) \in W_i^e \cap W_k$ for $z_k \in U_i^e \cap U_k$ by Lemma 3.10. Hence $h_{ik}(f_k(z_k) + \phi_k(z_k), s)$ is defined and is a point of W_i by Lemma 3.5. Moreover, it is clear that

$$|K(\phi, s)| < 2 + \varepsilon$$

if $|(\phi, s)| < \varepsilon$. Thus K maps $B_\varepsilon \times \Omega_\varepsilon$ into $C^1(F, |\ |)$.

Let

$$\beta: C^0(F, |\ |) \times T_oS \to T_oS$$

be the canonical projection. We put

$$M_1 = \{(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon \mid K(\phi, s) = 0\}$$

and

$$M = \{(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon \mid K(\phi, s) = 0, e\beta(\phi, s) = e(s) = 0\} \\ = \{(\phi, s) \in B_\varepsilon \times S_\varepsilon \mid K(\phi, s) = 0\} .$$

Now we take an element $(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon$ which satisfies the compatibility conditions (1) and (2) above. Let z_i be any fixed point of $U_i^e \cap U_k$. Let $z_k = g_{ki}(z_i, o)$. By Lemma 3.6, $(z_k, s) \in \eta_k(X_i \cap X_k)$. By Lemma 3.10 and Lemma 3.6, $(f_k(z_k) + \phi_k(z_k), s) \in \xi_k(Y_i \cap Y_k)$. Hence, by (2),

$$K(\phi, s)_{ik}(z_i) = 0 .$$

Since $z_i \in U_i^e \cap U_k$ is arbitrary,

$$K(\phi, s) = 0 .$$

Hence $(\phi, s) \in M$. Conversely, let $(\phi, s) \in M$. (1) of the compatibility conditions is automatically satisfied. Let z_k be a point of U_k. We assume that $(z_k, s) \in \eta_k(X_i^{e'} \cap X_k^{e'})$ and $(f_k(z_k) + \phi_k(z_k), s) \in \xi_k(Y_i^{e'} \cap Y_k^{e'})$. Then, by Lemma 3.7, $z_k \in U_i^e \cap U_k$. Since $K(\phi, s) = 0$,

$$h_{ik}(f_k(z_k) + \phi_k(z_k), s) = f_i(g_{ik}(z_k, s)) + \phi_i(g_{ik}(z_k, s)) .$$

Hence the equations

$$w_i = f_i(z_i) + \phi_i(z_i) ,$$

for $z_i \in U_i^{e'}$ and $i \in I$, define a holomorphic map f' of $\pi^{-1}(s)$ into $\mu^{-1}(s)$. Thus, by the principle of analytic continuation, equations

$$w_i = f_i(z_i) + \phi_i(z_i) ,$$

for $z_i \in U_i$ and $i \in I$, define f'. Hence (ϕ, s) satisfies (2) of the compatibility conditions. Thus the problem is reduced to analyze the set M.

PROPOSITION 5.1. *Let ε be sufficiently small. Then*

$$K: B_\varepsilon \times \Omega_\varepsilon \rightarrow C^1(F, | \ |)$$

is an analytic map and

$$K'(0, o) = \delta + \sigma: C^0(F, | \ |) \times T_oS \rightarrow C^1(F, | \ |)$$

where δ and σ are the continuous linear maps defined in §2 and §4 respectively and $\delta + \sigma$ is defined by

$$(\delta + \sigma)(\phi, s) = \delta\phi + \sigma s$$

for $(\phi, s) \in C^0(F, | \ |) \times T_oS$.

PROOF. The proof of the first half is similar to that of Lemma 3.4 of [8], so we we omit it. We prove the second half. Let $o(\phi, s)$ be some function of ϕ and s (and z_i) such that

$$|o(\phi, s)|/|(\phi, s)| \rightarrow 0$$

as $|(\phi, s)| \rightarrow 0$. Let $z_i \in U_i^e \cap U_k$. We put $z_k = g_{ki}(z_i, o)$. Then

$$\begin{aligned} K(\phi, s)_{ik}(z_i) &= K(\phi, s)_{ik}(z_i) - K(0, o)_{ik}(z_i) \\ &= (\partial h_{ik}/\partial w_k)(f_k(z_k), o)\phi_k(z_k) + (\partial h_{ik}/\partial s)(f_k(z_k), o)s \\ &\quad - \{f_i(g_{ik}(z_k, s)) - f_i(g_{ik}(z_k, o))\} \\ &\quad - \{\phi_i(g_{ik}(z_k, s)) - \phi_i(g_{ik}(z_k, o))\} - \phi_i(z_i) + o(\phi, s) \\ &= (\partial h_{ik}/\partial w_k)(f_k(z_k), o)\phi_k(z_k) + (\partial h_{ik}/\partial s)(f_k(z_k), o)s \\ &\quad - (\partial f_i/\partial z_i)(z_i)(\partial g_{ik}/\partial s)(z_k, o)s \\ &\quad - (\partial \phi_i/\partial z_i)(z_i)(\partial g_{ik}/\partial s)(z_k, o)s - \phi_i(z_i) + o(\phi, s). \end{aligned}$$

Since

$$|(\partial\phi_i/\partial z_i)(z_i)| , \quad z_i \in U_i^e ,$$

is estimated by $|\phi|$, we may put

$$- (\partial\phi_i/\partial z_i)(z_i)(\partial g_{ik}/\partial s)(z_k, o)s = o(\phi, s) .$$

Hence

$$\begin{aligned} K(\phi, s)_{ik}(z_i) &= (\delta\phi)_{ik}(z_i) + (\partial h_{ik}/\partial s)(f_k(z_k), o)s \\ &\quad - (\partial f_i/\partial z_i)(z_i)(\partial g_{ik}/\partial s)(z_k, o)s + o(\phi, s) \\ &= (\delta\phi)_{ik}(z_i) + (\sigma s)_{ik}(z_i) + o(\phi, s) . \end{aligned}$$

Hence

$$K(\phi, s) = \delta\phi + \sigma s + o(\phi, s) .$$

q.e.d.

Now we define a map

$$L\colon B_\varepsilon \times \Omega_\varepsilon \to C^0(F, |\ |) \times T_oS$$

by

$$L(\phi, s) = (\phi + E_0 B\Lambda K(\phi, s) - E_0\delta\phi, s)$$

where E_0, B, Λ, and δ are the continuous linear maps defined in §2. Then L is analytic by Proposition 5.1. We have $L(0, o) = (0, o)$ and

$$\begin{aligned} L'(0, o) &= \begin{pmatrix} 1 + E_0 B\Lambda\delta - E_0\delta & E_0 B\Lambda\sigma \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & E_0 B\sigma \\ 0 & 1 \end{pmatrix} . \end{aligned}$$

(We note that $B\Lambda\delta = \delta$ and $\Lambda\sigma = \sigma$.) Thus $L'(0, o)$ is a continuous linear isomorphism. Hence, by the inverse mapping theorem, there are a small positive number ε', an open neighborhood U of $(0, o)$ in $B_\varepsilon \times \Omega_\varepsilon$ and an analytic isomorphism Φ of $B_{\varepsilon'} \times \Omega_{\varepsilon'}$ onto U such that $L|U = \Phi^{-1}$. We put

$$\begin{aligned} T_1 &= L(M_1 \cap U) \quad \text{and} \\ T &= L(M \cap U) . \end{aligned}$$

Then $M_1 \cap U = \Phi(T_1)$ and $M \cap U = \Phi(T)$.

LEMMA 5.1. $T_1 \subset (H^0(F, |\ |) \cap B_{\varepsilon'}) \times \Omega_{\varepsilon'}$.

PROOF. Let $(\phi, s) \in M_1 \cap U$. Then

$$\begin{aligned} L(\phi, s) &= (\phi + E_0 B\Lambda K(\phi, s) - E_0\delta\phi, s) \\ &= (\phi - E_0\delta\phi, s) . \end{aligned}$$

We have

$$\delta(\phi - E_0\delta\phi) = \delta\phi - \delta\phi = 0 .$$

q.e.d.

COROLLARY 1. $T_1 = \{(\xi, s) \in (H^0(F, |\ |) \cap B_{\varepsilon'}) \times \Omega_{\varepsilon'} \mid K\Phi(\xi, s) = 0\}$.

COROLLARY 2. $T = \{(\xi, s) \in (H^0(F, |\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} \mid K\Phi(\xi, s) = 0\}$.

Corollary 1 follows from the definition of M_1 and Lemma 5.1. Corollary 2 follows from Corollary 1.

Now let $(\xi, s) \in (H^0(F, |\ |) \cap B_{\varepsilon'}) \times \Omega_{\varepsilon'}$. We put $(\phi, s) = \Phi(\xi, s)$. Then

$$\begin{aligned} 0 = \delta\xi &= \delta(\phi + E_0 B\Lambda K(\phi, s) - E_0\delta\phi) \\ &= B\Lambda K(\phi, s) = B\Lambda K\Phi(\xi, s) . \end{aligned}$$

Hence

$$\begin{aligned} K\Phi(\xi, s) &= H\Lambda K\Phi(\xi, s) + B\Lambda K\Phi(\xi, s) + E_1\delta K\Phi(\xi, s) \\ &= H\Lambda K\Phi(\xi, s) + E_1\delta K\Phi(\xi, s) \end{aligned}$$

where H and E_1 are the continuous linear maps defined in §2.

PROPOSITION 5.2. *Let ε' be sufficiently small. Then*

$$T = \{(\xi, s) \in (H^0(F, |\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} \mid H\Lambda K\Phi(\xi, s) = 0\}.$$

PROOF. The proof is almost similar to that of Lemma 3.6 of [8]. Only what we have to note are the following two points.

(A) By (2) of Lemma 3.8, if $(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon$, then

$$\zeta_j = g_{jk}(z_k, s) \in U_i^{e/2} \cap U_j^{e/2}$$

if $z_k = g_{ki}(z_i, o) \in U_i^e \cap U_j^e \cap U_k$.

(B) For $(\phi, s) \in B_\varepsilon \times \Omega_\varepsilon$, we put

$$\begin{aligned} &R^1(K(\phi, s), \phi, s) = \{R^1(K(\phi, s), \phi, s)_{ijk}\} \in C^2(F, |\ |) , \\ &R^1(K(\phi, s), \phi, s)_{ijk}(z_i) = h_{ij}(f_j(\zeta_j) + \phi_j(\zeta_j), s) \\ &\quad - h_{ij}(h_{jk}(f_k(z_k) + \phi_k(z_k), s), s) \\ &\quad + F_{ij}(z_j)K(\phi, s)_{jk}(z_j) \end{aligned}$$

where $z_j = g_{ji}(z_i, o)$, $z_k = g_{ki}(z_i, o)$ and $F_{ij}(z_j) = (\partial h_{ij}/\partial w_j)(f_j(z_j), o)$. Then, for $s \in S_\varepsilon$,

$$\begin{aligned} R^1(K(\phi, s), \phi, s)_{ijk}(z_i) &= h_{ij}(f_j(\zeta_j) + \phi_j(\zeta_j), s) \\ &\quad - f_i(g_{ij}(\zeta_j, s)) - h_{ik}(f_k(z_k) + \phi_k(z_k), s) \\ &\quad + f_i(g_{ik}(z_k, s)) + F_{ij}(z_j)K(\phi, s)_{jk}(z_j) . \end{aligned}$$

The rest goes pararell to the proof of Lemma 3.6 of [8]. q.e.d.

COROLLARY. *If $H^1(V, F) = 0$, then*

$$T = (H^0(F, |\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} .$$

Now, for each $t = (\xi, s) \in T$, we put

$$\Phi(t) = (\phi(t), b(t)) .$$

Then

$$\phi: T \to C^0(F, |\ |) \quad \text{and}$$
$$b: T \to S$$

are analytic maps. The map b is actually the projection map

$$t = (\xi, s) \to s .$$

If we write

$$\phi(t) = \{\phi_i(z_i, t)\}_{i \in I} ,$$

then it is easy to see that

$$\phi_i: U_i \times T \to C^r$$

is a holomorphic map. We define a holomorphic map

$$E: b^*X \to b^*Y$$

be the equations

$$w_i = f_i(z_i) + \phi_i(z_i, t), \quad \text{for} \quad z_i \in U_i, \quad \text{and} \quad t = t .$$

Then (E, T, b) is a family of holomorphic maps of (X, π, S) into (Y, μ, S) and satisfies

$$E_{(0,o)} = f .$$

We show that (E, T, b) is a maximal family. Let $t_o = (\xi_o, s_o)$ be a point of T. Let (G, R, h) be a family of holomorphic maps of (X, π, S) into (Y, μ, S) with a point r_o such that $h(r_o) = s_o$ and

$$G_{r_o} = E_{t_o}: \pi^{-1}(s_o) \to \mu^{-1}(s_o) .$$

The map $G_{r_o} = E_{t_o}$ is defined by the equations

$$w_i = f_i(z_i) + \phi_i(z_i, t_o)$$

for $z_i \in U_i$. Then it is easy to see that, there are a neighborhood R' of r_o, an ambient space $\tilde{R}'$ of R' and a vector valued holomorphic function ψ_i on $U_i \times \tilde{R}'$ such that, for each fixed $r \in R'$, G_r is defined by equations

$$w_i = f_i(z_i) + \phi_i(z_i, t_o) + \psi_i(z_i, r) ,$$

for $z_i \in U_i$. We put

$$\phi'_i(z_i, r) = \phi_i(z_i, t_o) + \psi_i(z_i, r) \quad \text{and}$$
$$\phi'(r) = \{\phi'_i(z_i, r)\}_{i \in I}$$

for $r \in \tilde{R}'$. We extend the map h to $\tilde{R}'$. Then

$$(\phi'(r), h(r)) \in C^0(F, |\ |) \times \Omega .$$

We note that

$$(\phi'(r_o), h(r_o)) = \Phi(t_o) .$$

It is easy to see that ϕ' is an analytic map of $\tilde{R}'$ into $C^0(F, |\ |)$.

We may assume that

$$(\phi'(r), h(r)) \in U = \Phi(B_{\varepsilon'} \times \Omega_{\varepsilon'})$$

for all $r \in \tilde{R}'$. Let $r \in R'$. Since the equations

$$w_i = \phi'_i(z_i, r) ,$$

for $z_i \in U_i$, define a holomorphic map of $\pi^{-1}(h(r))$ into $\mu^{-1}(h(r))$, $(\phi'(r), h(r)) \in U \cap M$ for each $r \in R'$. Hence $L(\phi'(r), h(r)) \in T$ for each $r \in R'$. We put

$$k(r) = L(\phi'(r), h(r))$$

for $r \in R'$. Then k is a holomorphic map of R' into T. We note that $k(r_o) = L\Phi(t_o) = t_o$. We have

$$\Phi(k(r)) = (\phi'(r), h(r)) .$$

Hence $h = bk$ and $\phi' = \phi k$. From these identities, we have

$$G_r = E_{k(r)} \colon \pi^{-1}(h(r)) \to \mu^{-1}(h(r))$$

for all $r \in R'$. Thus (E, T, b) is a maximal family.

This completes the proof of Theorem 1.

REMARK. Among maximal families, our maximal family (E, T, b) is a special one. It is so called effectively parametrized. In other words, the map k with properties

$$h = bk \quad \text{and}$$
$$G_r = E_{k(r)} \colon \pi^{-1}(h(r)) \to \mu^{-1}(h(r)) ,$$

for all $r \in R'$, is uniquely determined.

Appendix of § 5. Extensions of holomorphic maps.

DEFINITION. Let V be a compact complex manifold. Let W be a complex manifold. Let f be a holomorphic map of V into W. f is said to be *extendable* if and only if, for any families (X, π, S) and (Y, μ, S) of compact complex manifolds and of complex manifolds respectively with a point $o \in S$ such that $\pi^{-1}(o) = V$ and $\mu^{-1}(o) = W$, there are a neighborhood U of o in S and a holomorphic map H of $\pi^{-1}(U)$ into $\mu^{-1}(U)$ such that

(1) the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad H \quad} & \mu^{-1}(U) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & U & \end{array}$$

is commutative and

(2) $H|V = f$.

The following theorem is essentially due to Kodaira (Theorem 1, [4]). See also §6 of [9].

THEOREM. *Let V be a compact complex manifold. Let W be a complex manifold. Let f be a holomorphic map of V into W. Let f^*TW be the induced bondle over f of the holomorphic tangent bundle TW of W. If $H^1(V, f^*TW) = 0$, then f is extendable.*

PROOF. Let (X, π, S) and (Y, μ, S) be families of compact complex manifolds and of complex manifolds with a point $o \in S$ such that $\pi^{-1}(o) = V$ and $\mu^{-1}(o) = W$. Let (E, T, b) be the maximal family of holomorphic maps of (X, π, S) into (Y, μ, S) constructed in §5 with respect to f. If $H^1(V, F) = 0$, where $F = f^*TW$, then

$$T = (H^0(F, |\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'}$$

by the corollary of Proposition 5.2. We define a map

$$j: S_{\varepsilon'} \to T$$

by

$$j(s) = (0, s) .$$

Then j is a holomorphic injection. Using the notations in §5, we define a holomorphic map

$$H: \pi^{-1}(S_{\varepsilon'}) \to \mu^{-1}(S_{\varepsilon'})$$

by the equations

$$w_i = f_i(z_i) + \phi_i(z_i, j(s))$$

for $(z_i, s) \in U_i \times S_{\varepsilon'}$. Then H satisfies the requirement. q.e.d.

6. Theorem 2, Theorem 3 and their proofs. Let V be a compact complex manifold. Let W be a complex manifold. We denote by $H(V,$

W) the set of all holomorphic maps of V into W.

THEOREM 2. *Let* (X, π, S) *and* (Y, μ, S) *be families of compact complex manifolds and of complex manifolds respectively. We assume that* X *and* Y *satisfy the second axiom of countability. Then the disjoint union*

$$H = \coprod_{s \in S} H(\pi^{-1}(s), \mu^{-1}(s))$$

admits an analytic space structure such that

(1) (H, λ, S) *is a complex fiber space where*

$$\lambda\colon H \to S$$

is the canonical projection and

(2) *the map*

$$X \times_S H \to Y$$

defined by

$$(P, f) \to f(P)$$

is holomorphic, where

$$X \times_S H = \{(P, f) \in X \times H \,|\, \pi(P) = \lambda(f)\}\,,$$

the fiber product of X *and* H *over* S.

The proof of Theorem 2 below is essentially due to that of Theorem 2 of [8]. Let (X, π, S) and (Y, μ, S) be as above. Let o be a point of S. Let f be a holomorphic map of $\pi^{-1}(o)$ into $\mu^{-1}(o)$. Let (E, T, b) be the maximal family of holomorphic maps of (X, π, S) into (Y, μ, S) constructed in §5 with respect to f. By the construction of (E, T, b) in §5, for any two different point t_1 and t_2 of T, the corresponding maps

$$E_{t_1}\colon \pi^{-1}(b(t_1)) \to \mu^{-1}(b(t_1))$$

and

$$E_{t_2}\colon \pi^{-1}(b(t_2)) \to \mu^{-1}(b(t_2))$$

are different, (even if $b(t_1) = b(t_2)$). Thus there is a unique injective map

$$T \to H$$

defined by

$$t \to E_t\,.$$

We take this map as a local chart around $f \in H$. Using the maximality of (E, T, b) and Remark at the end of §5, these local charts patch up to give a (locally finite dimensional) analytic space structure in H. We have to show that the underlying topological space of H is a Hausdorff space.

Since X and Y are locally compact and satisfy the second axiom of countability by the assumption, they are metrizable. We denote by d and d' metrics in X and Y respectively. Let f and g be two elements of H. We define a distance

$$\tilde{d}(f, g)$$

by

$$\tilde{d}(f, g) = \sup_{P \in \pi^{-1}(\lambda(f))} \inf_{Q \in \pi^{-1}(\lambda(g))} \{d(P, Q) + d'(f(P), g(Q))\}$$
$$+ \sup_{Q \in \pi^{-1}(\lambda(g))} \inf_{P \in \pi^{-1}(\lambda(f))} \{d(P, Q) + d'(f(P), g(Q))\} .$$

LEMMA 6.1. *$\tilde{d}$ is a metric in H.*

PROOF. It is easy to check that $\tilde{d}$ satisfies the three axioms for metric. q.e.d.

LEMMA 6.2. *Let (E, T, b) be a family of holomorphic maps of (X, π, S) into (Y, μ, S). Let t_o be a point of T. Then $\tilde{d}(E_t, E_{t_o})$ is a continuous function of $t \in T$.*

PROOF. It suffices to prove that

$$\tilde{d}(E_t, E_{t_o}) \to 0 \quad \text{as} \quad t \to t_o .$$

It is known [7] that there are an open neighborhood T' of t_o in T and a continuous retraction

$$R: (b^*\pi)^{-1}(T') \to (b^*\pi)^{-1}(t_o)$$

such that $R_t = R \,|\, (b^*\pi)^{-1}(t)$ is a C^∞-diffeomorphism of $(b^*\pi)^{-1}(t)$ onto $(b^*\pi)^{-1}(t_o)$ for each $t \in T'$. We fix a point $t \in T'$. We identify $(b^*\pi)^{-1}(t)$ and $(b^*\pi)^{-1}(t_o)$ with $\pi^{-1}(b(t))$ and $\pi^{-1}(b(t_o))$ respectively in a canonical way (§1). Then R_t is regarded as a diffeomorphism of $\pi^{-1}(b(t))$ onto $\pi^{-1}(b(t_o))$. We have

$$\inf_{Q \in \pi^{-1}(b(t_o))} \{d(P, Q) + d'(E_t(P), E_{t_o}(Q))\}$$
$$\leqq d(P, R_t(P)) + d'(E_t(P), E_{t_o}(R_t(P)))$$

for any point $P \in \pi^{-1}(b(t))$. Hence

$$\sup_{P \in \pi^{-1}(b(t))} \inf_{Q \in \pi^{-1}(b(t_o))} \{d(P, Q) + d'(E_t(P), E_{t_o}(Q))\}$$
$$\leqq \sup_{P \in \pi^{-1}(b(t))} \{d(P, R_t(P)) + d'(E_t(P), E_{t_o}(R_t(P)))\} .$$

In a similar way, we get

$$\sup_{Q\in\pi^{-1}(b(t_o))}\inf_{P\in\pi^{-1}(b(t))}\{d(P, Q)+d'(E_t(P), E_{t_o}(Q))\}$$
$$\leqq \sup_{Q\in\pi^{-1}(b(t_o))}\{d(Q, R_t^{-1}(Q))+d'(E_t(R_t^{-1}(Q)), E_{t_o}(Q))\} .$$

Thus

$$\tilde{d}(E_t, E_{t_o}) \leqq 2 \sup_{P\in\pi^{-1}(b(t))}\{d(P, R_t(P))+d'(E_t(P), E_{t_o}(R_t(P)))\} .$$

Now it suffices to show that

$$\sup_{P\in\pi^{-1}(b(t))}\{d(P, R_t(P))+d'(E_t(P), E_{t_o}(R_t(P)))\}\to 0$$

as $t\to t_o$. We assume the converse. Then there are a positive number ε, a sequence $\{t_n\}_{n=1,2,\cdots}$ of points of T' converging to t_o and a sequence $\{P_n\}_{n=1,2,\cdots}$ of points of X such that $P_n\in\pi^{-1}(b(t_n))$, $n=1, 2, \cdots$, and

$$d(P_n, R_{t_n}(P_n))+d'(E_{t_n}(P_n), E_{t_o}(R_{t_n}(P_n)))\geqq\varepsilon$$

for $n=1, 2, \cdots$. Since each fiber $\pi^{-1}(s)$, $s\in S$, is compact, we may assume that $\{P_n\}_{n=1,2\cdots}$ converges to a point $P\in\pi^{-1}(b(t_o))$. Then

$$\begin{aligned}\varepsilon &\leqq d(P, R_{t_o}(P))+d'(E_{t_o}(P), E_{t_o}(R_{t_o}(P)))\\ &= d(P, P)+d'(E_{t_o}(P), E_{t_o}(P))\\ &=0 ,\end{aligned}$$

a contradiction. q.e.d.

Let $(H, \tilde{d})$ be the metric space H with the metric $\tilde{d}$ introduced above. Lemma 6.2 asserts that the identity map

$$I\colon H\to(H, \tilde{d})$$

is a continuous map. Since $(H, \tilde{d})$ is a Hausdorff space, H is also a Hausdorff space.

Next we prove (1) of Theorem 2. The map

$$\lambda\colon H\to S$$

is surjective, for $H(\pi^{-1}(s), \mu^{-1}(s))$ contains constant maps for any $s\in S$. In order to prove that λ is holomorphic, it is enough to prove it locally. Let o be a point of S. Let f be a holomorphic map of $\pi^{-1}(o)$ into $\mu^{-1}(o)$. Let (E, T, b) be the maximal family of holomorphic maps of (X, π, S) into (Y, μ, S) constructed in §5 with respect to f. Then it is clear that λ is locally given by the map b which is holomorphic.

Finally we prove (2) of Theorem 2. It is enough to prove it locally. Let o, f and (E, T, b) be as above. E is a holomorphic map of $b^*X = X\times_S T$ into $b^*Y=Y\times_S T$. It is written as

$$E(P, t) = (E_t(P), t)$$

for (P, t) with $\pi(P) = b(t)$, where E_t is the holomorphic map of $\pi^{-1}(b(t))$ into $\mu^{-1}(b(t))$ corresponding to t. $E_t(P)$ is holomorphic in (P, t), (see §5). It is clear that $E_t(P)$ is the local expression of the map in (2) of Theorem 2. This completes the proof of Theorem 2.

THEOREM 3. *Let (X, π, S) and (Y, μ, S) be as in Theorem 2. Then there is a maximal family (G, H, λ) of holomorphic maps of (X, π, S) into (Y, μ, S) with the following universal property: for any family (M, R, h) of holomorphic maps of (X, π, S) into (Y, μ, S), there is a unique holomorphic map k of R into H such that*

(1) $\lambda k = h$ *and*

(2) $M_r = G_{k(r)}: \pi^{-1}(h(r)) \to \mu^{-1}(h(r))$ *for all* $r \in R$.

PROOF. Let H and λ be as in Theorem 2. Let f be an element of H. Let (E, T, b) be the maximal family of holomorphic maps of (X, π, S) into (Y, μ, S) constructed in §5 with respect to f. E is a holomorphic map of b^*X into b^*Y. We took the map

$$t \in T \to E_t \in H$$

as a local chart around f. The canonical projection λ was locally given by b. We define a holomorphic map

$$G: \lambda^*X \to \lambda^*Y$$

by $G = E$ on $b^*X = (\lambda^*X)|T$. It is clear that G is well defined and has the universal property above. q.e.d.

7. Theorem 4 and its proof.

THEOREM 4. *Let (X, π, S) and (Y, μ, S) be families of compact complex manifolds. Let (Z, τ, S) be a family of complex manifolds. We assume that X, Y, and Z satisfy the second axiom of countability. Let*

$$H(X, Y; S) = \coprod_{s \in S} H(\pi^{-1}(s), \mu^{-1}(s)),$$

$$H(Y, Z; S) = \coprod_{s \in S} H(\mu^{-1}(s), \tau^{-1}(s)) \quad \text{and}$$

$$H(X, Z; S) = \coprod_{s \in S} H(\pi^{-1}(s), \tau^{-1}(s))$$

be the analytic spaces whose analytic structures are introduced by Theorem 2. Let λ_{XY}, λ_{YZ}, and λ_{XZ} be the canonical projections of $H(X, Y; S)$, $H(Y, Z; S)$, and $H(X, Z; S)$ respectively onto S. Then the map

$$H(X, Y; S) \times_S H(Y, Z; S) \to H(X, Z; S)$$

defined by

$$(f, g) \to gf ,$$

for (f, g) *with* $\lambda_{XY}(f) = \lambda_{YZ}(g)$, *is holomorphic.*

Let o be a point of S. We put $V = \pi^{-1}(o)$, $W = \mu^{-1}(o)$ and $N = \tau^{-1}(o)$. Then V and W are compact. Let

$$f: V \to W \quad \text{and}$$
$$g: W \to N$$

be holomorphic maps. Then similar arguments to those in §3 show that there are finite sets I and A and families of open sets $\{X_i\}_{i\in I}$ and $\{\tilde{X}_i\}_{i\in I}$ of X, $\{Y_i\}_{i\in I}$, $\{\tilde{Y}_i\}_{i\in I}$, $\{Y_\alpha\}_{\alpha\in A}$ and $\{\tilde{Y}_\alpha\}_{\alpha\in A}$ of Y and $\{Z_i\}_{i\in I}$, $\{\tilde{Z}_i\}_{i\in I}$, $\{Z_i\}_{\alpha\in A}$ and $\{\tilde{Z}_\alpha\}_{\alpha\in A}$ of Z satisfying the following conditions (1)-(7).

(1) $X_i \Subset \tilde{X}_i$, $Y_i \Subset \tilde{Y}_i$ and $Z_i \Subset \tilde{Z}_i$ for each $i \in I$ and $Y_\alpha \Subset \tilde{Y}_\alpha$ and $Z_\alpha \Subset \tilde{Z}_\alpha$ for each $\alpha \in A$,

(2) $\{X_i\}_{i\in I}$, $\{Y_i\}_{i\in I}$, $\{Z_i\}_{i\in I}$, $\{Y_i\}_{i\in I} \cup \{Y_\alpha\}_{\alpha\in A}$ and $\{Z_i\}_{i\in I} \cup \{Z_\alpha\}_{\alpha\in A}$ cover V, $f(V)$, $gf(V)$, W and $g(W)$ respectively,

(3) $Y_\alpha \cap f(V) = \emptyset$ for each $\alpha \in A$,

(4) there are an open neighborhood $\tilde{S}$ of o and holomorphic isomorphisms

$$\eta_i: \tilde{X}_i \to \tilde{U}_i \times \tilde{S},$$
$$\xi_i: \tilde{Y}_i \to \tilde{W}_i \times \tilde{S},$$
$$\xi_\alpha: \tilde{Y}_\alpha \to \tilde{W}_\alpha \times \tilde{S},$$
$$\zeta_i: \tilde{Z}_i \to \tilde{N}_i \times \tilde{S} \quad \text{and}$$
$$\zeta_\alpha: \tilde{Z}_\alpha \to \tilde{N}_\alpha \times \tilde{S}$$

such that diagrams

$$\begin{array}{ccc} \tilde{X}_i & \xrightarrow{\quad\eta_i\quad} & \tilde{U}_i \times \tilde{S} \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\text{proj}} & \\ & \tilde{S} & \end{array}$$

$$\begin{array}{ccc} \tilde{Y}_i & \xrightarrow{\quad\xi_i\quad} & \tilde{W}_i \times \tilde{S} \\ & {\scriptstyle\mu}\searrow \qquad \swarrow{\scriptstyle\text{proj}} & \\ & \tilde{S} & \end{array}$$

$$\begin{array}{ccc} \tilde{Y}_\alpha & \xrightarrow{\quad\xi_\alpha\quad} & \tilde{W}_\alpha \times \tilde{S} \\ & {\scriptstyle\mu}\searrow \qquad \swarrow{\scriptstyle\text{proj}} & \\ & \tilde{S} & \end{array}$$

$$\begin{array}{ccc} \tilde{Z}_i & \xrightarrow{\quad \zeta_i \quad} & \tilde{N}_i \times \tilde{S} \\ & {\scriptstyle \tau}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & \tilde{S} & \end{array}$$

and

$$\begin{array}{ccc} \tilde{Z}_\alpha & \xrightarrow{\quad \zeta_\alpha \quad} & \tilde{N}_\alpha \times \tilde{S} \\ & {\scriptstyle \tau}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & \tilde{S} & \end{array}$$

are commutative for each $i \in I$ and for each $\alpha \in A$, where $\tilde{U}_i$, $i \in I$, are open sets in $C^d (d = \dim V)$, $\tilde{W}_i$, $i \in I$, and $\tilde{W}_\alpha$, $\alpha \in A$, are open sets in $C^r (r = \dim W)$, and $\tilde{N}_i$, $i \in I$ and $\tilde{N}_\alpha$, $\alpha \in A$, are open sets in $C^q (q = \dim N)$,

(5) there are an open neighborhood S' of o with $S' \Subset \tilde{S}$ and open subsets U_i, W_i, W_α, N_i, and N_α of $\tilde{U}_i$, $\tilde{W}_i$, $\tilde{W}_\alpha$, $\tilde{N}_i$, and $\tilde{N}_\alpha$ respectively such that $U_i \Subset \tilde{U}_i$, $W_i \Subset \tilde{W}_i$, $W_\alpha \Subset \tilde{W}_\alpha$, $N_i \Subset \tilde{N}_i$, and $N_\alpha \Subset \tilde{N}_\alpha$ and such that

$$\begin{aligned} X_i &= \eta_i^{-1}(U_i \times S') , \\ Y_i &= \xi_i^{-1}(W_i \times S') , \\ Y_\alpha &= \xi_\alpha^{-1}(W_\alpha \times S') , \\ Z_i &= \zeta_i^{-1}(N_i \times S') \quad \text{and} \\ Z_\alpha &= \zeta_\alpha^{-1}(N_\alpha \times S') , \end{aligned}$$

for each $i \in I$ and for each $\alpha \in A$,

(6) there are coordinate systems

$$\begin{aligned} (z_i) &= (z_i^1, \cdots, z_i^d) , \\ (w_i) &= (w_i^1, \cdots, w_i^r) , \\ (w_\alpha) &= (w_\alpha^1, \cdots, w_\alpha^r) , \\ (y_i) &= (y_i^1, \cdots, y_i^q) , \\ (y_\alpha) &= (y_\alpha^1, \cdots, y_\alpha^q) \quad \text{and} \\ (s) &= (s^1, \cdots, s^k) \end{aligned}$$

in $\tilde{U}_i$, $\tilde{W}_i$, $\tilde{W}_\alpha$, $\tilde{N}_i$, $\tilde{N}_\alpha$, and $\tilde{\Omega}$ respectively, where $\tilde{S} \subset \tilde{\Omega}$ is a neat imbedding, such that

$$\begin{aligned} U_i &= \{z_i \in \tilde{U}_i \mid |z_i| < 1\} , \\ W_i &= \{w_i \in \tilde{W}_i \mid |w_i| < 1\} , \\ W_\alpha &= \{w_\alpha \in \tilde{W}_\alpha \mid |w_\alpha| < 1\} , \\ N_i &= \{y_i \in \tilde{N}_i \mid |y_i| < 1\} , \\ N_\alpha &= \{y_\alpha \in \tilde{N}_\alpha \mid |y_\alpha| < 1\} \quad \text{and} \\ S' &= \{s \in \tilde{S} \mid |s| < 1\} , \end{aligned}$$

$$(7)\qquad \begin{aligned} &f(\eta_i^{-1}(U_i \times o)) \subset \xi_i^{-1}(W_i \times o)\,,\\ &f(\eta_i^{-1}(\tilde{U}_i \times o)) \subset \xi_i^{-1}(\tilde{W}_i \times o)\,,\\ &g(\xi_i^{-1}(W_i \times o)) \subset \zeta_i^{-1}(N_i \times o)\,,\\ &g(\xi_i^{-1}(\tilde{W}_i \times o)) \subset \zeta_i^{-1}(\tilde{N}_i \times o)\,,\\ &g(\xi_\alpha^{-1}(W_\alpha \times o)) \subset \zeta_\alpha^{-1}(N_\alpha \times o) \quad \text{and}\\ &g(\xi_\alpha^{-1}(\tilde{W}_\alpha \times o)) \subset \zeta_\alpha^{-1}(\tilde{N}_\alpha \times o) \end{aligned}$$

for each $i \in I$ and for each $\alpha \in A$.

Henceforth, we identify $\eta_i^{-1}(U_i \times o)$, $\eta_i^{-1}(\tilde{U}_i \times o)$, $\xi_i^{-1}(W_i \times o)$, $\xi_i^{-1}(\tilde{W}_i \times o)$, $\xi_\alpha^{-1}(W_\alpha \times o)$, $\xi_\alpha^{-1}(\tilde{W}_\alpha \times o)$, $\zeta_i^{-1}(N_i \times o)$, $\zeta_i^{-1}(\tilde{N}_i \times o)$, $\zeta_\alpha^{-1}(N_\alpha \times o)$ and $\zeta_\alpha^{-1}(\tilde{N}_\alpha \times o)$ with U_i, $\tilde{U}_i$, W_i, $\tilde{W}_i$, W_α, $\tilde{W}_\alpha$, N_i, $\tilde{N}_i$, N_α and $\tilde{N}_\alpha$ respectively.

We put

$$\Omega = \{s \in \tilde{\Omega} \mid |s| < 1\}\,.$$

Then $S' = \tilde{S} \cap \Omega$.

Let e, $0 < e < 1$, be a positive number. We put

$$U_i^e = \{z_i \in U_i \mid |z_i| < 1 - e\} \quad \text{etc.}\,.$$

Then, by Lemmas 3.3 and 3.4, taking e sufficiently small, we may assume that

(8) $\{U_i^e\}_{i\in I}$, $\{W_i^e\}_{i\in I}$, $\{N_i^e\}_{i\in I}$, $\{W_i^e\}_{i\in I} \cup \{W_\alpha^e\}_{\alpha\in A}$ and $\{N_i^e\}_{i\in I} \cup \{N_\alpha^e\}_{\alpha\in A}$ cover V, $f(V)$, $gf(V)$, W, and $g(W)$ respectively and

(9) $f(U_i) \subset W_i^e$, $g(W_i) \subset N_i^e$, and $g(W_\alpha) \subset N_\alpha^e$ for each $i \in I$ and for each $\alpha \in A$.

We put $F = f^*TW$ and $G = g^*TN$. Let $C^p(F, |\ |)$ and $C^p(G, |\ |)$ be the Banach spaces defined in §2 with respect to coverings $\{U_i\}_{i\in I}$ of V and $\{W_i\}_{i\in I} \cup \{W_\alpha\}_{\alpha\in A}$ of W respectively.

Now we express the map f by the equations

$$w_i = f_i(z_i)$$

for $z_i \in U_i$ and $i \in I$. We also express the map g by the equations

$$y_i = g_i(w_i) \quad \text{and}$$
$$y_\alpha = g_\alpha(w_\alpha)$$

for $w_i \in W_i$, $i \in I$, and $w_\alpha \in W_\alpha$, $\alpha \in A$. Let s be a point of S'. Let

$$f'\colon \pi^{-1}(s) \to \mu^{-1}(s) \quad \text{and}$$
$$g'\colon \mu^{-1}(s) \to \tau^{-1}(s)$$

be holomorphic maps such that

$$f'(\pi^{-1}(s) \cap X_i) \subset \mu^{-1}(s) \cap Y_i\,,$$

$$g'(\mu^{-1}(s) \cap Y_i) \subset \tau^{-1}(s) \cap Z_i \quad \text{and}$$
$$g'(\mu^{-1}(s) \cap Y_\alpha) \subset \tau^{-1}(s) \cap Z_\alpha$$

for each $i \in I$ and for each $\alpha \in A$. We express the map f' by the equations

$$w_i = f'_i(z_i)$$

for $z_i \in U_i$, $i \in I$, using the isomorphisms

$$\eta_i: X_i \to U_i \times S' \quad \text{and}$$
$$\xi_i: Y_i \to W_i \times S' .$$

We express the map g' by the equations

$$y_i = g'_i(w_i) \quad \text{and}$$
$$y_\alpha = g'_\alpha(w_\alpha)$$

for $w_i \in W_i$, $i \in I$, and $w_\alpha \in W_\alpha$, $\alpha \in A$, using the isomorphisms

$$\xi_i: Y_i \to W_i \times S' ,$$
$$\xi_\alpha: Y_\alpha \to W_\alpha \times S' ,$$
$$\zeta_i: Z_i \to N_i \times S' \quad \text{and}$$
$$\zeta_\alpha: Z_\alpha \to N_\alpha \times S' .$$

Then the vector valued holomorphic functions f'_i, g'_i, and g'_α satisfy

$$f'_i(U_i) \subset W_i ,$$
$$g'_i(W_i) \subset N_i \quad \text{and}$$
$$g'_\alpha(W_\alpha) \subset N_\alpha .$$

We write

$$f'_i = f_i + \phi_i ,$$
$$g'_i = g_i + \psi_i \quad \text{and}$$
$$g'_\alpha = g_\alpha + \psi_\alpha$$

for each $i \in I$ and for each $\alpha \in A$. We consider elements

$$\phi = \{\phi_i\}_{i \in I} \in C^0(F, |\ |) \quad \text{and}$$
$$\psi = \{\psi_i\}_{i \in I} \cup \{\psi_\alpha\}_{\alpha \in A} \in C^0(G, |\ |) .$$

In §5, we have associated to f' and g',

$$(\phi, s) \in C^0(F, |\ |) \times T_oS \quad \text{and}$$
$$(\psi, s) \in C^0(G, |\ |) \times T_oS$$

respectively. Now the holomorphic map

$$g'f': \pi^{-1}(s) \to \tau^{-1}(s)$$

satisfies

$$g'f'(\pi^{-1}(s) \cap X_i) \subset \tau^{-1}(s) \cap Z_i$$

for each $i \in I$. The map $g'f'$ is expressed by the equations

$$y_i = g'_i(f'_i(z_i)) = g'_i f'_i(z_i)$$

for $z_i \in U_i$, $i \in I$, using the isomorphisms

$$\eta_i: X_i \to U_i \times S' \quad \text{and}$$
$$\zeta_i: Z_i \to N_i \times S' .$$

The vector valued holomorphic function $g'_i f'_i$ satisfies

$$g'_i f'_i(U_i) \subset N_i .$$

We put

$$g'_i f'_i = g_i f_i + \kappa_i$$

for each $i \in I$. Then

$$\kappa_i(z_i) = g_i(f_i(z_i) + \phi_i(z_i)) - g_i(f_i(z_i)) \\ + \psi_i(f_i(z_i) + \phi_i(z_i))$$

for $z_i \in U_i$, $i \in I$. We consider the element

$$\kappa = \{\kappa_i\}_{i \in I} \in C^0(H, |\ |)$$

where $H = (gf)^* TN = f^*G$ and $C^0(H, |\ |)$ is the Banach space defined in §2 with respect to the covering $\{U_i\}_{i \in I}$ of V. In §5, we have associated to the map $g'f'$

$$(\kappa, s) \in C^0(H, |\ |) \times T_o S .$$

Let ε, $0 < \varepsilon < 1$, be a small positive number satisfying Lemmas 3.5–3.10 with respect to all pairs

$$(\{X_i\}_{i \in I}, \{Y_i\}_{i \in I}) ,$$
$$(\{Y_i\}_{i \in I} \cup \{Y_\alpha\}_{\alpha \in A}, \{Z_i\}_{i \in I} \cup \{Z_\alpha\}_{\alpha \in A}) \quad \text{and}$$
$$(\{X_i\}_{i \in I}, \{Z_i\}_{i \in I})$$

(Lemma 3.9 for $A = \overline{f_k(U_k)}$ for all $k \in I$, etc.). Let $B_\varepsilon(F)$(resp. $B_\varepsilon(G)$) be the open ε-ball in $C^0(F, |\ |)$(resp. $C^0(G, |\ |)$) with the center the origin. We define a norm $|\ |$ in $C^0(F, |\ |) \times C^0(G, |\ |)$ by

$$|(\phi, \psi)| = \max(|\phi|, |\psi|)$$

for $(\phi, \psi) \in C^0(F, |\ |) \times C^0(G, |\ |)$. Then $C^0(F, |\ |) \times C^0(G, |\ |)$ is a Banach space and $B_\varepsilon(F) \times B_\varepsilon(G)$ is the open ε-ball in $C^0(F, |\ |) \times C^0(G, |\ |)$ with the center (0, 0).

We define a map

$$\kappa\colon B_\varepsilon(F) \times B_\varepsilon(G) \to C^0(H, |\ |)$$

by

$$\kappa(\phi, \psi)_i(z_i) = g_i(f_i(z_i) + \phi_i(z_i)) - g_i(f_i(z_i)) + \psi_i(f_i(z_i) + \phi_i(z_i))$$

for $z_i \in U_i$, $i \in I$. Then $\kappa(0, 0) = 0$.

LEMMA 7.1. *Let* $\varepsilon < e/2$. *Then*

$$\kappa\colon B_\varepsilon(F) \times B_\varepsilon(G) \to C^0(H, |\ |)$$

is an analytic map.

PROOF. We show that for any affine line L in $C^0(F, |\ |) \times C^0(G, |\ |)$, κ is an analytic map of $L \cap (B_\varepsilon(F) \times B_\varepsilon(G))$ into $C^0(H, |\ |)$. This implies that the map

$$\kappa\colon B_\varepsilon(F) \times B_\varepsilon(G) \to C^0(H, |\ |)$$

is analytic, (see e.g., Proposition 2 of [2]). We take a point $(\phi^0, \psi^0) \in L \cap (B_\varepsilon(F) \times B_\varepsilon(G))$. Then L is written as

$$L(t) = (\phi^0, \psi^0) + t(\phi^1, \psi^1)$$

for $t \in \boldsymbol{C}$ where $(\phi^1, \psi^1) \in C^0(F, |\ |) \times C^0(G, |\ |)$. We may assume that $(\phi^1, \psi^1) \in B_\varepsilon(F) \times B_\varepsilon(G)$ and $L(t) \in B_\varepsilon(F) \times B_\varepsilon(G)$ for all $t \in \Delta$, where

$$\Delta = \{t \in \boldsymbol{C} \mid |t| < 1\} .$$

Now

$$\begin{aligned}(\kappa L(t))_i(z_i) = g_i(f_i(z_i) &+ \phi_i^0(z_i) + t\phi_i^1(z_i)) \\ - g_i(f_i(z_i)) &+ \psi_i^0(f_i(z_i) + \phi_i^0(z_i) + t\phi_i^1(z_i)) \\ + t\psi_i^1(f_i(z_i) &+ \phi_i^0(z_i) + t\phi_i^1(z_i))\end{aligned}$$

for $z_i \in U_i$, $i \in I$, and $t \in \Delta$. We put

$$\begin{aligned}A(t)_i(z_i) &= g_i(f_i(z_i) + \phi_i^0(z_i) + t\phi_i^1(z_i)) - g_i(f_i(z_i)) , \\ B(t)_i(z_i) &= \psi_i^0(f_i(z_i) + \phi_i^0(z_i) + t\phi_i^1(z_i)) \quad \text{and} \\ C(t)_i(z_i) &= t\psi_i^1(f_i(z_i) + \phi_i^0(z_i) + t\phi_i^1(z_i)) .\end{aligned}$$

We put

$$\begin{aligned}A(t) &= \{A(t)_i\}_{i \in I} , \\ B(t) &= \{B(t)_i\}_{i \in I} \quad \text{and} \\ C(t) &= \{C(t)_i\}_{i \in I} .\end{aligned}$$

We show that $B(t)$ is an analytic map of Δ into $C^0(H, |\ |)$. Similar arguments show that $A(t)$ and $C(t)$ are analytic.
We put

$$w_i = f_i(z_i) + \phi_i^0(z_i)$$

and

$$x = x(t) = t\phi_i^1(z_i) .$$

By (9) above,

$$|f_i(z_i)| < 1 - e$$

for all $z_i \in U_i$. Hence

$$|w_i| < 1 - e + \frac{e}{2} = 1 - \frac{e}{2}$$

by the assumption that $\varepsilon < e/2$. By Cauchy's estimate,

$$\psi_i^0(w_i + x) - \psi_i^0(w_i) \ll \sum \varepsilon x_1^{\nu_1} \cdots x_r^{\nu_r} \Big/ \left(\frac{e}{2}\right)^{\nu_1 + \cdots \nu_r} = D(x)$$

if $|w_i| < 1 - e/2$ and $|x| < e/2$, where $\sum$ is extended over all non-negative integers with $\nu_1 + \cdots + \nu_r \geqq 1$ and $\ll$ means that the absolute values of the coefficients of $\psi_i^0(w_i + x) - \psi_i^0(w_i)$ in the formal power series in $x_1, \cdots, x_r$ are less than those of the corresponding coefficients of $D(x)$. Hence

$$B(t)_i(z_i) - B(0)_i(z_i) \ll \sum \varepsilon(t\varepsilon)^{\nu_1} \cdots (t\varepsilon)^{\nu_r} \Big/ \left(\frac{e}{2}\right)^{\nu_1 + \cdots + \nu_r} = E(t)$$

for $z_i \in U_i$, $i \in I$. Thus

$$B(t) - B(0) \ll E(t) .$$

$E(t)$ converges absolutely for $t \in \Delta$. This shows that $B(t)$ is an analytic map of Δ into $C^0(H, |\ |)$. q.e.d.

Let $\varepsilon < e/2$. Let Ω_ε be the open ε-ball of T_oS with the center o. We put $S_\varepsilon = S' \cap \Omega_\varepsilon$. By Lemma 7.1, the map

$$\tilde{\kappa}\colon B_\varepsilon(F) \times B_\varepsilon(G) \times \Omega_\varepsilon \to C^0(H, |\ |) \times \Omega_\varepsilon$$

defined by

$$\tilde{\kappa}(\phi, \psi, s) = (\kappa(\phi, \psi), s)$$

is an analytic map, where $B_\varepsilon(F) \times B_\varepsilon(G) \times \Omega_\varepsilon$ is the open ε-ball in the

Banach space $C^0(F, |\ |) \times C^0(G, |\ |) \times T_oS$ with the center the origin. Let

$$K_f: B_\varepsilon(F) \times \Omega_\varepsilon \to C^1(F, |\ |) ,$$
$$K_g: B_\varepsilon(G) \times \Omega_\varepsilon \to C^1(G, |\ |) \quad \text{and}$$
$$K_{gf}: B_\varepsilon(H) \times \Omega_\varepsilon \to C^1(H, |\ |)$$

be the maps defined in §5 with respect to f, g, and gf respectively. Let ε be sufficiently small. Then K_f, K_g, and K_{gf} are analytic by Proposition 5.1. We put

$$M_f = \{(\phi, s) \in B_\varepsilon(F) \times S_\varepsilon \mid K_f(\phi, s) = 0\} ,$$
$$M_g = \{(\psi, s) \in B_\varepsilon(G) \times S_\varepsilon \mid K_g(\psi, s) = 0\} \quad \text{and}$$
$$M_{gf} = \{(\kappa, s) \in B_\varepsilon(H) \times S_\varepsilon \mid K_{gf}(\kappa, s) = 0\} .$$

Now the set

$$C = (C^0(F, |\ |) \times T_oS) \times_{T_oS} (C^0(G, |\ |) \times T_oS)$$
$$= \{((\phi, s), (\psi, s')) \in (C^0(F, |\ |) \times T_oS) \times (C^0(G, |\ |) \times T_oS) \mid s = s'\}$$

is a closed subspace of the Banach space $(C^0(F, |\ |) \times T_oS) \times (C^0(G, |\ |) \times T_oS)$ and is isomorphic to the Banach space

$$C^0(F, |\ |) \times C^0(G, |\ |) \times T_oS$$

by the map

$$j: ((\phi, s), (\psi, s)) \to (\phi, \psi, s) .$$

The open ε-ball

$$C_\varepsilon = (B_\varepsilon(F) \times \Omega_\varepsilon) \times_{\Omega_\varepsilon} (B_\varepsilon(G) \times \Omega_\varepsilon)$$

in C with the center the origin contains $M_f \times_{S_\varepsilon} M_g$. By the definition of $\tilde{\kappa}$, $\tilde{\kappa}$ maps $j(M_f \times_{S_\varepsilon} M_g)$ into M_{gf}.

Let

$$\Phi_f: B_{\varepsilon'}(F) \times \Omega_{\varepsilon'} \to U_f \subset B_\varepsilon(F) \times \Omega_\varepsilon ,$$
$$\Phi_g: B_{\varepsilon'}(G) \times \Omega_{\varepsilon'} \to U_g \subset B_\varepsilon(G) \times \Omega_\varepsilon \quad \text{and}$$
$$\Phi_{gf}: B_{\varepsilon'}(H) \times \Omega_{\varepsilon'} \to U_{gf} \subset B_\varepsilon(H) \times \Omega_\varepsilon$$

be the analytic isomorphisms defined in §5 with respect to f, g and gf respectively. We may assume that $\tilde{\kappa}$ maps $j(C \cap (U_f \times U_g))$ into U_{gf}.

Let T_f, T_g, and T_{gf} be the analytic spaces defined in §5 with respect to f, g, and gf respectively. Then, by the definitions of T_f, T_g, and T_{gf},

$$\Phi_f(T_f) = M_f \cap U_f\,,$$
$$\Phi_g(T_g) = M_g \cap U_g \quad \text{and}$$
$$\Phi_{gf}(T_{gf}) = M_{gf} \cap U_{gf}.$$

Now we define a holomorphic map

$$c\colon T_f \underset{S_{\varepsilon'}}{\times} T_g \to T_{gf}$$

by

$$c((\xi, s), (\xi', s)) = \Phi_{gf}^{-1}(\tilde{\kappa} j(\Phi_f(\xi, s), \Phi_g(\xi', s)))\,.$$

Then the map

$$H(X, Y; S) \underset{S}{\times} H(Y, Z; S) \to H(X, Z; S)\,,$$

defined by

$$(f, g) \to gf$$

for (f, g) with $\lambda_{XY}(f) = \lambda_{YZ}(g)$, is locally given by the map c. This completes the proof of Theorem 4.

In order to prove Main Theorem, we will need the following lemma.

LEMMA 7.2. *The derivative $\kappa'(0, 0)$ at $(0, 0)$ of the analytic map κ in Lemma 7.1 is given by*

$$\kappa'(0, 0)(\phi, \psi) = (f^*J_g)\phi + f^*\psi$$

for $(\phi, \psi) \in C^0(F, |\ |) \times C^0(G, |\ |)$ where

$$((f^*J_g)\phi)_i(z_i) = (\partial g_i/\partial w_i)(f_i(z_i))\phi_i(z_i)$$

($(\partial g_i/\partial w_i)(f_i(z_i))$ is a matrix operating on the vector $\phi_i(z_i)$), and

$$(f^*\psi)_i(z_i) = \psi_i(f_i(z_i))\,, \quad \textit{for} \quad z_i \in U_i,\ i \in I\,.$$

PROOF. We note that $\kappa(0, 0) = 0$. Now, for $z_i \in U_i$,

$$\begin{aligned}\kappa(\phi, \psi)_i(z_i) &= g_i(f_i(z_i) + \phi_i(z_i)) - g_i(f_i(z_i)) \\ &\quad + \psi_i(f_i(z_i) + \phi_i(z_i)) - \psi_i(f_i(z_i)) + \psi_i(f_i(z_i)) \\ &= (\partial g_i/\partial w_i)(f_i(z_i))\phi_i(z_i) + (\partial \psi_i/\partial w_i)(f_i(z_i))\phi_i(z_i) \\ &\quad + \psi_i(f_i(z_i)) + o(\phi, \psi)\end{aligned}$$

where $o(\phi, \psi)$ is some function of (ϕ, ψ) (and of $z_i \in U_i$) such that

$$|o(\phi, \psi)|/|(\phi, \psi)| \to 0$$

as $|(\phi, \psi)| \to 0$. Since $f_i(z_i) \in W_i^\varepsilon$ for $z_i \in U_i$ by (9) above,

$$|(\partial \psi_i/\partial w_i)(f_i(z_i))|\,, \quad z_i \in U_i\,,$$

is estimated by $|\psi|$. Hence we may put

$$(\partial\psi_i/\partial w_i)(f_i(z_i))\phi_i(z_i) = o(\phi, \psi) .$$

Thus

$$\kappa(\phi, \psi)_i(z_i) = (\partial g_i/\partial w_i)(f_i(z_i))\phi_i(z_i) + \psi_i(f_i(z_i)) + o(\phi, \psi) .$$

q.e.d.

8. **Proof of Main Theorem.** Let (X, π, S) and (Y, μ, S) be families of compact complex manifolds. We assume that S satisfies the second axiom of countability. Since (X, π, S) and (Y, μ, S) are topological fiber bundles (see e.g., [7]), X and Y satisfy the second axiom of countability. By Theorem 2,

$$H = \coprod_{s\in S} H(\pi^{-1}(s), \mu^{-1}(s))$$

admits an analytic space structure such that (H, λ, S) is a complex fiber space where

$$\lambda: H \rightarrow S$$

is the canonical projection. Let $s \in S$. We denote by $I(\pi^{-1}(s), \mu^{-1}(s))$ the set of all holomorphic isomorphisms of $\pi^{-1}(s)$ onto $\mu^{-1}(s)$. (It may be empty.)

LEMMA 8.1. *The disjoint union*

$$I = \coprod_{s\in S} I(\pi^{-1}(s), \mu^{-1}(s))$$

is an open subset of H.

PROOF. Let o be a point of S. We put as before $V = \pi^{-1}(o)$ and $W = \mu^{-1}(o)$. Let f be a holomorphic isomorphism of V onto W. Let (E, T, b) be the maximal family of holomorphic maps of (X, π, S) into (Y, μ, S) constructed in §5 with respect to f. We use the notations in §5. For $t \in T$, E_t is a holomorphic map of $\pi^{-1}(b(t))$ into $\mu^{-1}(b(t))$. In particular, $E_{(0,o)} = f$. We write 0 instead of $(0, o)$ to simplify the notation. We show that there is an open neighborhood T' of 0 in T such that, for each $t \in T'$, E_t is a holomorphic isomorphism of $\pi^{-1}(b(t))$ onto $\mu^{-1}(b(t))$. Since T gives a local chart in H, this proves the lemma.

The map

$$E: b^*X \rightarrow b^*Y$$

is given by the equations

$$w_i = f_i(z_i) + \phi_i(z_i, t) ,$$
$$t = t ,$$

for $(z_i, t) \in U_i \times T$. Its Jacobian matrix at $(z_i, 0)$ is

$$\begin{pmatrix} (\partial f_i/\partial z_i)(z_i) & (\partial \phi_i/\partial t)(z_i, 0) \\ 0 & 1 \end{pmatrix}.$$

It is non-singular. Noting that V is compact, this implies that there is an open neighborhood T' of 0 in T such that

$$E\colon (b^*\pi)^{-1}(T') \to (b^*\mu)^{-1}(T')$$

is a local isomorphism. In particular, E_t is a local isomorphism of $\pi^{-1}(b(t))$ into $\mu^{-1}(b(t))$ for each $t \in T'$.

Next we show that E_t is surjective for each $t \in T'$ provided T' is sufficiently small. Since V is compact, the number of connected components of V is finite. We arrange them as follows:

$$V_1, \cdots, V_m .$$

Since f is a holomorphic isomorphism of V onto W, connected components of W are

$$W_1 = f(V_1), \cdots, W_m = f(V_m) .$$

On the other hand, it is known [7] that there are an open neighborhood T' of 0 in T and a continuous retraction

$$R_1\colon (b^*\pi)^{-1}(T') \to V$$

such that $R_{1t} = R_1 | (b^*\pi)^{-1}(t)$ is a C^∞-diffeomorphism of $(b^*\pi)^{-1}(t) = \pi^{-1}(b(t))$ onto V for each $t \in T'$. Hence $\pi^{-1}(b(t))$ has m connected components

$$V_1(t) = R_{1t}^{-1}(V_1), \cdots, V_m(t) = R_{1t}^{-1}(V_m) .$$

In a similar way, there is a continuous retraction

$$R_2\colon (b^*\mu)^{-1}(T') \to W$$

such that $R_{2t} = R_2 | (b^*\mu)^{-1}(t)$ is a C^∞-diffeomorphism of $\mu^{-1}(b(t))$ onto W for each $t \in T'$. Hence $\mu^{-1}(b(t))$ has m connected components

$$W_1(t) = R_{2t}^{-1}(W_1), \cdots, W_m(t) = R_{2t}^{-1}(W_m) .$$

We may assume that T' is connected. Then we show that connected components of $(b^*\pi)^{-1}(T')$ and $(b^*\mu)^{-1}(T')$ are

$$X_\alpha = \bigcup_{t \in T'} V_\alpha(t), \ \alpha = 1, \cdots, m$$

and

$$Y_\alpha = \bigcup_{t \in T'} W_\alpha(t), \ \alpha = 1, \cdots, m$$

respectively. We note that the map

$$\tilde{R}_1\colon (b^*\pi)^{-1}(T') \to V \times T'$$

defined by

$$\tilde{R}_1(P) = (R_1(P),\ b^*\pi(P)),$$

for $P \in (b^*\pi)^{-1}(T')$, is a homeomorphism, as is easily seen. In order to show that X_α is connected, it is enough to show that any point $P \in V_\alpha(t)$ is connected to $R_{1t}(P) \in V_\alpha$ by a curve in X_α. Let $c(\tau)$ be a continuous curve in T' such that $c(0) = t$ and $c(1) = 0$. Then the curve

$$d(\tau) = \tilde{R}_1^{-1}(R_{1t}(P),\ c(\tau))$$

belongs in X_α and $d(0) = P$ and $d(1) = R_{1t}(P)$. Hence X_α is connected. We show that any points $P \in X_\alpha$ and $Q \in X_\beta$, $\alpha \neq \beta$, can not be connected by a curve in $(b^*\pi)^{-1}(T')$. If it is so, then the above argument shows that some points $P \in V_\alpha$ and $Q \in V_\beta$, $\alpha \neq \beta$, is connected by a curve $d(\tau)$ in $(b^*\pi)^{-1}(T')$. Then P and Q are connected by the curve $R_1(d(\tau))$ in V, a contradiction. Hence X_α, $\alpha = 1, \cdots, m$ are connected components of $(b^*\pi)^{-1}(T')$. In a similar way, we see that Y_α, $\alpha = 1, \cdots, m$ are connected components of $(b^*\mu)^{-1}(T')$. Now we take T' sufficiently small so that E_t is a local isomorphism of $\pi^{-1}(b(t))$ into $\mu^{-1}(b(t))$ for each $t \in T'$. Then $E_t(V_\alpha(t))$ coincides with a connected component of $\mu^{-1}(b(t))$ for each $t \in T'$ and for each α. Since $E_t(z) = E(z, t)$ is holomorphic (and hence continuous) in both variables, $E_t(V_\alpha(t))$ and $f(V_\alpha) = W_\alpha$ belong to the same connected component Y_α of $(b^*\mu)^{-1}(T')$. Thus $E_t(V_\alpha(t)) = W_\alpha(t)$. This shows that E_t is surjective for each $t \in T'$.

Finally we show that E_t is injective for each $t \in T'$ provided T' is sufficiently small. We assume the converse. Then there are a sequence $\{t_n\}$ in T' converging to 0 and a sequence of pairs of *different* points $\{(P_n, Q_n)\}_{n=1,2,\cdots}$ of $\pi^{-1}(b(t_n))$ such that $E_{t_n}(P_n) = E_{t_n}(Q_n)$, $n = 1, 2, \cdots$. Since π is a proper map, we may assume that

$$P_n \to P \in V \quad \text{and}$$
$$Q_n \to Q \in V$$

as $n \to +\infty$. Then $f(P) = f(Q)$ so that $P = Q$. Since

$$E\colon (b^*\pi)^{-1}(T') \to (b^*\mu)^{-1}(T')$$

is a local isomorphism, there is an open neighborhood X' of P in $(b^*\pi)^{-1}(T')$ such that E is an isomorphism on X'. If n is sufficiently large, P_n and Q_n belong to X'.
Thus

$$E(P_n) = E_{t_n}(P_n) = E_{t_n}(Q_n) = E(Q_n)$$

implies that $P_n = Q_n$, a contradiction. q.e.d.

Let (X, π, S) be a family of compact complex manifolds. We assume that S satisfies the second axiom of countability. Then, by Lemma 8.1,

$$A = \coprod_{s \in S} \mathrm{Aut}(\pi^{-1}(s))$$

is an open subset of the analytic space

$$H = \coprod_{s \in S} H(\pi^{-1}(s), \pi^{-1}(s)) .$$

Hence A is an analytic space. The canonical projection

$$\lambda: A \to S$$

is holomorphic by Theorem 2. For each $s \in S$, $\mathrm{Aut}(\pi^{-1}(s))$ contains the identity map I_s. Hence λ is surjective. This shows (1) of Main Theorem.

$X \times_S A$ is an open subset of $X \times_S H$. By Theorem 2, the map

$$X \times_S A \to X$$

defined by

$$(P, f) \to f(P) ,$$

where $\pi(P) = \lambda(f)$, is holomorphic. This shows (2) of Main Theorem.

Now, we show (3) of Main Theorem. Let o be a point of S. Let I_o be the identity map of $V = \pi^{-1}(o)$. We review the considerations in §3–§5 replacing f, (Y, μ, S), (w_i), h_{ik} and ξ_i in §3 to I_o, (X, π, S), (z_i), g_{ik} and η_i respectively. We may assume that open sets W_i and $\widetilde{W}_i$ in §3 satisfy

$$U_i \Subset W_i \Subset \widetilde{U}_i \Subset \widetilde{W}_i$$

in the present case. We may also assume that

$$W_i = \{z_i \in \widetilde{U}_i \mid |z_i| < 1 + e'\}$$

and

$$W_i^e = \{z_i \in \widetilde{U}_i \mid |z_i| < 1 + e' - e\}$$

where e and e' are small positive numbers such that $0 < e < e' < 1$. The holomorphic vector bundle F in §4 becomes TV (the holomorphic tangent bundle) in the present case. Now let $s \in S'$ and let f' be a holomorphic map of $\pi^{-1}(s)$ into itself. We assume that

$$f'(\pi^{-1}(s) \cap X_i) \Subset \pi^{-1}(s) \cap Y_i$$

where $X_i = \eta_i^{-1}(U_i \times S')$ and $Y_i = \eta_i^{-1}(W_i \times S')$. Then f' is expressed locally as vector valued holomorphic functions $f_i'(z_i)$, $z_i \in U_i$. We put

$$\phi_i(z_i) = f'_i(z_i) - z_i \quad \text{and}$$
$$\phi = \{\phi_i\}_{i \in I} \in C^0(F, |\ |) .$$

We have associated $(\phi, s) \in C^0(F, |\ |) \times T_oS$ to f' in §5.

Now we assume that $f' = I_s$, the identity map of $\pi^{-1}(s)$. Then the local expression $f'_i(z_i)$ of f' must be the identity function: $f'_i(z_i) = z_i$. Hence the corresponding ϕ must be zero. We use the notations in §5. We put

$$M = \{(\phi, s) \in B_\varepsilon \times S_\varepsilon \,|\, K(\phi, s) = 0\} .$$

Then the above consideration shows that

$$(0, s) \in M$$

for all $s \in S_\varepsilon$. On the other hand, the map L in §5 was defined by

$$L(\phi, s) = (\phi + E_0 B \Lambda K(\phi, s) - E_0 \delta\phi, s) .$$

Thus

$$L(0, s) = (0 + E_0 B \Lambda K(0, s) - E_0 \delta 0, s) = (0, s) .$$

Hence the set

$$\{(0, s) \in (H^0(F|\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} \,|\, s \in S_{\varepsilon'}\}$$

is contained in

$$T = \{(\xi, s) \in (H^0(F, |\ |) \cap B_{\varepsilon'}) \times S_{\varepsilon'} \,|\, H \Lambda K \Phi(\xi, s) = 0\} .$$

Each $(0, s) \in T$, $s \in S_{\varepsilon'}$, corresponds to the identity map I_s of $\pi^{-1}(s)$. The map

$$s \in S_{\varepsilon'} \rightarrow (0, s) \in T$$

is holomorphic. The proof of Lemma 8.1 shows that there is an open neighborhood T' of $(0, o)$ in T such that T' gives a local chart in A around I_o. This proves (3) of Main Theorem.

Finally we prove (4) of Main Theorem.

LEMMA 8.2. *Let (X, π, S) be a family of compact complex manifolds. We assume that S satisfies the second axiom of countability. Let*

$$A = \coprod_{s \in S} \mathrm{Aut}\,(\pi^{-1}(s))$$

be the analytic space whose analytic space structure is introduced above. Then the map

$$f \in A \rightarrow f^{-1} \in A$$

is holomorphic.

PROOF. Let o be a point of S. Let f be an automorphism of $V = \pi^{-1}(o)$. We replace f and g in the proof of Theorem 4 to f^{-1} and f respectively. Thus, in the present case, we replace (Z, τ, S), (y_i) and ζ_i to (X, π, S), (z_i), and η_i respectively. We may assume that the open sets N_i and $\tilde{N}_i$ in §7 satisfy

$$U_i \Subset N_i \Subset \tilde{U}_i \Subset \tilde{N}_i$$

in the present case. We may assume that

$$N_i = \{z_i \in \tilde{U}_i \,|\, |z_i| < 1 + e'\}$$

and

$$N_i^e = \{z_i \in \tilde{U}_i \,|\, |z_i| < 1 + e' - e\}$$

where e and e' are small positive numbers such that $0 < e < e' < 1$. We note that the set A of indices in §7 is empty in the present case. Now we put

$$h = f^{-1} .$$

Let s be a point of S'. Let h' and f' be holomorphic maps of $\pi^{-1}(s)$ into itself such that

$$h'(\pi^{-1}(s) \cap X_i) \Subset \pi^{-1}(s) \cap Y_i \quad \text{and}$$
$$f'(\pi^{-1}(s) \cap Y_i) \Subset \pi^{-1}(s) \cap Z_i$$

where $Z_i = \eta_i^{-1}(N_i \times S')$. We express the maps h' and f' by the equations

$$w_i = h_i'(z_i) ,$$

for $z_i \in U_i$, and

$$z_i = f_i'(w_i) ,$$

for $w_i \in W_i$, respectively. We write

$$h_i' = h_i + \phi_i \quad \text{and}$$
$$f_i' = f_i + \psi_i .$$

We consider the elements

$$\phi = \{\phi_i\}_{i \in I} \in C^0(G, |\ |) \quad \text{and}$$
$$\psi = \{\psi_i\}_{i \in I} \in C^0(F, |\ |)$$

where $G = h^*TV = (f^{-1})^*TV$ and $F = f^*TV$.
As in §7, We associate

$$(\phi, s) \in C^0(G, |\ |) \times T_oS \quad \text{and}$$
$$(\psi, s) \in C^0(F, |\ |) \times T_oS$$

to h' and f' respectively. Then the composition $f'h'$ corresponds to

$$(\kappa, s) \in C^0(H, |\ |) \times T_o S$$

where $H = TV$ and $\kappa = \{\kappa_i\}_{i\in I}$ where

$$\kappa_i(z_i) = f_i(h_i(z_i) + \phi_i(z_i)) - z_i + \psi_i(h_i(z_i) + \phi_i(z_i))$$

for $z_i \in U_i$. We define a map

$$\kappa: B_\varepsilon(G) \times B_\varepsilon(F) \to C^0(H, |\ |)$$

by

$$\kappa(\phi, \psi)_i(z_i) = f_i(h_i(z_i) + \phi_i(z_i)) - z_i + \psi_i(h_i(z_i) + \phi_i(z_i))$$

for $z_i \in U_i$. By Lemma 7.1, κ is analytic, provided ε is sufficiently small. By Lemma 7.2,

$$\kappa'(0, 0)(\phi, \psi) = (h^* J_f)\phi + h^*\psi ,$$

for $(\phi, \psi) \in C^0(G, |\ |) \times C^0(F, |\ |)$, where

$$((h^* J_f)\phi)_i(z_i) = (\partial f_i/\partial w_i)(h_i(z_i))\phi_i(z_i)$$

for $z_i \in U_i$ and

$$(h^*\psi)_i(z_i) = \psi_i(h_i(z_i))$$

for $z_i \in U_i$. We consider an analytic map

$$\beta: B_\varepsilon(G) \times B_\varepsilon(F) \to C^0(H, |\ |) \times C^0(F, |\ |)$$

defined by

$$\beta(\phi, \psi) = (\kappa(\phi, \psi), \psi) .$$

Then

$$\beta'(0, 0) = \begin{pmatrix} h^* J_f & h^* \\ 0 & 1 \end{pmatrix} .$$

It is easy to see that

$$h^* J_f: C^0(G, |\ |) \to C^0(H, |\ |)$$

is a continuous linear isomorphism. Hence $\beta'(0, 0)$ is a continuous linear isomorphism. By the inverse mapping theorem, there are a small positive number ε', an open neighborhood U of $(0, 0)$ in $B_\varepsilon(G) \times B_\varepsilon(F)$ and an analytic isomorphism

$$\alpha: B_{\varepsilon'}(H) \times B_{\varepsilon'}(F) \to U$$

such that $\beta | U = \alpha^{-1}$. We write

$$\alpha(\kappa, \psi) = (\gamma(\kappa, \psi), \psi) .$$

Then the map

$$\psi \in B_{\varepsilon'}(F) \longrightarrow \gamma(0, \psi) \in B_{\varepsilon}(G)$$

is an analytic map. Hence the map

$$(\psi, s) \in B_{\varepsilon'}(F) \times \Omega_{\varepsilon'} \longrightarrow (\gamma(0, \psi), s) \in B_{\varepsilon}(G) \times \Omega_{\varepsilon'}$$

is analytic, where $\Omega_{\varepsilon'}$ is the open ε'-ball in T_oS with the center o. Now it is clear that if $(\psi, s) \in B_{\varepsilon'}(F) \times S_{\varepsilon'}$ corresponds to an automorphism f' of $\pi^{-1}(s)$, then $(\gamma(0, \psi), s)$ corresponds to $(f')^{-1}$. Let T_f and $T_{f^{-1}}$ be the analytic spaces constructed in §5 with respect to f and $h = f^{-1}$ respectively. Let Φ_f and $L_{f^{-1}}$ be the analytic maps defined in §5 with respect to f and f^{-1} respectively. The proof of Lemma 8.1 shows that if we take a sufficiently small open neighborhood T' of $(0, o)$ in T_f, then each $t = (\xi, s) \in T'$ corresponds to an automorphism E_t of $\pi^{-1}(s)$. We put $\Phi_f(t) = (\psi, s)$. Then the above argument shows that $L_{f^{-1}}(\gamma(0, \psi), s)$ belongs to $T_{f^{-1}}$ and corresponds to E_t^{-1}. Now the map

$$T' \longrightarrow T_{f^{-1}}$$

defined by

$$t = (\xi, s) \xrightarrow{\Phi_f} (\psi, s) \longrightarrow L_{f^{-1}}(\gamma(0, \psi), s)$$

is holomorphic. This proves Lemma 8.2. q.e.d.

Now, $A \times_S A$ is an open subset of $H \times_S H$. Hence Theorem 4 and Lemma 8.2 imply (4) of Main Theorem.

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