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ON AUTOMORPHISM GROUPS OF II_1-FACTORS

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1. **Introduction.** In the present paper, we shall study groups of *-automorphisms on II_1-factors, using various topologies.

One of the main purposes is to attack the problem whether every II_1-factor has outer *-automorphisms. In [7], we proved that the symmetry on $M \overline{\otimes} M$ is outer for every non-atomic factor M; therefore, it is very plausible that every II_1-factor may have outer *-automorphisms.

Now let M be a II_1-factor with the separable predual. Let $C(M)$ (resp. $H(M)$ and $T(M)$) be the set of all central (resp. hyper-central and trivial-central) sequences in M. If $H(M) \subsetneqq C(M)$, then by McDuff's theorem [2], M is *-isomorphic to $M \overline{\otimes} U$, where U is the hyperfinite II_1-factor, so that M has outer *-automorphisms.

Among other things, we shall show that if $T(M) \subsetneqq C(M)$, then M has outer *-automorphisms (Corollary 7).

2. **Theorems.** Let M be a W^*-algebra, and let $A^*(M)$ be the group of all *-automorphisms on M. Let $B(M)$ be the Banach algebra of all bounded linear operators on M and let M_* be the predual of M. By the standard theory of Banach spaces, $B(M)$ is the dual Banach space of $M \otimes_\gamma M_*$, where γ is the greatest cross norm. We shall consider the topology $\sigma(B(M), M \otimes_\gamma M_*)$ on $A^*(M)$. We call this topology on $A^*(M)$ the weak *-topology and denote it by w^*.

PROPOSITION 1. *Suppose that a directed set* (ρ_α) *of elements in* $A^*(M)$ *converges to* $\rho_0 \in A^*(M)$ *in the* w^**-topology; then for any* $a \in M$, $(\rho_\alpha(a))$ *converges to* $\rho_0(a)$ *in the* $s(M, M_*)$*-topology.*

PROOF. Let M_* be the set of all normal positive linear functionals on M; then for $\varphi \in M_*$,

$$\varphi((\rho_\alpha(a) - \rho_0(a))^*(\rho_\alpha(a) - \rho_0(a))) = \varphi(\rho_\alpha(a^*a) + \rho_0(a^*a) - \rho_\alpha(a^*)\rho_0(a) - \rho_0(a^*)\rho_\alpha(a)) \to \varphi(\rho_0(a^*a) + \rho_0(a^*a) - \rho_0(a^*)\rho_0(a) - \rho_0(a^*)\rho_0(a)) = 0 .$$

Similarly,

$$\varphi((\rho_\alpha(a) - \rho_0(a))(\rho_\alpha(a) - \rho_0(a))^*) \to 0 .$$

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This completes the proof.

Let $I^*(M)$ be the subgroup of all inner *-automorphisms on M; then it is a normal subgroup of $A^*(M)$. Henceforward we shall assume that M is a II_1-factor and let τ be the unique tracial state on M. Define a scalar product (,) in M as follows: $(a, b) = \tau(b^*a)$ $(a, b \in M)$. Let $\mathscr{H}_\tau$ be the completion of M with respect to this scalar product and denote the norm of $\mathscr{H}_\tau$ by $\|\cdot\|_2$. For $\rho \in A^*(M)$, $\tau(\rho(a)) = \tau(a)(a \in M)$ and so the mapping $a \rightarrow \rho(a)$ will define a unitary operator $U(\rho)$ on $\mathscr{H}_\tau$ such that $U(\rho)a = \rho(a)(a \in M)$. Then the mapping $\rho \rightarrow U(\rho)$ is a unitary representation of the group $A^*(M)$, when $A^*(M)$ is considered as a discrete group. Since $U(\rho)bU(\rho^{-1})a = U(\rho)b\rho^{-1}(a) = \rho(b\rho^{-1}(a)) = \rho(b)a$, $U(\rho)bU(\rho^{-1}) = \rho(b)$ $(a, b \in M)$.

PROPOSITION 2. *The mapping $\rho \rightarrow U(\rho)$ of $A^*(M)$ with the w^*-topology into $B^u(\mathfrak{H}_\tau)$ with the strong operator topology is homeomorphic, where $B^u(\mathfrak{H}_\tau)$ is the group of all unitary elements in $B(\mathfrak{H}_\tau)$.*

PROOF. Suppose that (ρ_α) converges to ρ_0 in the w^*-topology; then by Proposition 1, $\|\rho_\alpha(a) - \rho_0(a)\|_2 \rightarrow 0$ for $a \in M$. Therefore, $(U(\rho_\alpha))$ converges to $U(\rho_0)$ strongly. Conversely suppose that $(U(\rho_\alpha))$ converges to $U(\rho_0)$ strongly; then $((\rho_\alpha(a) - \rho_0(a))b, b) \rightarrow 0$ for $a, b \in M$. Since $(\rho_\alpha(a))$ is uniformly bounded, this implies that (ρ_α) converges to ρ_0 in the w^*-topology. This completes the proof.

PROPOSITION 3. *$\{U(\rho) \mid \rho \in A^*(M)\}$ is closed in $B^u(\mathfrak{H}_\tau)$ with respect to the strong operator topology.*

PROOF. Suppose that $\{U(\rho_\alpha)\}$ converges to v in $B^u(\mathscr{H}_\tau)$; then $U(\rho_\alpha)bU(\rho_\alpha^{-1}) \rightarrow vbv^*(b \in M)$ (strongly), for $U(\rho_\alpha) \rightarrow v$ (strongly) implies $U(\rho_\alpha)^* \rightarrow v^*$ (strongly), since $U(\rho_\alpha)$ and v are unitary, and the multiplication is jointly strongly continuous on bounded spheres of $B(\mathscr{H}_\tau)$. It is easily seen that the mapping $b \rightarrow vbv^*(b \in M)$ is an automorphism on M. This completes the proof.

Now let $\overline{A^*(M)}^w$ be the w^*-closure of $A^*(M)$ in $B(M)$. It is an interesting problem to investigate mappings belonging to $\overline{A^*(M)}^w$. For example, if M is asymptotically abelian, then it contains the $\natural$-mapping $a \rightarrow \tau(a)1$ $(a \in M)$ ([5]). If M is inner asymptotically abelian, then $\overline{I^*(M)}^w$ contains the $\natural$-mapping ([8]). $\overline{A^*(M)}^w$ may contain another important class of mappings "into-*-isomorphisms". We shall show an example of II_1-factors having this property.

EXAMPLE. Let $\mathscr{H}$ be a Hilbert space and let $\mathfrak{A}(\mathscr{H})$ be the canonical anti-commutation relation algebra (the C A R algebra) over $\mathscr{H}$ ([3]); then $\mathfrak{A}(\mathscr{H})$ is a uniformly hyperfinite C^*-algebra of $\{2, 4, 8, \cdots\}$. Let u be a unitary operator on $\mathscr{H}$; then u will define a *-automorphism ρ^u of $\mathfrak{A}(\mathscr{H})$ by the relation $\rho^u(a(f)) = a(u(f))$ for $f \in \mathscr{H}$, where $f \to a(f)$ is a linear mapping, from $\mathscr{H}$ into $\mathfrak{A}(\mathscr{H})$ such that $\{a(f), a(g)\}_+ = 0$, $\{a(f)^*, a(g)\}_+ = (g, f)$ for $g, f \in \mathscr{H}$. Now let (u_n) be a sequence of unitary operators on $\mathscr{H}$ such that it converges strongly to an isometry v such that $vv^* < 1_{\mathscr{H}}$, and let τ be the unique tracial state on $\mathfrak{A}(\mathscr{H})$. Since v is an isometry of $\mathscr{H}$ into $\mathscr{H}$, it will define a *-isomorphism ρ^v of $\mathfrak{A}(\mathscr{H})$ into $\mathfrak{A}(\mathscr{H})$, by the relation $\rho^v(a(f)) = a(v(f))$ for $f \in \mathscr{H}$. Since $\tau(a(f)^*a(g)) = 1/2(g, f)$, $\| \rho^{u_n}(a(f)) - \rho^v(a(f)) \|_2 = (1/\sqrt{2}) \| a(u_n(f)) - a(v(f)) \|_2 = (1/\sqrt{2}) \| u_n(f) - v(f) \|_2 \to 0 (n \to \infty)$. Hence $\rho^{u_n}(a(f_1)a(f_2) \cdots a(f_n)) \to \rho^v(a(f_1)a(f_2) \cdots a(f_n))$ (strongly) on $\mathscr{H}_\tau$. This implies that (ρ^{u_n}) converges to an into-*-isomorphism ρ^v in the w^*-topology.

PROBLEM 1. Can we conclude that $\overline{A^*(M)}^w$ contains "into-*-isomorphisms" for all II_1-factors M?

If we use the strong *-operator topology, then $B^u(\mathscr{H}_\tau)$ is a complete topological group, so that by Proposition 2, $\{U(\rho) \mid \rho \in A^*(M)\}$ is also a complete topological group; therefore by identifying $A^*(M)$ with $\{U(\rho) \mid \rho \in A^*(M)\}$, we can introduce a complete topological group structure into $A^*(M)$. We shall call this topological structure of $A^*(M)$ the strong *-topology of $A^*(M)$ and is denoted by s^*. It is clear that the s^*-topology is stronger than the w^*-topology. If M has the separable predual, then $B^u(\mathscr{H}_\tau)$ is a separable complete metric group with respect to the strong *-operator topology; hence $A^*(M)$ is a separable complete metric topological group.

THEOREM 4. *Let M be the hyperfinite* II_1*-factor; then $I^*(M)$ is s^*-dense in $A^*(M)$.*

PROOF. Take a uniformly hyperfinite C^*-subalgebra $\mathfrak{A}$ of M such that $\mathfrak{A}$ is σ-dense in M. For $\rho \in A^*(M)$, the restriction of ρ to $\mathfrak{A}$ will give a *-isomorphism of $\mathfrak{A}$ onto $\rho(\mathfrak{A})$. Hence by Powers' theorem [4], there is a unitary element u in M such that $u^*\rho(\mathfrak{A})u = \mathfrak{A}$. Now put $\rho'(x) = u^*\rho(x)u$ $(x \in \mathfrak{A})$. Then ρ' is a *-automorphism of $\mathfrak{A}$. Suppose that $\mathfrak{A} =$ the uniform closure of $\bigcup_{n=1}^\infty \mathfrak{A}_n$, where $\mathfrak{A}_n$ is a type I_{i_n}-factor $(i_n < +\infty)$, $\mathfrak{A}_n \subset \mathfrak{A}_{n+1}$ $(n = 1, 2, \cdots)$ and $1 \in \mathfrak{A}_n$ $(n = 1, 2, \cdots)$. For $\mathfrak{A}_n$, let $\{e_{ij}^n \mid i, j = 1, 2, \cdots, i_n\}$ be a matrix unit of $\mathfrak{A}_n$; then for $\varepsilon > 0$, there is an $\mathfrak{A}_m$ and a finite family of elements $\{a_{ij} \mid i, j = 1, 2, \cdots, i_n\}$ in $\mathfrak{A}_m$ such that $n \leqq m$ and $\| \rho'(e_{ij}^n) - a_{ij} \| < \varepsilon$ $(i, j = 1, 2, \cdots, i_n)$. Then there is a matrix unit $(f_{ij} \mid i, j = 1, 2, \cdots, i_n)$ in $\mathfrak{A}_m$ and a positive number $\delta > 0$ such that $\|\rho'(e_{ij}^n) - f_{ij}\| < \delta$, where $\delta \to 0$, when $\varepsilon \to 0$ ([1]).

Let $\mathfrak{A}_m = \mathfrak{A}_n \otimes (\mathfrak{A}_n' \cap \mathfrak{A}_m)$, and let $\mathfrak{B}$ be the C^*-subalgebra of $\mathfrak{A}$ generated by $\{f_{ij} \mid i, j = 1, 2, \cdots, i_n\}$; put $\mathfrak{A}_m = \mathfrak{B} \otimes (\mathfrak{B}' \cap \mathfrak{A}_m)$ and $\zeta_1(e_{ij}^n) = f_{ij}$ $(i, j = 1, 2, \cdots, i_n)$. Then ζ_1 can be uniquely extended to a *-isomorphism, denoted again by ζ_1, of $\mathfrak{A}_n$ onto $\mathfrak{B}$. Let ζ_2 be a *-isomorphism of $(\mathfrak{A}_n' \cap \mathfrak{A}_m)$ onto $(\mathfrak{B}' \cap \mathfrak{A}_m)$; then $\zeta_1 \otimes \zeta_2$ will define a *-automorphism of $\mathfrak{A}_m$ onto $\mathfrak{A}_m$. Since $\mathfrak{A}_m$ is a type I-factor, every automorphism of it is inner; hence there is a unitary element v in such that $\zeta_1 \otimes \zeta_2(x) = v^*xv$ for $x \in \mathfrak{A}_m$.

$$\|\rho'(e_{ij}^n) - v^*e_{ij}^n v\| \leqq \delta \qquad (i, j = 1, 2, \cdots, i_n) .$$

Hence,

$$\begin{aligned}\|\rho(e_{ij}^n) - uv^*e_{ij}^n vu^*\| &= \|u^*\rho(e_{ij}^n)u - v^*e_{ij}v\| \\ &= \|\rho'(e_{ij}^n) - v^*e_{ij}^n v\| < \delta \qquad (i, j = 1, 2, \cdots, i_n) .\end{aligned}$$

From this, we can easily conclude that for $a_1, a_2, \cdots, a_m \in \mathfrak{A}$,

$$\inf_{\zeta \in I^*(M), 1 \leqq j \leqq m} \max \|\rho(a_j) - \zeta(a_j)\| = 0$$

and so we can easily conclude that for $\xi_1, \xi_2, \cdots, \xi_m \in \mathscr{H}_\tau$,

$$\inf_{\zeta \in I^*(M), 1 \leqq j \leqq m} \max \|(U(\rho) - U(\zeta))\xi_j\| = 0 .$$

This implies that there is a sequence (ρ_n) in $I^*(M)$ such that $U(\rho_n)$ $U(\rho)$ (strongly) and so $U(\rho_n)^* \to U(\rho)^*$ (strongly). This completes the proof.

THEOREM 5. *Suppose that M is a II_1-factor with the separable predual and $T(M) = C(M)$. Then the set $\{U(\rho) \mid \rho \in I^*(M)\}$ is complete with respect to the strong operator topology. In particular, $I^*(M)$ is complete with respect to the s^*-topology and $I^*(M)$ is closed in $A^*(M)$ with respect to the w^*-topology and s^*-topology respectively.*

PROOF. Let (ρ_n) be a Cauchy sequence of inner *-automorphisms on M. Put $\rho_n(a) = u_n{}^*au_n (a \in M)$, where u_n are unitary in M. Then,

$$\|u_n^*au_n - u_m^*au_m\|_2 = \|au_nu_m^* - u_nu_m^*a\|_2 = \|[a, u_nu_m^*]\|_2 \to 0 \ (m, n \to \infty) .$$

Since $T(M) = C(M)$, $\|u_nu_m^* - \tau(u_nu_m^*)1\|_2 \to 0$ $(n, m \to \infty)$.

$$\|u_nu_m^* - \tau(u_nu_m^*)1\|_2 = \|u_n - \tau(u_nu_m^*)u_m\|_2 \to 0 \ (n, m \to \infty) .$$

Since $\|u_nu_m^*\|_2 = 1$, $|\tau(u_nu_m^*)| \to 1$ and so there is a double sequence of complex numbers $(\mu_{n,m})$ with $|\mu_{n,m}| = 1$ such that $\|u_n - \mu_{n,m}u_m\|_2 \to 0$ $(n, m \to \infty)$. We can choose a subsequence (n_j) of (n) such that

$$\sum_{j=1}^{\infty} \|u_{n_j} - \mu_{n_j, n_{j+1}} \cdot u_{n_{j+1}}\|_2 < +\infty .$$

By induction, we shall define a sequence of unitary elements (v_j) in M. Put $v_1 = u_{n_1}$; suppose that (v_i) $(i \leqq j)$ are defined; then take a complex number $\lambda_{j+1}(|\lambda_{j+1}| = 1)$ such that $\inf_{|\lambda|=1} \| v_j - \lambda u_{n_{j+1}} \|_2 = \| v_j - \lambda_{j+1} u_{n_{j+1}} \|_2$ and define $v_{j+1} = \lambda_{j+1} u_{n_{j+1}}$. Let $j \geqq k$; then

$$\begin{aligned} \| v_j - v_k \|_2 &\leqq \sum_{i=1}^{j-k} \| v_{k+i} - v_{k+i-1} \|_2 = \sum_{i=1}^{j-k} \| v_{k+i-1} - \lambda_{k+i} u_{n_{k+i}} \|_2 \\ &\leqq \sum_{i=1}^{j-k} \| v_{k+i-1} - \mu_{n_{k+i-1}, n_{k+i}} \lambda_{k+i-1} u_{n_{k+i}} \| \\ &= \sum_{i=1}^{j-k} \| u_{k+i-1} - \mu_{n_{k+i-1}, n_{k+i}} u_{n_{k+i}} \|_2 \longrightarrow 0 \qquad (j, k \longrightarrow \infty) . \end{aligned}$$

Therefore, there is an isometry v belonging to M such that $\| v_n - v \|_2 \longrightarrow 0$. Since M is finite, v is unitary. Moreover,

$$\| v_j^* a v_j - v^* a v \|_2 = \| u_{n_j}^* a u_{n_j} - v^* a v \|_2 \longrightarrow 0 \; (a \in M) .$$

$$\| u_m^* a u_m - u_{n_j}^* a u_{n_j} \|_2 + \| u_{n_j}^* a u_{n_j} - v^* a v \|_2 \longrightarrow 0 \; (m \longrightarrow \infty,\ n_j \geqq m) .$$

This completes the proof.

REMARK 1. This theorem implies that a sequence of inner *-automorphisms on a II_1-factor M satisfying $T(M) = C(M)$ can not approach to an into-*-isomorphism on M in the w^*-topology. In fact, suppose that a sequence (ρ_n) of inner *-automorphisms on M converges to an into- *-isomorphism ρ in the w^*-topology; then $U(\rho)a = \rho(a)$ $(a \in M)$ will define an isometry of $\mathscr{H}_\tau$ into $\mathscr{H}_\tau$. Since $(U(\rho_n)a, b) = (\rho_n(a), b) = \tau(b^* \rho_n(a)) \longrightarrow (\rho(a), b)$ $(a, b \in M)$ and $\| \rho_n(a) \|_2^2 = (\rho_n(a), \rho_n(a)) = \tau(\rho_n(a^*a)) \longrightarrow \tau(\rho(a^*a)) = \tau(\rho(a)^*\rho(a)) = \| \rho(a) \|_2^2$, $\| U(\rho_n)a - U(\rho)a \|_2 \longrightarrow 0$ $(n \longrightarrow \infty)$. Therefore, $\{U(\rho_n)\}$ is a Cauchy sequence in the strong operator topology.

Next we shall show the converse of Theorem 5 to be true.

THEOREM 6. *Suppose that M is a II_1-factor with the separable predual. If $I^*(M)$ is closed in $A^*(M)$ with respect to the s^*-topology, then $T(M) = C(M)$.*

PROOF. It is easily seen that if every central sequence of unitary elements in M is trivial, then $T(M) = C(M)$ ([2]). Suppose that $I^*(M)$ is s^*-closed in $A^*(M)$. Then $I^*(M)$ is a separable complete metric topological group with respect to the s^*-topology. Let M^u be the group of all unitary elements in M. For $u, v \in M^u$, we shall define a metric $d(u, v)$ as follows; $d(u, v) = \| u - v \|_2$. This is equivalent to the s^*-topology on M^u, for M is finite. Hence M^u is a separable complete metric topological group. For $u \in M^u$, define $\rho_u(a) = uau^*$ $(a \in M)$; then ρ_u is an inner *-automorphism on M. Consider a group homomorphism

$\Phi: u \rightarrow \rho_u$ of M^u onto $I^*(M)$. Then clearly Φ is a continuous mapping of M^u with the s^*-topology onto $I^*(M)$ with the s^*-topology. Hence the kernel K of Φ is closed in M^u and so the quotient group M^u/K is again a topological group with the second countability axiom, and the canonical isomorphism $\tilde{\Phi}$ of M^u/K onto $I^*(M)$ defined by Φ is continuous. Let (a_n) be a set of elements in M^u which is s^*-dense in M^u. For an arbitrary positive number ε, let V_ε be the set of elements in M^u such that $V_\varepsilon = \{u \mid d(1, u) < \varepsilon, u \in M^u\}$. Since $d(xa, ya) = \|(x-y)a\|_2 = \|x-y\|_2$ for $a, x, y \in M^u$, $V_\varepsilon a_n = \{w \mid d(a_n, w) < \varepsilon, w \in M^u\}$. Clearly $\bigcup_{n=1}^\infty V_\varepsilon a_n = M^u$ and so $\bigcup_{n=1}^\infty \Phi(V_\varepsilon a_n) = \bigcup_{n=1}^\infty \Phi(V_\varepsilon)\Phi(a_n) = I^*(M)$. Since $I^*(M)$ is a complete metric space, it is of the second category. Hence $\Phi(V_\varepsilon a_{n_0})$ is not nowhere dense for some n_0. Hence the s^*-closure $\overline{\Phi(V_\varepsilon)\Phi(a_{n_0})}$ of $\Phi(V_\varepsilon)\Phi(a_{n_0})$ contains an open set. Since $\overline{\Phi(V_\varepsilon)\Phi(a_{n_0})} = \overline{\Phi(V_\varepsilon)}\Phi(a_{n_0})$, $\overline{\Phi(V_\varepsilon)}$ contains an open set. For V_ε there is a V_{ε_1} such that $V_{\varepsilon_1}V_{\varepsilon_1} \subset V_\varepsilon$. Since $V_{\varepsilon_1} = V_{\varepsilon_1}^{-1}$, $\overline{\Phi(V_{\varepsilon_1})}\,\overline{\Phi(V_{\varepsilon_1})}$ contains an open set containing the unit 1. Since $\overline{\Phi(V_{\varepsilon_1})}\,\overline{\Phi(V_{\varepsilon_1})} \subset \overline{\Phi(V_\varepsilon)}$, $\overline{\Phi(V_\varepsilon)}$ contains an open set G containing the unit. Let $\rho_v \in G$; then there is a sequence (u_n) in V_ε such that $\rho_{u_n} \rightarrow \rho_v(s^*)$. Therefore, $\|u_n^* a u_n - v^* a v\|_2 \rightarrow 0 \ (n \rightarrow \infty)$. Since $u_n \in V_\varepsilon$, $\|u_n - 1\|_2 < \varepsilon$; hence for $a \in M^u$,

$$\|u_n^* a u_n - v^* a v\|_2 = \|(u_n v^*)^* a (u_n v) - a\|_2 = \|a(u_n v^*) - (u_n v^*)a\|_2 \rightarrow 0 .$$

Therefore,

$$\begin{aligned} \|av^* - v^*a\|_2 &= \|a(1-u_n)v^* - (1-u_n)v^*a + au_nv^* - u_nv^*a\|_2 \\ &\leqq \|a(1-u_n)v^* - (1-u_n)v^*a\|_2 + \|au_nv^* - u_nv^*a\|_2 \\ &\leqq 2\|a\|\,\|1-u_n\|_2 + \|au_nv^* - u_nv^*a\|_2 \leqq 2\varepsilon + \|au_nv^* - u_nv^*a\|_2 . \end{aligned}$$

Hence $\|av^* - v^*a\|_2 \leqq 2\varepsilon$. Therefore, $\|av^*a^* - v^*\|_2 \leqq 2\varepsilon$ for $a \in M^u$. Hence $\|\tau(v^*)1 - v^*\|_2 \leqq 2\varepsilon$ and $\|\tau(v)1 - v\|_2 \leqq 2\varepsilon$. Since

$$\begin{aligned} \|\tau(v)1 - v\|_2^2 &= \tau((v^* - \overline{\tau(v)}1)(v - \tau(v)1)) \\ &= \tau(1 + |\tau(v)|^2 1 - |\tau(v)|^2 1 - |\tau(v)|^2 1) = (1 - |\tau(v)|^2) \leqq 4\varepsilon^2 , \\ \left\|\frac{\tau(v)1}{|\tau(v)|} - v\right\|_2^2 &= 1 + 1 - |\tau(v)| - |\tau(v)| = 2(1 - |\tau(v)|) \\ &= 2(1 - |\tau(v)|^2)/1 + |\tau(v)| \leqq 8\varepsilon^2 . \end{aligned}$$

Hence $\inf_{|\lambda|=1} d(v, \lambda 1) < 3\varepsilon$. This implies that $\tilde{\Phi}^{-1}$ is continuous; hence $\tilde{\Phi}$ is homeomorphic. Especially, if $\rho_{v_n} \rightarrow \rho_1(s^*)$ $(v_n \in M^u)$, then there is a sequence of complex numbers (λ_n) with $|\lambda_n| = 1$ such that $d(v_n, \lambda_n 1) = \|v_n - \lambda_n 1\|_2 \rightarrow 0$. If (w_n) is a central sequence of unitary elements in M, then $\|[x, w_n]\|_2 = \|xw_n - w_nx\|_2 = \|w_n^* x w_n - x\|_2 \rightarrow 0 \ (n \rightarrow \infty)$ and $\|w_n x w_n^* - x\|_2 = \|x - w_n^* x w_n\|_2$; hence $\rho_{w_n} \rightarrow \rho_1(s^*)$. Therefore, we have $T(M) = C(M)$. This completes the proof.

COROLLARY 7. *If $T(M) \subsetneqq C(M)$ for a II_1-factor M with the separable predual, then M has outer *-automorphisms belonging to the s^*-closure of $I^*(M)$.*

Now let $\overline{I^*(M)}$ be the s^*-closure of $I^*(M)$ in $A^*(M)$. Since $I^*(M)$ is a normal subgroup of $A^*(M)$, $\overline{I^*(M)}$ is also a normal subgroup of $A^*(M)$. Let Π be the group of all finite permutations of all positive integers N onto itself, and let $U(\Pi)$ be the II_1-factor generated by the left regular representation; then by Theorem 4, $I^*(U(\Pi)) \subsetneqq \overline{I^*(U(\Pi))} = A^*(U(\Pi))$. Next let G_2 be the free group of two generators; then by Theorem 5, $I^*(U(G_2)) = \overline{I^*(U(G_2))}$. Moreover, it is well known that $I^*(U(G_2)) \subsetneqq A^*(U(G_2))$. Now we shall show the following theorem.

THEOREM 8. *There are three II_1-factors M_1, M_2, M_3 such that*
1. $I^*(M_1) \subsetneqq \overline{I^*(M_1)} = A^*(M_1)$;
2. $I^*(M_2) = \overline{I^*(M_2)} \subsetneqq A^*(M_2)$;
3. $I^*(M_3) \subsetneqq \overline{I^*(M_3)} \subsetneqq A^*(M_3)$.

PROOF. It is enough to show the 3. Let $\Gamma_j = G_2\ (j = 1, 2, \cdots)$; then $U(\sum_{j=1}^{\infty} \oplus \Gamma_j)$ is asymptotically abelian ([5]), but by Zeller-Meier's theorem it is not inner asymptotically abelian ([8]). Let (ρ_n) be a sequence of *-automorphisms on $U(\sum_{j=1}^{\infty} \oplus \Gamma_j)$ such that $\| [\rho_n(a), b] \|_2 \to 0\ (a, b \in U(\sum_{j=1}^{\infty} \oplus \Gamma_j))$. Now suppose that $\overline{I^*(U(\sum_{j=1}^{\infty} \oplus \Gamma_j))} = A^*(U(\sum_{j=1}^{\infty} \oplus \Gamma_j))$. Then for each ρ_n, there is a sequence of inner *-automorphisms $\{\zeta_{n,m}\}$ on $U(\sum_{j=1}^{\infty} \oplus \Gamma_j)$ such that

$$\| \zeta_{n,m}(x) - \rho_n(x) \|_2 \to 0 \ \left(m \to \infty,\ x \in U\left(\sum_{j=1}^{\infty} \oplus \Gamma_j\right)\right).$$

Let $\{x_i \mid i = 1, 2, \cdots\}$ be a subset of $U(\sum_{j=1}^{\infty} \oplus \Gamma_j)$ which is dense in $l^2(\sum_{j=1}^{\infty} \oplus \Gamma_j)$. For each positive integer k, there is a positive integer m_k such that $\| \zeta_{n,m}(x_i) - \rho_n(x_i) \|_2 < 1/k$ for all $m \geqq m_k$ and $n = 1, 2, \cdots, k$, $i = 1, 2, \cdots, k$. We may assume that $m_k \leqq m_{k+1}$. Put $\gamma_k = \zeta_{k,m_k}$; then we can easily see

$$\begin{aligned} \| [\gamma_k(a), b] \|_2 &\leqq \| [\gamma_k(a) - \rho_k(a), b] \|_2 + \| [\rho_k(a), b] \|_2 \\ &\leqq 2 \| b \| \| \gamma_k(a) - \rho_k(a) \|_2 + \| [\rho_k(a), b] \|_2 \to 0 \\ &\qquad \left(a, b \in U\left(\sum_{j=1}^{\infty} \oplus \Gamma_j\right)\right). \end{aligned}$$

This implies that $U(\sum_{j=1}^{\infty} \oplus \Gamma_j)$ is inner asymptotically abelian, a contradiction. Hence $\overline{I^*(U(\sum_{j=1}^{\infty} \oplus \Gamma_j))} \subsetneqq A^*(U(\sum_{j=1}^{\infty} \oplus \Gamma_j))$. On the other hand, $T(U(\sum_{j=1}^{\infty} \oplus \Gamma_j)) \subsetneqq C(U(\sum_{j=1}^{\infty} \oplus \Gamma_j))$ and so $I^*(U(\sum_{j=1}^{\infty} \oplus \Gamma_j)) \subsetneqq \overline{I^*(U(\sum_{j=1}^{\infty} \oplus \Gamma_j))}$ by Theorem 6. This completes the proof.

Finally we shall state some problems. Let $Q(M)$ be the w^*-closed convex subset of $B(M)$ generated by $A^*(M)$. Then it is easily seen that $Q(M)$ can be identified with the weakly closed convex subset of $B(\mathscr{H}_\tau)$ generated by $\{U(\rho) \mid \rho \in A^*(M)\}$. Under this identification, the w^*-topology on $Q(M)$ is equivalent to the weak operator topology. Therefore, if M has the separable predual, $Q(M)$ is a compact metric space with the w^*-topology; hence we can apply the Choquet theory to $Q(M)$. Therefore, it is an interesting question to determine extreme points in $Q(M)$. It is clear that all *-automorphisms and all into-*-isomorphisms belonging to $Q(M)$ are extreme. Are there other extreme points in $Q(M)$? Let φ be a state on M and consider a positive linear mapping Ψ of M into M as follows: $\Psi(a) = \varphi(a)1$ for $a \in M$. Then it is easily seen that Ψ belongs to $Q(M)$ if and only if $\varphi = \tau$.

Let N be a W^*-subalgebra of M; then there is the unique canonical conditional expectation P of M onto N such that $\tau(P(a)x) = \tau(ax)$ for $a \in M$ and $x \in N$. If $N = M \cap (N' \cap M)'$, then it is easily seen that P belongs to $Q(M)$. Can we conclude that P belongs to $Q(M)$ without the assumption $N = M \cap (N' \cap M)'$? If $Q(M)$ contains an into-*-isomorphism, then by the standard theory of locally convex spaces $\overline{A^*(M)}^w$ contains it. Therefore, if $I^*(M) = A^*(M)$, then $Q(M)$ can not contain an into-*-automorphism, by Theorems 5 and 6. Hence it is a quite interesting question whether $Q(M)$ contains an into-*-isomorphism for every II_1-factor M.

In conclusion, we notice that much part of this paper can be extended to infinite factors with minor modifications.

REFERENCES

[1] J. GLIMM, On a certain class of operator algebras, Trans. Amer. Math. Soc., 95 (1960), 216-244.

[2] D. MCDUFF, Central sequences and the hyperfinite factor, Proc. London Math. Soc., 21 (1970), 443-461.

[3] R. POWERS, Dissertation.

[4] R. POWERS, Representations of uniformly hyperfinite algebras and the associated von Neumann algebras, Ann. of Math., 86 (1967), 138-171.

[5] S. SAKAI, Asymptotically abelian II_1-factors, Publ. Res. Inst. Sci., 4 (1968), 299-307.

[6] S. SAKAI, C^*-algebras and W^*-algebras, Springer-Verlag, 1971.

[7] S. SAKAI, Automorphisms and tensor products of operator algebras, to appear in American Journal of Mathematics.

[8] G. ZELLER-MEIER, Deux autres facteurs de type II_1, Invent. Math., 17 (1969), 235-242.

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