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EQUIVARIANT CHARACTERISTIC NUMBERS AND INTEGRALITY THEOREM FOR UNITARY T^n-MANIFOLDS

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1. Introduction. Let G be a compact Lie group and M a compact unitary (i.e., weakly complex) G-manifold. Thus G acts on M by diffeomorphisms preserving the given complex structure of the stable tangent bundle of M. The stable tangent bundle, with this G-action, defines an element $\bar{\tau}M$ in $\widetilde{K}_G(M)$ where $\widetilde{K}_G(M)$ denotes the kernel of the augmentation $K_G(M) \xrightarrow{\dim} H_0(X, \boldsymbol{Z})$. If $\boldsymbol{t} = (t_1, t_2, \cdots)$ is a sequence of indeterminates and V is a complex G-vector bundle V over M, we define $\gamma_t(V - \dim V)$ in $K_G(M)[[\boldsymbol{t}]]$ by

$$\gamma_t(V - \dim V) = \prod_{i=1}^{\dim V} (1 + t_1(V_j - 1) + t_2(V_i - 1)^2 + \cdots)$$

where V is written formally as

$$V = \sum_{i=1}^{\dim V} V_i \ .$$

γ_t extends to a map

$$\gamma_t \colon \widetilde{K}_G(M) \to K_G(M)[[\boldsymbol{t}]]$$

such that

$$\gamma_t(x + y) = \gamma_t(x)\gamma_t(y) \ .$$

Suppose that M is closed (i.e., compact and without boundary) and let $p_{M!} \colon K_G(M) \to K_G^* = K_G^*(\text{point})$ be the Gysin homomorphism of $p_M \colon M \to$ point. The element $p_{M!}(\gamma_t(\bar{\tau}M))$ in $K_G^*[[\boldsymbol{t}]]$ turns out to be an invariant of the G-equivariant bordism class the of unitary G-manifold M so that the assignment $[M] \mapsto p_!(\gamma_t(\bar{\tau}M))$ defines a homomorphism

$$\rho \colon U_*^G \to K_G^*[[\boldsymbol{t}]] \ ,$$

where U_*^G is the bordism ring of closed unitary G-manifolds. The homomorphism ρ also preserves the ring structure. The coefficients of the formal power series $\rho[M]$ are called equivariant K-theory characteristic numbers of $[M] \in U_*^G$. Note that the coefficient ring K_G^* has trivial odd-dimensional component K_G^{-1} and $K_G^* = K_G$ is canonically isomorphic to the representation ring $R(G)$ of G [4].

Let S denote the multiplicative set in K_G generated by the K-theory Euler classes $\lambda_{-1}(W)=\sum(-1)^i\lambda^i(W)$ of the nontrivial irreducible G-modules W. By the localization theorem of Atiyah-Segal [2], [3], the image of $\rho[M]=p_{M!}(\gamma_t(\tau M))$ in the localized ring $S^{-1}K_G[[t]]$ equals[1)]

$$p_{X!}\left(\gamma_t(\tau X)\frac{\gamma_t(V-\dim V)}{\lambda_{-1}(V)}\right)$$

where X is the fixed point set[2)] of the given G-action and V is the normal bundle of X in M with the induced G-action. This fits in with the following commutative diagram within the bordism context:

$$\begin{tikzcd} U_*^G \arrow[r, "\rho"] \arrow[d, "f"] & K_G[[t]] \arrow[d] \\ B_*^G \arrow[r, "\bar\rho"] & S^{-1}K_G[[t]] \end{tikzcd} \tag{1.1}$$

Here B_*^G is the bordism group derived from the pairs (X, V) of closed unitary manifolds X and complex vector bundles[3)] V over X having G-action by automorphisms without trivial irreducible factors. f is obtained by taking the bordism class of the fixed point set and its normal bundle. $\bar\rho$ is given by

$$\bar\rho[X, V]=p_{X!}\left(\gamma_t(\tau X)\frac{\gamma_t(V-\dim V)}{\lambda_{-1}(V)}\right). \tag{1.2}$$

The second vertical homomorphism is the canonical map.

Now consider the case where G is topologically cyclic,[4)] i.e., G is isomorphic to $Z_l\times T^n$, the product of a cyclic group of order l and an n-dimensional torus. It is well known that the canonical map $K_G\to S^{-1}K_G$ is an inclusion if $G=T^n$. The fixed point homomorphism f is also monic for $G=T^n$ (Hamrick-Ossa [13]). In this paper we shall prove

THEOREM (1.3). *If G is topologically cyclic then the homomorphism*

$$\bar\rho\colon B_*^G\to S^{-1}K_G[[t]]$$

1) Strictly speaking the statement is slightly more general than the theorem of Atiyah Segal. However, their proof can also be applied to our situation.

2) A point x of M is called fixed point of the G-action if the isotropy subgroup at x coincides with G itself. The set of all fixed points is called the fixed point set.

3) We make the convention that a manifold may have components of various dimension and a vector bundle over it may have various fiber dimensions over each component.

4) It can be shown that, if G is not topologically cyclic, then $S^{-1}K_G=\{0\}$. Thus $\bar\rho$ is non-trivial only if G is topologically cyclic.

is monic.

THEOREM (1.4) (*Integrality theorem*). *We have*

$$\bar{\rho}^{-1}(K_{T^n}[[t]]) = f(U_*^{T^n}) .$$

As an immediate corollary of (1.3) we obtain

COROLLARY (1.5). *The homomorphism*

$$\rho\colon U_*^{T^n} \to K_{T^n}[[t]]$$

is monic, i.e., the bordism classes of $U_*^{T^n}$ are determined by their equivariant K-theory characteristic numbers.

Theorem (1.4) has also the following implications. Let $U_*^G(\mathscr{F}(1))$ be the bordism group of closed unitary G-manifolds without fixed points. In the exact triangle of Conner-Floyd [7]

$$\begin{tikzcd} U_*^G \arrow[rr, "f"] & & B_*^G \arrow[dl, "\partial"] \\ & U_*^G(\mathscr{F}(1)) \arrow[ul] & \end{tikzcd} .$$

f is monic and hence ∂ is epic if $G = T^n$. Therefore, $U_*^{T^n}(\mathscr{F}(1))$ is canonically identified with $B_*^{T^n}/f(U_*^{T^n})$ and diagram (1.1) induces

$$\bar{\bar{\rho}}\colon U_*^{T^n}(\mathscr{F}(1)) \to S^{-1}K_{T^n}[[t]]/K_{T^n}[[t]] = (S^{-1}K_{T^n}/K_{T^n})[[t]].$$

The coefficients of $\bar{\bar{\rho}}[M]$ in $(S^{-1}K_{T^n}/K_{T^n})[[t]]$ are the v-invariants of $[M] \in U_*^{T^n}(\mathscr{F}(1))$ of Atiyah-Singer [5, §7]. Clearly, Theorem (1.4) implies

COROLLARY (1.6).

$$\bar{\bar{\rho}}\colon U_*^{T^n}(\mathscr{F}(1)) \to (S^{-1}K_{T^n}/K_{T^n})[[t]]$$

is monic.

Finally, we define the equivariant U^*-theory by $U_G^*(Y) = U^*(EG \times_G Y)$ where $EG \to BG$ is the universal G-bundle and Y is a G-space. If W is a G-module then $EG \times_G W \to BG$ is a complex vector bundle over BG whose Euler class in $U^*(BG)$ will be denoted by $e(W)$. Let S be the multiplicative set in $U_G^* = U^*(BG)$ generated by the U_G^*-theory Euler classes $e(W)$ of non-trivial irreducible G-modules W. Then diagram (1.1) for $G = T^n$ factors through

$$\begin{tikzcd} U_*^{T^n} \arrow[r, "\vartheta"] \arrow[d] & U_{T^n}^* \arrow[r, "B"] \arrow[d] & \hat{K_{T^n}}[[t]] \arrow[d] \\ B_*^{T^n} \arrow[r, "\bar{\vartheta}"] & S^{-1}U_{T^n}^* \arrow[r, "\bar{B}"] & S^{-1}\hat{K_{T^n}}[[t]] , \end{tikzcd} \qquad \begin{aligned} B \circ \vartheta &= \rho \\ \bar{B} \circ \bar{\vartheta} &= \bar{\rho} , \end{aligned}$$

where B is the Boardman map, $\bar{B}$ is induced from B (see § 5) and $K_G^{\wedge}$ denotes the completion of $K_G = R(G)$[5] [4], [18]. For the precise definition of ϑ and $\bar{\vartheta}$, see § 5. From (1.3) and (1.4) follows immediately

THEOREM (1.7). *The homomorphisms*

$$\vartheta: U_*^{T^n} \to U_{T^n}^*, \ \bar{\vartheta}: B_*^{T^n} \to S^{-1}U_{T^n}^* \quad \text{and} \quad \bar{\bar{\vartheta}}: U_*^{T^n}(\mathscr{F}(1)) \to S^{-1}U_{T^n}^*/U_{T^n}^*$$

are monic.

Equivariant bordism and equivariant characteristic numbers are extensively studied by tom Dieck [8], [9], [10], [11], [12]. In [12] he proved that, if G is the cyclic group of prime power order, then ϑ', $\bar{\vartheta}$ and $\bar{\bar{\vartheta}}$ are monic. He also proved the corresponding results for unoriented $(\boldsymbol{Z}_2)^k$-manifolds [10]. In his theory tom Dieck used an equivariant cohomology theory based on an equivariant Thom spectrum. Our method is more direct; only knowledge of ordinary K-theory and U^*-theory is needed. In the proof of integrality theorem (1.4), Propositions (3.2) and (3.3) are crucial. If G is a torus a geometric construction, essentially due to Ossa [16] (cf. also [13] and [15]) yields a G-manifold bounded by a given G-manifold without fixed points. That construction is used to define the homomorphism Q_α in (3.2).

The proof of (1.3) is given in § 2. Theorem (1.4) is proved in § 3. In § 4 the definition of ϑ, $\bar{\vartheta}$ and their relation to ρ, $\bar{\rho}$ are briefly discussed.

The results of the present paper have been announced in [14].

2. **Proof of Theorem** (1.3). Let G be a compact Lie group and let $\{W_\mu\}_{\mu \in \Lambda}$ be the set of non-trivial irreducible G-modules. Let $\mathscr{K} = \mathscr{K}(G)$ be the set of sequences $\boldsymbol{k} = (k_\lambda)_{\lambda \in \Lambda}$ of non-negative integers indexed by Λ with almost all zero terms. A complex vector bundle V with an G-action over a trivial G-space of the form

$$V = \sum E_\lambda \otimes W_\lambda, \ \dim E_\lambda = k_\lambda \,,$$

will be called a G-vector bundle of type $\boldsymbol{k} = (k_\lambda) \in \mathscr{K}$. If V is a complex G-vector bundle over a trivial G-space X such that the action of G on the fibers has no trivial irreducible factors, then V is a finite disjoint union of the $V_{\boldsymbol{k}}$ where $V_{\boldsymbol{k}}$ is of type $\boldsymbol{k}$ with base space $X_{\boldsymbol{k}}$, X being the disjoint union of the $X_{\boldsymbol{k}}$, cf. [2]. The correspondence $[X, V] \mapsto \sum_{\boldsymbol{k}} [X_{\boldsymbol{k}}, V_{\boldsymbol{k}}]$

[5] The canonical map $K_G \to K_G^{\wedge}$ is not monic in general [18]. Suppose that G is topologically cyclic; $G = \boldsymbol{Z}_l \times T^n$. Then it can be shown that $S^{-1}K_G$ is monic if l is a power of prime, but $S^{-1}K_G^{\wedge} = 0$ otherwise.

yields a canonical isomorphism

$$B_*^G = \sum_{k \in \mathscr{K}} U_*(BU(\boldsymbol{k}))$$

where $BU(\boldsymbol{k})$ denotes $\prod_\mu BU(k_\mu)$, $BU(k_\mu)$ being a classifying space of complex $\boldsymbol{k}_\mu$-vector bundle ($BU(0) = $ point).

If $\boldsymbol{k} = (k_\mu) \in \mathscr{K}$, then the degree of $\boldsymbol{k}$ is by definition $\sum_\lambda k_\lambda$, and will be denoted by $|\boldsymbol{k}|$. A partition of type $\boldsymbol{k}$ is a function p defined on Λ with values in the set of subsets of $\{1, 2, \cdots, |\boldsymbol{k}|\}$, such that the cardinal number of the set $p(\lambda)$ equals k_λ and the intersection $p(\lambda) \cap p(\mu)$ is empty for $\lambda \neq \mu$. The totality of the partitions of type $\boldsymbol{k}$ will be denoted by $\mathscr{P}(\boldsymbol{k})$. If $p \in \mathscr{P}(\boldsymbol{k})$, then let τ_λ: $\{1, \cdots, k_\lambda\} \to p(\lambda)$ be the unique monotone bijection. Suppose that a triple $(V, p, \boldsymbol{m})$ is given, where $V = \sum E_\mu \otimes W_\mu$ is a G-vector bundle of type $\boldsymbol{k}$ over X, p is an element of $\mathscr{P}(\boldsymbol{k})$ and $\boldsymbol{m} = (m_1, \cdots, m_k)$, $k = |\boldsymbol{k}|$, is a sequence of non-negative integers. Then we define $\lambda^{(p,\boldsymbol{m})}(V) \in K(X)$ to be the coefficient of $\prod_\mu \prod_{j=1}^{k_\mu} t_{\mu m_{\tau_\mu(j)}}$ in

$$\prod_\mu \prod_{j=1}^{k_\mu} (t_{\mu 0} + E_{\mu j} t_{\mu 1} + E_{\mu j}^2 t_{\mu 2} + E_{\mu j}^3 t_{\mu 3} + \cdots)$$

where each E_μ is written formally as

$$E_\mu = \sum_{j=1}^{k_\mu} E_{\mu j} \quad \text{(sum of line bundles)} .$$

Similarly $\gamma^{(p,\boldsymbol{m})}(V - |\boldsymbol{k}|) \in K(X)$ is defined to be the coefficient of

$$\prod_\mu \prod_{j=1}^{k_\mu} t_{\mu m_{\tau_\mu(j)}}$$

in

$$\prod_\mu \prod_{j=1}^{k_\mu} (t_{\mu 0} + (E_{\mu j} - 1) t_{\mu 1} + (E_{\mu j} - 1)^2 t_{\mu 2} + \cdots) .$$

For a fixed $p \in \mathscr{P}(\boldsymbol{k})$, every $\lambda^{(p,\boldsymbol{m})}(V)$ is a linear combination of the $\gamma^{(p,\boldsymbol{n})}(V - |\boldsymbol{k}|)$. Specifically we have the following.

LEMMA (2.1). *Suppose that $\boldsymbol{m} = (m_1, \cdots, m_k)$ is such that $m_i \neq m_j$ for $i \neq j$. Then, for any $p \in \mathscr{P}(\boldsymbol{k})$, we have*

$$\lambda^{(p,\boldsymbol{m})}(V) = \sum_{\boldsymbol{n}} \binom{\boldsymbol{m}}{\boldsymbol{n}} d(p, \boldsymbol{n}) \gamma^{(p,\boldsymbol{n})}(V - k) ,$$

where $d(p, \boldsymbol{n})$ is the order of the subgroup of the symmetric group of degree k consisting of the elements σ such that $\sigma p = p$, $\boldsymbol{n} = \boldsymbol{n}^\sigma = (n_{\sigma(1)}, \cdots, n_{\sigma(k)})$ and

$$\binom{m}{n} = \prod_{j=1}^{k} \binom{m_j}{n_j} .$$

NOTE. *If σ is a permutation of $\{1, 2, \cdots, k\}$ such that $\sigma p = p$, then $\gamma^{(p,n)}(V-k) = \gamma^{(p,n^\sigma)}(V-k)$.*

The proof is straightforward and will be left to the reader.

LEMMA (2.2). *Let $\boldsymbol{k} \in \mathscr{K}$ with $k = |\boldsymbol{k}|$. A bordism class $[X, V] \in U_*(BU(\boldsymbol{k}))$ vanishes if and only if we have*

$$p_{X!}(\gamma_t(\bar{\tau}X)\gamma^{(p,n)}(V-k)) = 0 \tag{2.3}$$

in $K(point) = Z$ for all $p \in \mathscr{P}(k)$ and all $\boldsymbol{n} = (n_1, \cdots, n_k)$.

PROOF. The necessity is trivial. Suppose conversely that (2.3) holds for all p and $\boldsymbol{n}$. We note the Riemann-Roch relation

$$p_{X!}(x) = \mathrm{ch}(x)\mathscr{T}(x)[X] , \quad x \in K(X) ,$$

and the fact that

$$\mathrm{ch}\, \gamma^{(p,n)}(V-k) = c^{(p,n)}(V) + \text{higher terms}$$

and

$$\mathrm{ch}\, \gamma_t(\bar{\tau}X) = c_t(\bar{\tau}X) + \text{higher terms} ,$$

where $c^{(p,n)}$ are defined in a manner similar to $\lambda^{(p,n)}$, replacing complex line bundles by their Euler classes, and c_t is given by $c_t(\bar{\tau}X) = \prod_i (1 + x_i t_1 + x_i^2 t_2 + \cdots)$ where $c(\bar{\tau}X) + \prod_i (1 + x_i)$ is the Chern class of $\bar{\tau}X$. Using the above relations, we deduce from (2.3) that

$$c_t(\bar{\tau}X)c^{(p,n)}(V) = 0 \tag{2.4}$$

for all p and $\boldsymbol{n}$. Since the space $BU(\boldsymbol{k}) = \prod BU(k_i)$ has no torsion in ordinary homology, vanishing of all the Chern numbers of the form (2.4) implies, by a theorem of Conner-Floyd [6], that $[X, V] = 0$. Thus (2.2) is proved.

Now suppose that G is $Z_l \times T^n$. Let W_0 be the standard Z_l-module, W the standard S^1-module and W_i the pull-back of W by the projection $p_i: T^n \to S^1$ on the i-th factor, $1 \leqq i \leqq n$. It is well known that the character rings are given by $R(Z_l) = Z[W_0]/(1 - W_0^l)$, $R(T^n) = Z[W_1, W_1^{-1}, \cdots, W_n, W_n^{-1}]$ and $R(G) = R(Z_l) \otimes R(T^n)$. We shall denote by W^λ the element $W_0^{\lambda_0} W_1^{\lambda_1} \cdots W_n^{\lambda_n} \in R(G)$, where $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n) \in Z_l \times Z^n$. The ring $R(G)$ is additively a free abelian group generated by $\{W^\lambda\}$. Let S be the multiplicative set in $R(G)$ generated by $\{1 - W^\lambda \mid \lambda \neq 0\}$ and S_{T^n} the multiplicative set in $R(T^n)$ generated by $\{1 - W^{\lambda'} \mid \lambda' = (0, \lambda_1, \cdots, \lambda_n),$

$\lambda' \neq 0\}$. The following lemma is standard and the proof is omitted.

LEMMA (2.5). *The homomorphism*

$$\varepsilon\colon S^{-1}R(G) \to \boldsymbol{C} \otimes S_{T^n}^{-1}R(T^n) = S_{T^n}^{-1}(\boldsymbol{C} \otimes R(T^n))$$

given by

$$\varepsilon\left(\frac{\sum_{i=0}^{l-1} W_0^i \chi_i}{\prod_\lambda (1 - W^\lambda)^{a_\lambda}}\right) = \frac{\sum_{i=0}^{l-1} \omega^i \chi_i}{\prod_\lambda (1 - \omega^{\lambda_0} W^{\lambda'})^{a_\lambda}}, \; \chi_i \in R(T^n),$$

is well defined and injective. Here $\omega = \exp(2\pi\sqrt{-1}/l)$ *and* $\lambda' = (0, \lambda_1, \cdots, \lambda_n)$ *if* $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$.

We shall now proceed to the proof of Theorem (1.3). Suppose that there are given elements $\boldsymbol{k}_i \in \mathscr{K}$, $1 \leqq i \leqq s$, and $[X_i, V_i] \in U_*(BU(\boldsymbol{k}_i))$ such that $\bar{\rho}(\sum_{i=1}^s [X_i, V_i]) = 0$. We wish to prove that $[X_i, V_i] = 0$ for all i. Set $k_i = |\boldsymbol{k}_i|$. We may assume, without loss of generality, that $k_i = k$ for $1 \leqq i \leqq r$ and $k_j < k$ for $r + 1 \leqq j \leqq s$. Recall that each V_i has the form

$$V_i = \sum_{\lambda \neq 0} E_{i\lambda} \otimes W^\lambda, \quad \dim E_{i\lambda} = k_{i\lambda},$$

where $\boldsymbol{k}_i = (k_{i\lambda})_\lambda$. Writing $E_{i\lambda}$ formally as Whitney sum of complex line bundles $E_{i\lambda j}$, $1 \leqq j \leqq k_{i\lambda}$, we get

$$\text{(2.6)} \quad \frac{\gamma_t(V_i - k_i)}{\lambda_{-1}(V_i)} = (-1)^{k_i} \prod_\lambda \prod_{j=1}^{k_{i\lambda}} \left(\frac{1}{E_{i\lambda j} W^\lambda - 1} + t_1 + t_2(E_{i\lambda j} W^\lambda - 1) + \cdots\right).$$

Given a partition $p \in \mathscr{P}(\boldsymbol{k})$ and a sequence $\boldsymbol{n} = (n_1, \cdots, n_k)$, $k = |\boldsymbol{k}|$, we define $L(p, \boldsymbol{n}) \in \boldsymbol{Z}_l \times \boldsymbol{Z}^n$ by

$$L(p, \boldsymbol{n}) = \sum_{\lambda \neq 0} \Big(\sum_{j \in p(\lambda)} n_j\Big)\lambda.$$

Note that the sum is essentially a finite sum because $p(\lambda) = \phi$ for almost all $\lambda \in \boldsymbol{Z}_l \times \boldsymbol{Z}^n - 0$. If $\boldsymbol{m} = (m_1, \cdots m_k)$ is a sequence of mutually distinct positive integers, then the coefficient $c_{i,\boldsymbol{m}}$ of $t_{m_1+1} t_{m_2+1} \cdots t_{m_k+1}$ in (2.6) is zero if $r < i$ and is of the form

$$\text{(2.7)} \quad c_{i,\boldsymbol{m}} = (-1)^{k+\Sigma m_j} \sum_{\boldsymbol{n}} (-1)^{\Sigma n_j} \binom{\boldsymbol{m}}{\boldsymbol{n}} \Big(\sum_{p \in \mathscr{P}(\boldsymbol{k}_i)} d(p, \boldsymbol{n}) \lambda^{(p,\boldsymbol{n})}(V_i) W^{L(p,\boldsymbol{n})}\Big)$$

for $1 \leqq i \leqq r$, where

$$\binom{\boldsymbol{m}}{\boldsymbol{n}} = \prod_{j=1}^{k} \binom{m_j}{n_j}.$$

Now since all the base spaces X_i, $1 \leqq i \leqq s$, are compact manifolds, there are no non-zero terms in $\gamma_t(\bar{\tau} X_i)$ involving $t_{m_1+1}, \cdots, t_{m_k+1}$ provided $m_1, \cdots, m_k$ are sufficiently large (see e.g. [1]). Therefore, equating to zero the terms involving $t_{m_1+1}, \cdots, t_{m_k+1}$ in $\varepsilon\bar{\rho}(\sum [X_i, V_i])$, we obtain

$$\varepsilon\Big(\sum_{i=1}^{r} p_{X_i!}(\gamma_t(\bar{\tau} X_i) c_{i.\boldsymbol{m}})\Big) = 0 \tag{2.8}$$

for sufficiently large $m_1, \cdots, m_k$, where $c_{i.\boldsymbol{m}}$ is given by (2.7) when $m_1, \cdots, m_k$ are mutually distinct.

Note that $\binom{\boldsymbol{m}}{\boldsymbol{n}}$ is a polynomial function of $m_1, \cdots, m_k$ for a fixed $\boldsymbol{n}$. The following lemma can be proved easily be induction on k.

LEMMA (2.9). *Let A be the set of all sequences $\boldsymbol{n} = (n_1, \cdots, n_k)$ of nonnegative integers. Then the $\left\{\binom{\boldsymbol{m}}{\boldsymbol{n}}\right\}_{\boldsymbol{n} \in A}$ are linearly independent as functions of $m_1, \cdots, m_k$.*

From (2.9) it follows that the coefficient of $\binom{\boldsymbol{m}}{\boldsymbol{n}}$ in (2.8) must vanish identically for each $\boldsymbol{n} \in A$. Using the expression (2.7), we obtain

$$\varepsilon\Big(\sum_{i=1}^{r} p_{X_i!}\Big(\gamma_t(\bar{\tau} X_i)\Big(\sum_{p \in \mathscr{S}(\boldsymbol{k}_i)} d(p, \boldsymbol{n}) \lambda^{(p,\boldsymbol{n})}(V_i) W^{L(p,\boldsymbol{n})}\Big)\Big)\Big) = 0 \tag{2.10}$$

for any $\boldsymbol{n} \in A$.

Next, in addition to $L(p, \boldsymbol{n})$, we define $L_0(p, \boldsymbol{n}) \in \boldsymbol{Z}_l$ and $L'(p, \boldsymbol{n}) \in \boldsymbol{Z}^n$ by

$$L_0(p, \boldsymbol{n}) = \sum_{\lambda \neq 0} \Big(\sum_{j \in p(\lambda)} n_j\Big) \lambda_0$$

and

$$L'(p, \boldsymbol{n}) = \sum_{\lambda \neq 0} \Big(\sum_{j \in p(\lambda)} n_j\Big) \lambda'$$

where $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ and $\lambda' = (0, \lambda_1, \cdots, \lambda_n)$. Thus

$$L(p, \boldsymbol{n}) = (L_0(p, \boldsymbol{n}), L'(p, \boldsymbol{n})).$$

We then introduce an equivalence relation $p \equiv q$ on the set $\bigcup_{1 \leqq i \leqq r} \mathscr{P}(\boldsymbol{k}_i)$ in the following way. Two elements p and q are defined to be equivalent if and only if $L'(p, \boldsymbol{n})$ and $L'(q, \boldsymbol{n})$ are identical functions of $\boldsymbol{n}$. We shall denote the equivalence class of p by $[p]$. Thus $L'(p, \boldsymbol{n})$ is actually a function of $[p]$ which we denote by $L'([p], \boldsymbol{n})$. The set of equivalence classes $[p]$ will be denoted by Q. Let $\pi: A \to (\boldsymbol{Z}_l)^k$ be the natural projection. Then $L_0(p, \boldsymbol{n})$ depends only on $\pi(\boldsymbol{n}) = g$, and will be denoted by $L_0(p, g)$. From the definition we obtain immediately

LEMMA (2.11). *If $p \equiv q$ and $p \neq q$ then $L_0(p, g)$ and $L_0(q, g)$ are not identically equal as functions of $g \in (\boldsymbol{Z}_l)^k$.*

Let then B be the subset of A consisting of the elements $\boldsymbol{m} = (m_1, \cdots, m_k)$ such that the m_i are mutually distinct and $L'([p], \boldsymbol{m}) \neq L'([q], \boldsymbol{m})$ for all pairs $([p], [q]) \in Q^2$ with $[p] \neq [q]$. B is obtained from A by removing a finite number of hyperplanes. Note that, if $\boldsymbol{m} \in B$, then $\{W^{L'([p],\boldsymbol{m})}\}_{[p]\in Q}$ form a linearly independent set in $R(T^n)$. Observing that

$$\varepsilon\Big(\sum_{p\in\xi} W^{L(p,\boldsymbol{m})}\Big) = \Big(\sum_{p\in\xi} \omega^{L_0(p,\boldsymbol{m})}\Big) W^{L'(\xi,\boldsymbol{m})}$$

for every class $\xi \in Q$, we see from (2.10) that, for any $\boldsymbol{m} \in B$ and any class ξ, the identity

$$\sum_{p\in\xi} p_{X_{i(p)}!}(\gamma_t(\bar{\tau} X_{i(p)})\lambda^{(p,\boldsymbol{m})}(V_{i(p)}))\omega^{L_0(p,\boldsymbol{m})} = 0 \tag{2.12}$$

holds, where $i(p)$ denotes the unique i such that $p \in \mathscr{P}(\boldsymbol{k}_i)$.

At this point we use Lemma (2.1). Fix an element $g \in (\boldsymbol{Z}_l)^k$. If $\boldsymbol{m} \in (\pi\,|\,B)^{-1}(g)$ then, from (2.1) and (2.12) we have

$$\sum_{\boldsymbol{n}}\Big\{\sum_{p\in\xi} p_{X_{i(p)}!}(\gamma_t(\bar{\tau} X_{i(p)})d(p, \boldsymbol{n})\gamma^{(p,\boldsymbol{n})}(V_{i(p)} - k))\omega^{L_0(p,g)}\Big\}\binom{\boldsymbol{m}}{\boldsymbol{n}} = 0 \tag{2.13}$$

for any $\xi \in Q$. From the fact that B is obtained from A by removing a finite number of hyperplanes, it follows easily that the set $(\pi\,|\,B)^{-1}(g)$ contains arbitrarily large $\boldsymbol{m}$ (i.e., $\min m_j$ is arbitrarily large). Therefore, the coefficient of each $\binom{\boldsymbol{m}}{\boldsymbol{n}}$ in (2.13) must vanish, by (2.9). Thus, we have proved that

$$\sum_{p\in\xi} p_{X_{i(p)}!}(\gamma_t(\bar{\tau} X_{i(p)})d(p, \boldsymbol{n})\gamma^{(p,\boldsymbol{n})}(V_{i(p)} - k))\omega^{L_0(p,g)} = 0 \tag{2.14}$$

for any $\xi \in Q$, $\boldsymbol{n} \in A$, and $g \in (\boldsymbol{Z}_l)^k$.

For a fixed p, the function $\omega_p: (\boldsymbol{Z}_l)^k \to \boldsymbol{C}$ given by $\omega_p(g) = \omega^{L_0(p,g)}$ is clearly an irreducible character of the group $(\boldsymbol{Z}_l)^k$, and hence can be considered as an element of $R((\boldsymbol{Z}_l)^k)$. Thus, (2.14) is translated into

$$\sum_{p\in\xi} p_{X_{i(p)}!}(\gamma_t(\bar{\tau} X_{i(p)})d(p, \boldsymbol{n})\gamma^{(p,\boldsymbol{n})}(V_{i(p)} - k))\omega_p = 0 \tag{2.15}$$

which holds in $R((\boldsymbol{Z}_l)^k)[[t]]$ for any $\xi \in Q$ and $\boldsymbol{n} \in A$. Moreover, Lemma (2.11) implies that, for a fixed $\xi \in Q$, $\{\omega_p\}_{p\in\xi}$ is a linearly independent set in $R((\boldsymbol{Z}_l)^k)$, since they are mutually distinct irreducible characters. Hence the coefficient of each ω_p in (2.15) must vanish and we obtain

$$p_{X_{i(p)}!}(\gamma_t(\bar{\tau}X_{i(p)})\gamma^{(p,n)}(V_{i(p)}-k))=0\,,$$

for any p and n. In other words, for any i, $1\leqq i\leqq r$, any $p\in\mathscr{P}(k_i)$ and any $n\in A$, the relation

$$p_{X_i!}(\gamma_t(\bar{\tau}X_i)\gamma^{(p,n)}(V_i-k))=0 \tag{2.16}$$

holds. In virtue of Lemma (2.2) this implies

$$[X_i,\ V_i]=0 \tag{2.17}$$

in $U_*(BU(k_i))$ for $1\leqq i\leqq r$.

Finally, the induction on $\max|k_i|$ shows that (2.17) holds also for all i, $1\leqq i\leqq s$. This completes the proof of (1.3).

In passing, we note the following proposition which is proved in a manner entirely similar to (2.16).

PROPOSITION (2.18). *Let $G=Z_l\times T^n$. Suppose that there are given $[X_i,\ V_i]\in U_*(BU(k_i))$ and $y_i\in K(X_i)$, $1\leqq i\leqq s$, such that*

$$\sum_i p_{X_i!}\left(y_i\gamma_t(\bar{\tau}X_i)\frac{\gamma_t(V_i-|k_i|)}{\lambda_{-1}(V_i)}\right)=0$$

in $S^{-1}K_G[[t]]$. Then the relation

$$p_{X_i!}(y_i\gamma_t(\bar{\tau}X_i)\gamma^{(p,n)}(V_i-|k_i|))=0$$

holds for all i, $p\in\mathscr{P}(k_i)$ and $n\in A$.

3. **Proof of Theorem** (1.4). The integrality Theorem (1.4) follows from more precise facts which will be stated in Propositions (3.1), (3.2) and (3.3).

Let G_λ be the stabilizer of the irreducible T^n-module W^λ, where $\lambda=(\lambda_1,\cdots,\lambda_n)\in Z^n$. G_λ is the subgroup of T^n consisting of the elements $(e^{it_1},\cdots,e^{it_n})$ such that $\sum\lambda_it_i\equiv 0 \bmod 2\pi$. If W^λ is non-trivial (i.e., $\lambda\neq 0$), then G_λ is isomorphic to $T^{n-1}\times Z_d$, where d is the greatest common divisor of $\lambda_1,\cdots,\lambda_n$, which we shall call the order of λ and denote by $d(\lambda)$. A subcircle of T^n is of the form

$$S^1=\{(e^{i\alpha_1t},\cdots,e^{i\alpha_nt})\}\,,$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)\in Z^n-0$ with $d(\alpha)=1$.

Given a subcircle S^1_α of T^n, we define $U_*^{T^n}(\mathscr{F}(1),\alpha)$ to be the U_*-submodule of $U_*^{T^n}(\mathscr{F}(1))$ consisting of the elements $[M]$ admitting a representing T^n-manifold M on which the induced S^1_α-action has no fixed points.

There are exactly two isomorphisms from S^1 onto S_α. We arbitrarily choose one of these, which we denote by θ. If W^λ is a non-trivial irreducible T^n-module such that $G_\lambda \not\supset S^1_\alpha$, then the composite homomorphism

$$S^1 \xrightarrow{\theta} S^1_\alpha \subset T^n \xrightarrow{h_\lambda} S^1$$

is of the form $z \mapsto z^l$, with $l \neq 0$, where h_λ is the homomorphism corresponding to W_λ. We shall call W^λ *positive* if l is positive. In this case l will be denoted by $\bar{d}(\lambda)$. Note that $\bar{d}(\lambda)$ is a multiple of $d(\lambda)$.

Given a subcircle S^1_α we define $B^{T^n}_*(\alpha)$ to be the U_*-submodule of $B^{T^n}_*$ generated by the elements $[X, V]$ such that V is of the form

$$V = \sum_\mu E_\mu \otimes W^\mu \oplus E_\lambda \otimes W^\lambda$$

where $S^1_\alpha \not\subset G_\lambda$, W^λ is positive and $\dim E_\lambda = 1$, while the sum is taken over $\{\mu\}$ such that either W^μ is positive and $\bar{d}(\mu) < \bar{d}(\lambda)$, or $S^1_\alpha \subset G_\mu$. We define the subset $\mathscr{K}_\alpha$ of $\mathscr{K}(T^n)$ as follows. $\boldsymbol{k} = (k_\mu) \in \mathscr{K}(T^n)$ belongs to $\mathscr{K}_\alpha$ if and only if there exists a positive λ_k such that

1) $k_{\lambda_k} = 1$,
2) if $k_\mu \neq 0$, $\mu \neq \lambda_k$, then either W^μ is positive and $\bar{d}(\mu) < \bar{d}(\lambda_k)$, or $S^1_\alpha \subset G_\mu$.

Itis clear then that

$$B^{T^n}_*(\alpha) = \sum_{k \in \mathscr{K}_\alpha} U_*(BU(\boldsymbol{k})) .$$

PROPOSITION (3.1). *$U^{T^n}_*(\mathscr{F}(1))$ is the union of the $U^{T^n}_*(\mathscr{F}(1), \alpha)$.*

PROOF. Let M be a compact T^n-manifold without fixed points. Let $\{H_1, \cdots, H_m\}$ be the totality of isotropy subgroups of the T^n-action on M. They are proper subgroups of T^n, and are finite in number. For a subcircle S^1_α, the condition that the induced action of S^1_α has no fixed points is equivalent to the condition that S^1_α is contained in none of the H_i. But it is easy to see that a circle S^1_α satisfying this condition exists.

PROPOSITION (3.2). *There is a U_*-module map $Q_\alpha: U^{T^n}_*(\mathscr{F}(1), \alpha) \to B^{T^n}_*$ such that ∂Q_α equals the identity and the image of Q_α is contained in $B^{T^n}_*(\alpha)$. In particular, if $n = 1$ then $Q = Q_\alpha$ is a splitting map for ∂.*

Proof of (3.2) will be postponed until the last part of this section. It is not hard to see that Theorem (1.4) is an easy consequence of the following Proposition (3.3), together with (3.1) and (3.2).

PROPOSITION (3.3). *Let $S^1_\alpha \subset T^n$ be a subcircle. Let $\bar{\rho}_\alpha$ denote*

$$\bar{\rho} \,|\, B^{T^n}_*(\alpha) .$$

Then we have

$$\bar{\rho}_\alpha^{-1}(K_{T^n}[[t]]) = 0 .$$

PROOF. Suppose that we are given elements $[X_i, V_i] \in U_*(BU(k_i))$, $1 \leqq i \leqq s$, where $k_i \in \mathscr{K}_\alpha$, such that

$$\bar{\rho}(\sum [X_i, V_i]) \in K_{T^n}[[t]] . \tag{3.4}$$

We wish to show that $[X_i, V_i] = 0$ for all i. Let $l = \max_i \bar{d}(\lambda_{k_i})$. We shall proceed by induction on l and on the number of i which satisfy $\bar{d}(\lambda_{k_i}) = l$.

First, we make a few preparatory remarks. Let $\pi: T^n \to T^n$ be a finite covering map. It is easy to see that the induced map $\pi^*: \Lambda \to \Lambda$ is injective, where $\Lambda = \boldsymbol{Z}^n - 0$ denotes the non-trivial irreducible representations of T^n as before. It follows that the induced homomorphism $\pi^*: R(T^n) \to R(T^n)$ is also injective. Hence, it induces a homomorphism $\pi^*: S^{-1}R(T^n) \to S^{-1}R(T^n)$ which is also injective. Similarly, if $[X, V] \in B_*^{T^n}$, where V is of the form

$$V = \sum_{\mu \in \Lambda} E_\mu \otimes W^\mu ,$$

then

$$\pi^* V = \sum_{\mu \in \Lambda} E_\mu \otimes \pi^* W^\mu$$

is well-defined and the homomorphism $\pi^*: B_*^{T^n} \to B_*^{T^n}$ defined by

$$\pi^*[X, V] = [X, \pi^* V]$$

is injective. Moreover, if $S_{\alpha'}^1$ is the component group of $\pi^{-1}(S_\alpha^1)$ and q is the degree of $\pi: S_{\alpha'}^1 \to S_\alpha^1$, then it is easy to see that $\pi^* W^\mu = W^{\pi^*\mu}$ is positive with respect to the isomorphism $\tilde{\theta}: S^1 \to S_{\alpha'}^1$, which is uniquely determined by $\pi\tilde{\theta}(z) = \theta(z^q)$ if and only if W^μ is positive, and that $\bar{d}(\pi^*\mu)$ equals $q\bar{d}(\mu)$.

We return to the proof of (3.3). Clearly we may assume that $\lambda_{k_i} = \lambda$ if $1 \leqq i \leqq r$ and $\bar{d}(\lambda) = l$, and that $\lambda_{k_j} \neq \lambda$ and $\bar{d}(\lambda_{k_j}) \leqq l$ if $r + 1 \leqq j \leqq s$. It will suffice to show that $[X_i, V_i] = 0$ for $1 \leqq i \leqq r$. For then we shall have fewer k_j such that $\bar{d}(k_j) = l$, and the inductive argument can be applied.

Let T_λ be the connected component group of the stabilizer G_λ of W^λ. Since W^λ is positive, the group $H = S_\alpha^1 \cap T_\lambda$ is finite and $S_\alpha^1 \cup T_\lambda$ generates T^n. Fixing an isomorphism $\theta': T^{n-1} \to T_\lambda$, we define $\pi: T^n = S^1 \times T^n \to T^n$ by

$$\pi(u, v) = \theta(u)\theta'(v) .$$

Then π is a finite covering map, furthermore, the connected component group of $\pi^{-1}(S_\alpha^1)$ is $S_{\alpha'}^1 = S^1 \times 1$ and the degree of $\pi\colon S_{\alpha'}^1 \to S_\alpha^1$ equals 1. The component group $T_{\pi^*\lambda}$ of the stabilizer of $\pi^* W^\lambda = W^{\pi^*\lambda}$ is $1 \times T^{n-1}$, hence $S_{\alpha'}^1 \cap T_{\pi^*\lambda} = 1$. Now to prove that $[X_i, V_i] = 0$ it suffices to prove that $\pi^*[X_i, V_i] = 0$, since π^* is a monomorphism. But the collection $\{\pi^*[X_i, V_i]\}$ satisfies conditions similar to $\{[X_i, V_i]\}$, i.e.,

$$\bar{\rho}(\sum \pi^*[X_i, V_i]) \in K_{T^n}[[t]] ,$$

and if $\pi^*[X_i, V_i] \in U_*(BU(k_i'))$, then $k_i' \in K_{\alpha'}$ and $\lambda_{k_i'} = \pi^*\lambda$ whenever $1 \leqq i \leqq r$ and $\bar{d}(\pi^*\lambda) = l$, while $\lambda_{k_j'} \neq \pi^*\lambda$ and $\bar{d}(\lambda_{k_j'}) \leqq l$ for $r+1 \leqq j \leqq s$, as is easily seen from the above remarks.

Therefore, replacing $[X_i, V_i]$ by $\pi^*[X_i, V_i]$, α by α' and λ by $\pi^*\lambda$, we may assume from the first that $S_\alpha^1 \cap T_\lambda = 1$, so that $S_\alpha^1 \times T_\lambda = T^n$. Then changing coordinates through $\theta \times \theta'\colon T^n = S_\alpha^1 \times T_\lambda \to T^n$, we may assume that $S_\alpha^1 = T^1 \times 1$ and $T_\lambda = 1 \times T^{n-1}$, so that $\lambda = (l, 0, \cdots, 0) \in \boldsymbol{Z}^n$.

Let J be the subset of $\boldsymbol{Z}^n - 0$ given by

$$J = \{\theta = (\theta_1, \theta_2, \cdots, \theta_n) \mid 0 \leqq \theta_1 \leqq l,\ \theta \neq (l, 0, \cdots, 0) = \lambda\} .$$

If $k_{i\theta} \neq 0$, where $k_i = (k_{i\theta})_\theta$, $1 \leqq i \leqq s$, and $\theta \neq \lambda$, then it is easy to see that $\theta \in J$. Let S_J be the multiplicative set in $R(T^n)$ generated by $\{\lambda_{-1}(W^\theta) = 1 - W^\theta\}_{\theta \in J}$. Define R_J to be the localized ring $S_J^{-1}R(T^n)$. This is a subring of $S^{-1}R(T^n)$. If $j\colon \boldsymbol{Z}_l \times T^{n-1} \subset T^1 \times T^{n-1} = T^n$ is the usual inclusion and $j^*\colon R(T^n) \to R(\boldsymbol{Z}_l \times T^{n-1})$ is the induced homomorphism, then it is easy to see that $0 \notin j^*S_J$ and hence $j^*S_J \subset S \subset R(\boldsymbol{Z}_l \times T^{n-1})$. Thus, we have the induced homomorphism $j^*\colon R_J \to S^{-1}R(\boldsymbol{Z}_l \times T^{n-1})$.

By assumption, V_i, $1 \leqq i \leqq r$, is of the form

$$V_i = V_i' \oplus E_i \otimes W^\lambda,\ (E_i \text{ is a line bundle}) ,$$

with

$$V_i' = \sum_{\mu \in J'} E_{i\mu} \otimes W^\mu$$

where $J' = \{\mu = (\mu_1, \mu_2, \cdots, \mu_n) \mid 0 \leqq \mu_1 < l,\ \mu \neq 0\} \subset J$. Now set $e = 1 - W^\lambda$, $E_i = 1 + x_i$, $E_i^* = (1 + x_i)^{-1}$. We have a formula

$$\begin{aligned}\frac{1}{1 - E_i \otimes W^\lambda} &= \frac{E_i^*}{e} + x_i\left(\frac{E_i^*}{e}\right)^2 + \cdots + x_i^N\left(\frac{E_i^*}{e}\right)^{N+1} \\ &= \frac{E_i^*}{e} + x_i\left(\frac{E_i^*}{e}\right)^2 + \cdots + x_i^{N-1}\left(\frac{E_i^*}{e}\right)^N + \frac{x_i^N}{e^{N+1}} ,\end{aligned}$$

where N is a constant such that $x_i^j = 0$ for all i whenever $j > N$. Using this formula, we obtain from (3.4) the relation

$$\frac{b_{N+1}}{e^{N+1}} + \cdots + \frac{b_1}{e} \in R_J[[t]] , \tag{3.5}$$

where $b_j \in R_J[[t]]$ is given by

$$b_j = \sum_{i=1}^{r} p_{X_i!}\left(x_i^{j-1} E_i^{*j} \gamma_t(\bar{\tau} X_i) \frac{\gamma_t(V_i' - k_i')}{\lambda_{-1}(V_i')}\right), \tag{3.6}$$

with $k_i' = \dim V_i'$. In particular, b_{N+1} takes the form

$$b_{N+1} = \sum_{i=1}^{r} p_{X_i!}\left(x_i^N \gamma_t(\bar{\tau} X_i) \frac{\gamma_t(V_i' - k_i')}{\lambda_{-1}(V_i')}\right). \tag{3.7}$$

From (3.5), it follows that

$$b_{N+1} \in eR_J[[t]] .$$

Applying j^*, we get

$$j^*(b_{N+1}) = 0 \quad \text{in} \quad S^{-1}K_{Z_l \times T^{n-1}}[[t]] , \tag{3.8}$$

since $j^*(e) = 0$.

We now regard V_i, $1 \leqq i \leqq r$, as a $(Z_l \times T^{n-1})$-vector bundle via j. Since j^* maps J' injectively into the non-trivial irreducible $(Z_l \times T^{n-1})$-modules, $[X_i, V_i]$ belongs to $U_*(BU(k_i'))$ by virtue of the identification

$$B_*^{Z_l \times T^{n-1}} = \sum_{k' \in \mathscr{K}(Z_l \times T^{n-1})} U_*(BU(k')) ,$$

where k_i' is induced from k_i via the mapping j. With this understanding (3.7) and (3.8) mean that

$$\sum_{i=1}^{r} p_{X_i!}\left(x_i^N \gamma_t(\bar{\tau} X_i) \frac{\gamma_t(V_i' - k_i')}{\lambda_{-1}(V_i')}\right) = 0$$

in $S^{-1}K_{Z_l \times T^{n-1}}[[t]]$ for $[X_i, V_i] \in B_*^{Z_l \times T^{n-1}}$. From this and Proposition (2.18), we infer that

$$p_{X_i!}(x_i^N \gamma_t(\bar{\tau} X_i) \gamma^{(p,n)}(V_i' - k_i')) = 0$$

for all $i, 1 \leqq i \leqq r$, $p \in \mathscr{P}(k_i')$ and n. This implies that, if we write the coefficients of t^α in b_{N+1} as linear combinations of elements of $S^{-1}K_{Z_l \times T^{n-1}}[[t]]$ according to (3.7), then they all vanish, and hence $b_{N+1} = 0$.

Then, by an inductive argument on j, using (3.6), we see that

$$p_{X_i!}(x_i^j \gamma_t(\bar{\tau} X_i) \gamma^{(p,n)}(V_i' - k_i') = 0$$

for all i $(1 \leqq i \leqq r)$, j $(0 \leqq j \leqq N)$, $p \in \mathscr{P}(k_i')$ and n. But this is equivalent to say that

$$p_{X_i!}(\gamma_t(\bar{\tau} X_i) \gamma^{(q,m)}(V_i - k_i)) = 0$$

for all i, $1 \leqq i \leqq r$, $q \in \mathscr{P}(k_i)$ and m. From Lemma (2.2) it then follows that

$$[X_i, V_i] = 0 \quad \text{for} \quad 1 \leqq i \leqq r\,.$$

This proves Proposition (3.3).

It remains to prove (3.2). Let M be a closed unitary T^n-manifold such that the induced action of S^1_α has no fixed points. Thus $[M] \in U^{T^n}_*(\mathscr{F}(1), \alpha)$. There are only a finite number of isotropy subgroups of the S^1_α-action and these are all finite cyclic groups. Let l be the maximum order of these cyclic group . Let $Y_1, \cdots, Y_m$ be the totality of connected components of the fixed point set under the action of $Z_l \subset S^1_\alpha$. We shall make the following construction for each Y_i. First suppose that l is greater than 1. Let Y denote one of the Y_i. The group Z_l acts on the normal bundle U of Y by automorphisms and therefore yields a unique decomposition

$$U = \sum_{0<i<l} U_i\,,$$

where $g \in Z_l$ acts on U_i by the scalar multiplication on $\theta^{-1}(g^i)$, which we denote by $\psi'\theta^{-1}(g^i)$. Thus, if we denote by $\psi(g)$ the action of $g \in T^n$, then $\psi(g) = \psi'\theta^{-1}(g^i)$ on U_i for $g \in Z_l \subset S^1_\alpha \subset T^n$. It follows that there exists a unique S^1-action ψ'' on U such that

$$\psi\theta(g) = \psi'(g)^i \psi''(g)^l\,, \qquad g \in S^1, \text{ on } U_i\,. \tag{3.9}$$

The action ψ'' commutes with the T^n-action ψ and is principal. Set $Y' = Y/\psi''(S^1)$. Since ψ'' is a principal action, Y' is a smooth manifold and the projection $Y \to Y'$ is an S^1-bundle. Let U'_0 denote the complex line bundle over Y' associated with it. There is a unique weakly complex structure on Y' such that, if U'_0 is endowed with a weakly complex structure as the total space of a complex line bundle over the weakly complex manifold Y', then the weakly complex structure on Y as the boundary of the disk bundle $D(U'_0)$ of U'_0 coincides with the original structure on Y induced from that of M. Furthermore, $U' = U/\psi''(S^1)$ is a complex vector bundle over Y'. Define $\tilde{U}$ to be the lift of U' over $D(U'_0)$. Then clearly $\tilde{U}\,|\,Y = U$, and $D(U) \cup S(\tilde{U}) = \partial D(\tilde{U})$ can be identified with the sphere bundle $S(U'_0 \oplus U')$ of $U'_0 \oplus U'$, where $S(\tilde{U})$ denotes the sphere bundle of $\tilde{U}$. Define M' to be the weakly complex manifold $(M - \operatorname{int} D(U)) \cup -S(\tilde{U})$ where $-S(\tilde{U})$ is the weakly complex manifold $S(\tilde{U})$ with the opposite structure. Since the action ψ of T^n commutes with ψ'', it induces an action of T^n on $\tilde{U}$, which we shall also denote by ψ.

ASSERTION (3.10). *The restricted action of S^1_α on $\tilde{U}$ has no fixed points outside of $Y' = Y/\psi''(S^1)$. On $S(\tilde{U})$, the isotropy subgroups of S^1_α are cyclic groups of order less than l, so that the fixed point set of the* *restricted action of* $Z_l \subset S^1_\alpha$ *on* M' *equals* $\bigcup Y_j - Y$. *Hence,* $S(U'_0 \oplus U')$ *and* M' *represent elements of* $U^{T^n}_*(\mathscr{F}(1), \alpha)$, *in which we have*

$$[M] = [S(U'_0 \oplus U')] + [M'] .$$

For the proof, see [16] or [15].

Since T^n is a connected abelian group, the submanifold Y is invariant under the action of T^n. Let $H_1, \cdots, H_s$ be the totality of isotropy subgroups of codimension 1 of the T^n-action on Y and let $F_1, \cdots, F_s$ be the corresponding fixed point sets. From the maximality of l, it follows that $S^1_\alpha \cap H_j = Z_l$ for all j. We set $X_j = F_j/\psi''(S^1)$.

ASSERTION (3.11). *The fixed point set of the induced* T^n*-action* ψ *on* $D(U'_0 \oplus U')$ *is precisely the disjoint union of* $X_1, \cdots, X_s$, *so that each* X_j *has a natural weakly complex structure. Let* V_j *be the normal bundle of* X_j *in* $D(U'_0 \oplus U')$. *Then* V_j *decomposes as a direct sum of complex vector bundles*

$$V_j = U'_0 | X_j \oplus U' | X_j \oplus V_{jY'} ,$$

where $V_{jY'}$ *is the normal bundle of* X_j *in* Y'. *The bundles* $U'_0 | X_j$, $U' | X_j$ *and* $V_{jY'}$ *are invariant under the* T^n*-action* ψ. $U'_0 | X_j$ *is a line bundle and if* W^λ *is the unique irreducible* T^n*-module contained in it then* W^λ *is positive and* $\bar{d}(\lambda) = l$. *Irreducible* T^n*-modules contained in* $U' | X_i$ *are all positive and the* $\bar{d}(\mu)$ *are less than* l. *Finally, for irreducible* T^n*-modules* W^μ *contained in* $V_{jY'}$, *we have* $S^1_\alpha \subset G_\mu$. *In particular,* $[X_j, V_j]$ *belongs to* $B^{T^n}_*(\alpha)$.

PROOF. The first part is easy, hence we prove the only second part. On U'_0, the scalar multiplication $\psi'_1(g)$ of $g \in S^1$ is given by $\psi''(g)$, and the action ψ'' of S^1 restricted to Y, the sphere bundle of U'_0, is related to ψ by $\psi(\theta(g)) = \psi''(g)^l$, so that we have $\psi(\theta(g)) = \psi'_1(g)^l$. If W^λ is the irreducible T^n-module contained in $U'_0 | X_j$, then this means that W^λ is positive and $\bar{d}(\lambda) = l$. As in (3.9), the actions ψ, ψ' and ψ'' are related by $\psi(\theta(g)) = \psi'(g)^i \psi''(g)^l$ on U_i, where $g \in S^1$. Therefore, on $U'_i = U_i/\psi''(S^1)$, we have the relation

$$\psi(\theta(g)) = \psi'_1(g)^i , \quad g \in S^1 .$$

Because of the inequalities $0 < i < l$, this shows that if W^μ is an irreducible T^n-module contained in $U'_i \subset U'$, then W^μ is positive and $\bar{d}(\mu) < l$. Finally, S^1_α keeps fixed every point of $V_{jY'}$. Hence, if W^μ is contained in $V_{jY'}$, then it is clear that $S^1_\alpha \subset G_\mu$.

ASSERTION (3.12). *The notations being as above, let* $X = \bigcup X_j$ *and* V *be the normal bundle of* X *in* Y'. *Thus* $V | X_j = V_j$ *and* $[X, V] =$

$\sum [X_j, V_j] \in B_*^{T^n}(\alpha)$. *If $S(V)$ is the sphere bundle of V then we have*

$$[M] = [S(V)] + [M'] . \tag{3.13}$$

To prove this it is sufficient, by virtue of (3.10) and (3.11), to show that $[S(V)] = [S(U_0' \oplus U')]$. But the manifold $D(U_0' \oplus U') - \operatorname{int} D(V)$ is T^n-invariant and, there are no fixed points of the induced T^n-action on that manifold. This provides the bordism between $S(U_0' \oplus U')$ and $S(V)$ in $U_*^{T^n}(\mathscr{F}(1))$.

Now, we have started from an element $[M] \in U_*^{T^n}(\mathscr{F}(1))$ and arrived at a pair $[X, V] \in B_*^{T^n}(\alpha)$ and $[M'] \in U_*^{T^n}(\mathscr{F}(1), \alpha)$ such that the number of components of the fixed point set under the induced Z_l-action on M' is decreased by one, and such that (3.13) is satisfied. We can proceed by induction on l and the number of components of the fixed point set of $Z_l \subset S_\alpha^1$, repeating the same construction as above, to get $[X^{(1)}, V^{(1)}] \in B_*^{T^n}(\alpha)$ and $[M^{(1)}] \in U_*^{T^n}(\mathscr{F}(1), \alpha)$ such that

$$[M] = \partial[X^{(1)}, V^{(1)}] + [M^{(1)}] , \tag{3.14}$$

and the induced S_α^1-action on $[M^{(1)}]$ is principal. Moreover, it is not hard to see that the above construction is canonical and that the assignments $[M] \mapsto [X^{(1)}, V^{(1)}]$ and $[M] \mapsto [M^{(1)}]$ are well-defined U_*-module homomorphisms $U_*^{T^n}(\mathscr{F}(1), \alpha) \to B_*^{T^n}(\alpha)$ and $U_*^{T^n}(\mathscr{F}(1), \alpha) \to U_*^{T^n}(\mathscr{F}(1), \alpha)$, respectively.

At this final stage, the S_α^1-action ψ on $M^{(1)}$ being principal, $M^{(1)} \to M^{(1)}/\psi(S_\alpha^1)$ is a differentiable S^1-principal bundle, where S^1 is identified with S_α^1 via θ. Let V' be the associated complex line bundle. The manifold $Y^{(0)} = M^{(1)}/\psi(S_\alpha^1)$ can be given a unique weakly complex structure such that, if V' is endowed with the weakly complex structure as the total space of complex line bundle, then the weakly complex structure on $M^{(1)}$ as the boundary of $D(V')$ coincides with the given one. The T^n-action ψ extends uniquely over V'. Let $X^{(0)}$ be the fixed point set of the T^n-action on $D(V')$; $X^{(0)}$ is contained in $Y^{(0)}$. Let $V^{(0)}$ denote the normal of $X^{(0)}$ in $D(V')$. Then we have a direct sum decomposition

$$V^{(0)} = V' | X^{(0)} \oplus V_{Y^{(0)}}$$

where $V_{Y^{(0)}}$ is the normal bundle in $Y^{(0)}$. $V' | X^{(0)}$ and $V_{Y^{(0)}}$ are T^n-invariant, and we see, exactly as in (3.11), that if W^λ is the unique irreducible T^n-module contained in $V' | X^{(0)}$, then W^λ is positive and $\bar{d}(\lambda) = 1$, and that $[X^{(0)}, V^{(0)}]$ belongs to $B_*^{T^n}(\alpha)$. We see also, as in (3.12), that

$$[M^{(1)}] = \partial[X^{(0)}, V^{(0)}] . \tag{3.15}$$

Moreover, the assignment $[M^{(1)}] \mapsto [X^{(0)}, V^{(0)}]$ is well-defined homomorphism. Then we define $Q_\alpha \colon U_*^{T^n}(\mathscr{F}(1), \alpha) \to B_*^{T^n}(\alpha)$ by

$$Q_\alpha[M] = [X^{(0)}, V^{(0)}] + [X^{(1)}, V^{(1)}] .$$

This is a well-defined homomorphism and satisfies $\partial Q_\alpha[M] = [M]$, by (3.14) and (3.15). This proves (3.2) and hence completes the proof of (1.4).

4. Equivariant U^*-theory characteristic numbers. The contents of this section are not new (cf. [8]). However, they are included here because it seems appropriate to give a brief description of the material in terms of ordinary U^*-theory and K-theory.

U^*-cohomology theory is extended over infinite CW-complexes with finite skeletons by defining

$$U^*(Y) = \varprojlim U^*(Y_\alpha) ,$$

where Y_α ranges over all finite subcomplexes of Y.

Let G be a compact Lie group and $\pi: EG \to BG$ a universal G-bundle. We may assume that BG is an inductive limit of compact smooth manifolds BG_q. We then define an equivariant U^*-theory on the category of compact G-spaces and G-maps by letting $U_G^*(\ \ ) = U^*(EG \times_G \ \ )$. In this theory, products are defined in an obvious way. If X and Y are closed weakly complex manifolds with smooth G-action preserving weakly complex structures, and $g: X \to Y$ is a G-map, then g can be given a natural complex orientation in the sense of Quillen [17]. If $EG_q \to BG_q$ is the restriction of E on BG_q, then $1 \times_G g: EG_q \times_G X \to EG_q \times_G Y$ is also a complex oriented map and hence induces a Gysin homomorphism $(1 \times_G g)_{!q}: U^*(EG_q \times_G X) \to U^*(EG_q \times_G Y)$. It is not hard to show that the Gysin homomorphisms $(1 \times_G g)_{!q}$ commute with the limiting process as $q \to \infty$, and thus define a map $g_!: U_G^*(X) \to U_G^*(Y)$, which may be called the Gysin homomorphism of g in the U_G^*-theory. It is easy to see that $g_!$ is a U^*-module map.

Next, let $V \to X$ be a G-complex vector bundle over a compact G-space X. Then $EG \times_G V \to EG \times_G X$ is also a complex vector bundle. Let $e(V) \in U_G^*(X) = U^*(EG \times_G X)$ denote its U^*-theory Euler class [17]; we then have the usual product formula $e(V_1 \oplus V_2) = e(V_1)e(V_2)$. In particular, if W is a G-module, then its Euler class $e(W)$ is defined in $U_G^* = U_G^*(\text{point})$. Let S denote the multiplicative set in U_G^* generated by Euler classes of non-trivial irreducible G-modules, and let $S^{-1}U_G^*$ denote the localized ring of U_G^* with respect to S. More generally, $S^{-1}U_G^*(X)$ can be defined, since $U_G^*(X)$ is a U_G^*-module.

We now define $\vartheta: U_*^G \to U_G^*$ by

$$\vartheta[M] = p_{M!}(1) ,$$

where $p_{M!}: U_G^*(M) \to U_G^*$ is the Gysin homomorphism of $p_M: M \to \text{point}$.

The verification of the following proposition is easy and will be left to the reader.

PROPOSITION (4.1). *$\vartheta: U_*^G \to U_G^*$ is a well-defined ring homomorphism, and in particular, a U^*-module map. The product in U_*^G is given by*

$$[M_1][M_2] = [M_1 \times M_2] ,$$

where the G-action on $M_1 \times M_2$ is the diagonal action.

Next we require the following lemma.

LEMMA (4.2). *Let $[X, V] \in B_*^G$. Then $e(V)$ is invertible when it is regarded as an element in $S^{-1}U_G^*(X)$.*

PROOF. Clearly, we may assume that X is connected. By the product formula for Euler classes we may assume that V contains only one irreducible G-module W, i.e., that V is of the form $E \otimes W$. If $\dim E = k$ and degree $W = d$ then, applying the formal group law in U^*-theory [17], we have

$$e(V) = e(W)^k + \sum_{\substack{q>0, j\geq 0 \\ -i+j+q=dk}} a_{ijq} , \qquad a_{ijq} \in U^{-2i} \cdot U_G^{2j} \cdot U^{2q}(X) ,$$

where $U^{-2i} \cdot U_G^{2j} \cdot U^{2q}(X)$ stands for the image of the cross-product $U^{-2i} \times U_G^{2j} \times U^{2q}(X) \to U^*(BG \times X) = U_G^*(X)$. Notice that X is a trivial G-space. Therefore, in the ring $S^{-1}U_G^*(X)$, we have

$$\frac{e(V)}{e(W)^k} = 1 + \sum_{\substack{q>0, j\geq 0 \\ -i+j+q=dk}} b_{ijq} , \qquad b_{ijq} \in U^{-2i} \cdot U_G^{2(j-dk)} \cdot U^{2q}(X) .$$

Since X is a finite CW-complex, elements in $\sum_{q>0} U^{2q}(X)$ are nilpotent [17]. Hence $b = \sum b_{ijq}$ is nilpotent and $(1+b)^{-1}$ exists in $S^{-1}U_G^*(X)$. Thus, $e(V)/e(W)^k$ is invertible in $S^{-1}U_G^*(X)$.

As is readily seen, the Gysin homomorphism $p_{X!}: U_G^*(X) \to U_G^*$ is a U_G^*-module map and hence induces $p_{X!}: S^{-1}U_G^*(X) \to S^{-1}U_G^*$. We define $\bar{\vartheta}: B_*^G \to S^{-1}U_G^*$ by

$$\bar{\vartheta}[X, V] = p_{X!}(e(V)^{-1}) .$$

PROPOSITION (4.3). *$\bar{\vartheta}: B_*^G \to S^{-1}U_G^*$ is a well-defined ring homomorphism, and in particular, a U^*-module map. The product in B_*^G is given by*

$$[X_1, V_1][X_2, V_2] = [X_1 \times X_2, V_1 \times V_2] .$$

The verification is left to the reader.

PROPOSITION (4.4). *The following diagram is commutative:*

$$\begin{tikzcd} U_*^G \arrow[r, "\vartheta"] \arrow[d, "f"] & U_G^* \arrow[d] \\ B_*^G \arrow[r, "\bar{\vartheta}"] & S^{-1}U_G^* . \end{tikzcd}$$

Here the second vertical arrow represents the canonical map. Moreover f is a ring homomorphism.

PROOF. An argument similar to that of [2] and [3] applies here. Given a closed unitary G-manifold M, let X be its fixed point set, V the normal bundle of X in M, and $i: X \to M$ the inclusion. By the definition of Euler class [17], we have

$$i^* i_!(x) = e(V)x$$

for $x \in U_G^*(X)$. Passing to $S^{-1}U_G^*$, we obtain

$$e(V)^{-1} i^* i_!(x) = x .$$

On the other hand, it was shown by tom Dieck [11] that

$$i^*: S^{-1}U_G^*(M) \to S^{-1}U_G^*(X)$$

is an isomorphism, hence it follows that $i_!(e(V)^{-1}) = 1$. Thus in $S^{-1}U_G^*$ we have

$$p_{M!}(1) = p_{M!} i_! (e(V)^{-1}) = p_{X!}(e(V)^{-1}) .$$

This proves the commutativity. The statement regarding f is clear.

Finally, we define the Boardman map

$$B: U_G^* \to K_G^{*\wedge}[[t]]$$

as the limit of the maps $B: U^*(BG_q) \to K^*(BG_q)[[t]]$ defined by

$$B(g_!(1)) = g_!(\gamma_t(-\nu_g)) ,$$

where g is a complex oriented map $(Y, \partial Y) \to (BG_q, \partial BG_q)$ from a compact smooth manifold Y and ν_g is the stable complex normal bundle of g.[6] It can be shown that B is a well-defined ring homomorphism. Moreover, the following proposition is an immediate consequences of the definition of ϑ, B and ρ.

PROPOSITION (4.5). *The composition $B \cdot \vartheta$ coincides with the composition $U_*^G \xrightarrow{\rho} K_G[[t]] \to K_G^{\wedge}[[t]]$.*

LEMMA (4.6). *If W is a G-module, then we have*

[6] Usual convention seems to adopt the definition

$$B(g_!(1)) = g_!(\gamma_t(\nu_g)) .$$

$$B(e(W)) = \lambda_{-1}(W)(\gamma_t(W - \dim W))^{-1} .$$

PROOF. Let $\tilde{W}_q$ denote the vector bundle $EG_q \times_G W \to BG_q$. Under the canonical map α_q: $K_G = R(G) \to K(BG_q)$, W is mapped into $\tilde{W}_q$. Let Y be a self-intersection of BG_q in $\tilde{W}_q$ and i: $Y \subset BG_q$ the inclusion. By definition, we have $e(\tilde{W}_q) = i_!(1)$. Similarly, in K-theory, $\lambda_{-1}(\tilde{W}_q) = i_!(1)$. Moreover, $\nu_i = i^*(\tilde{W}_q)$. Using the Riemann-Roch relation, we deduce that

$$\begin{aligned} B(e(\tilde{W}_q)) &= i_! i^*(\gamma_t(\tilde{W}_q - \dim \tilde{W}_q))^{-1} \\ &= i_!(1)(\gamma_t(\tilde{W}_q - \dim \tilde{W}_q))^{-1} \\ &= \lambda_{-1}(\tilde{W}_q)(\gamma_t(\tilde{W}_q - \dim \tilde{W}_q))^{-1} \\ &= \alpha_q\{\lambda_{-1}(W)(\gamma_t(W - \dim W))^{-1}\} . \end{aligned}$$

Taking the limit and using the canonical identification α: $K_G^{\wedge} \cong \varprojlim K(BG_q)$ [4], we obtain the desired relation.

By virtue of (4.6), the ring homomorphism B passes to the localization and induces

$$\bar{B}: S^{-1}U_G^* \to S^{-1}K_G^{\wedge}[[t]] .$$

The following proposition can be proved in a manner similar to (4.6). Details are left to the reader.

PROPOSITION (4.7). *The composition* $\bar{B} \circ \bar{\vartheta}$ *coincides with the composition* $B_*^G \xrightarrow{\bar{\rho}} S^{-1}K_G[[t]] \to S^{-1}K_G^{\wedge}[[t]]$.

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