Tôhoku Math. Journ.
26 (1974), 407-410.

ON THE DIVERGENCE OF REARRANGED WALSH SERIES II

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(Received April 12, 1973)

Concerning the rearrangement of Walsh series, the author [2] proved the following

THEOREM A. *If* $\{\rho(n)\}$ *is a sequence of positive numbers with* $\rho(n) = o(\sqrt[4]{\log n})$, *then there exists a sequence of real numbers* $\{a_n\}$ *for which*

$$\sum_{n=1}^{\infty} a_n^2 \rho^2(n) < \infty$$

and such that the Walsh series

$$\sum_{n=1}^{\infty} a_n w_n(x) \tag{1}$$

can be rearranged into an almost everywhere divergent series.

This is a generalization of a theorem in [3], and also can be considered as a Walsh series analogue of a theorem in [1] which concerns the rearrangement of trigonometric series. In the present note, we intend to sharpen further this theorem. We write $L_1(n) = \log n$ and $L_s(n) = L_1(L_{s-1}(n))$ $(s = 2, 3, \cdots)$.

THEOREM. *For any natural number s, there exists a sequence of real numbers* $\{a_n\}$ *for which*

$$\sum_{n=N+1}^{\infty} a_n^2 \sqrt{L_1(n)}\, L_2(n) L_3(n) \cdots L_s(n) < \infty^{1)}$$

such that the Walsh series (1) *can be rearranged into an almost everywhere divergent series.*

COROLLARY. *For any natural number s, there exists a sequence of real numbers* $\{a_n\} \in l_2$ *such that the Walsh series* (1) *can be rearranged to satisfy*

$$\limsup_{N\to\infty} \left| \sum_{j=1}^{N} a_{n(j)} w_{n(j)}(x) \right| \{L_1(N)\}^{-1/4} \{L_2(N) L_3(N) \cdots L_s(N)\}^{-1/2} > 0$$

almost everywhere.

1) N is a natural number depending s such that $L_s(N) > 0$.

The method of the proof of the theorem is that of [2] and of [4].

1. Lemmas. Denoting by E the set of all dyadic irrational numbers in the interval $(0, 1)$, we set $E_i = \{x \in E; r_i(x) = 1\}$, where $r_i(x) = \operatorname{sgn} \sin 2^i \pi x$ $(i = 1, 2, \cdots)$. It is obvious that $\{E_i\}$ is stochastically independent and $\operatorname{mes} E_i = 2^{-1}$.

We set $g(i) = 2^{-1}(i-1)i + 1$ $(i = 1, 2, \cdots)$, and define a function $n = n(i, k, j)$ $(j = 1, \cdots, 2^k; k = 0, 1, \cdots; i = 1, 2, \cdots)$ as follows: For each pair (i, k), consider all integers

$$2^{\nu_0} + 2^{\nu_1} + \cdots + 2^{\nu_l} \tag{2}$$

satisfying $g(i+k) \leqq \nu_0 < \nu_1 < \cdots < \nu_l = g(i+k) + k$. Then label (2) as $n(i, k, j)$ $(j = 1, \cdots, 2^k)$ so that

$$2^{g(i+k)+k} = n(i, k, 1) < n(i, k, 2) < \cdots < n(i, k, 2^k) = (2^{k+1} - 1)2^{g(i+k)} .$$

The following lemmas can be proved by the same argument as in [2] which gives the special case $i = 1$.

LEMMA 1. *For each natural number i, set $\psi_1^{(i)}(0; x) = 1$ and*

$$\psi_{2j-1}^{(i)}(k+1; x) = 2^{-1}\psi_j^{(i)}(k; x)(1 + w_{n(i,k,j)}(x)) ,$$
$$\psi_{2j}^{(i)}(k+1; x) = 2^{-1}\psi_j^{(i)}(k; x)(1 - w_{n(i,k,j)}(x))$$

$(j = 1, \cdots, 2^k; k = 0, 1, \cdots)$; *then the following* (i)–(iii) *hold.*

(i) *Each $\psi_j^{(i)}(k; x)$ $(j = 1, \cdots, 2^k; k = 0, 1, \cdots)$ is a linear combination of Walsh functions with indices $2^{g(i)}p$ $(< 2^{g(i+k-1)+k})$.*

(ii) $$\psi_j^{(i)}(k; x) = 0 \text{ or } 1 \ (x \in E).$$

(iii) *Set $E(i, k, j) = \{x \in E_i; \psi_j^{(i)}(k; x) = 1\}$ $(j = 1, \cdots, 2^k; k = 0, 1, \cdots)$, then*

$$E(i, k, j) = E(i, k+1, 2j-1) \cup E(i, k+1, 2j) ,$$
$$E(i, k, j) \cap E(i, k, j') = \varnothing \qquad (1 \leqq j < j' \leqq 2^k) ,$$
$$\bigcup_{j=1}^{2^k} E(i, k, j) = E_i ,$$
$$\operatorname{mes} E(i, k, j) = 2^{-k-1} .$$

LEMMA 2. *For each natural number i, set*

$$\Psi_{2j-1}^{(i)}(k+1; x) = \psi_j^{(i)}(k; x) w_{n(i,k,j)}(x) ,$$
$$\Psi_{2j}^{(i)}(k+1; x) = -r_i(x)\Psi_{2j-1}^{(i)}(k+1; x)$$

$(j = 1, \cdots, 2^k; k = 0, 1, \cdots)$; *then the following* (iv)–(vii) *hold.*

(iv) *Each $\Psi_j^{(i)}(k; x)$ $(j = 1, \cdots, 2^k; k = 1, 2, \cdots)$ is a linear combination of Walsh functions with indices p such that*

$$p \in I_{ik} = [2^{g(i+k-1)+k-1}, 2^{g(i+k-1)+k}) .$$

$$I_{ik} \cap I_{i'k'} = \emptyset \quad \textit{for} \quad (i, k) \neq (i', k') .$$

(v) *If* $1 \leqq j < j' \leqq 2^k$, *then* $\Psi_j^{(i)}(k; x)$ *and* $\Psi_{j'}^{(i)}(k; x)$ *have no term of the same index.*

(vi)
$$\begin{aligned} &\Psi_{2j-1}^{(i)}(k; x) = 1 && \textit{for } x \in E(i, k, 2j-1) , \\ &\Psi_{2j-1}^{(i)}(k; x) + 2\Psi_{2j}^{(i)}(k; x) = 1 && \textit{for } x \in E(i, k, 2j) , \\ &\Psi_{2j-1}^{(i)}(k; x) = \Psi_{2j}^{(i)}(k; x) = 0 && \textit{for } x \in E_i - E(i, k-1, j) . \end{aligned}$$

(vii) $\int_0^1 \{\Psi_j^{(i)}(k; x)\}^2 dx = 2^{-k+1} \qquad (j = 1, \cdots, 2^k; k = 1, 2, \cdots) .$

2. Proof of the theorem. Set

$$R(i, k; x) = \{(i+k-1)L_1(i+k-1) \cdots L_s(i+k-1)\}^{-1} \sum_{j=1}^{2^{k-1}} \{\Psi_{2j-1}^{(i)}(k; x) + 2\Psi_{2j}^{(i)}(k; x)\}$$

($k = 1, 2, \cdots; i = N+1, N+2, \cdots$). Arrange $\{\Psi_j^{(i)}(k; x)\}$ ($j = 1, 2, \cdots, 2^k$; $k = 1, \cdots, m$) into

(3) $\qquad U_1^{(i)}(m; x), U_2^{(i)}(m; x), \cdots, U_{h(m)}^{(i)}(m; x) \qquad (h(m) = 2^{m+1} - 2)$

as follows: For the case $m = 1$, set $U_1^{(i)}(1; x) = \Psi_1^{(i)}(1; x)$, $U_2^{(i)}(1; x) = \Psi_2^{(i)}(1; x)$. Supposing the case $m = m$ defined, define the case $m+1$ by inserting $\Psi_{2j-1}^{(i)}(m+1; x)$, $\Psi_{2j}^{(i)}(m+1; x)$ after $\Psi_j^{(i)}(m; x)$ in (3).

Define m_i so that

$$\sum_{n=i}^{i+m_i-1} \{nL_1(n)L_2(n) \cdots L_s(n)\}^{-1} \geqq 1 ,$$

and set

$$\sum_{k=1}^{m_i} R(i, k; x) = \sum_{j=1}^{h(m_i)} b_j^{(i)} U_j^{(i)}(m_i; x) \qquad (i = N+1, N+2, \cdots) .$$

Then, by (iii) and (vi), there exists $\mu_i(x)$ ($\leqq h(m_i)$) such that

$$\sum_{j=1}^{\mu_i(x)} b_j^{(i)} U_j^{(i)}(m_i; x) \geqq 1 \qquad (x \in E_i; i = N+1, N+2, \cdots) .$$

Writing $\lambda(n) = \sqrt{L_1(n)}\, L_2(n)L_3(n) \cdots L_s(n)$, and using (vii), we get

$$\sum_{k=1}^{m_i} \lambda(2^{g(i+k-1)+k}) \int_0^1 R^2(i, k; x) dx$$
$$= 5 \sum_{k=1}^{m_i} \lambda(2^{g(i+k-1)+k}) \{(i+k-1)L_1(i+k-1) \cdots L_s(i+k-1)\}^{-2}$$

$$\leqq 5 \sum_{n=i}^{i+m_i-1} \lambda(2^{n^2})\{nL_1(n) \cdots L_s(n)\}^{-2}$$

$$\leqq C \sum_{n=i}^{\infty} \{nL_1(n) \cdots L_{s-1}(n)\}^{-1}\{L_s(n)\}^{-2}$$

$$\leqq C\{L_s(i-1)\}^{-1} \qquad (i = N+1, N+2, \cdots) .$$

Define a sequence of integers $(N <)\, i_1 < i_2 < \cdots$ such that

$$L_s(i_l - 1) \geqq l^2 \qquad (l = 1, 2, \cdots) .$$

Now, define the rearranged Walsh series

$$\sum_{j=1}^{\infty} a_{n(j)} w_{n(j)}(x) \tag{4}$$

by considering

$$\sum_{i \in \{i_l\}} \sum_{j=1}^{h(m_i)} b_j^{(i)} U_j^{(i)}(m_i; x) .$$

Then the series (4) diverges for $x \in \limsup_{l\to\infty} E_{i_l}$; and applying Borel-Cantelli lemma, we get $\mathrm{mes}(\limsup_{l\to\infty} E_{i_l}) = 1$. Further we have

$$\sum_{n=N+1}^{\infty} a_n^2 \lambda(n) = \sum_{i \in \{i_l\}} \sum_{k=1}^{m_i} \sum_{n \in I_{ik}} a_n^2 \lambda(n)$$

$$\leqq \sum_{i \in \{i_l\}} \sum_{k=1}^{m_i} \lambda(2^{g(i+k-1)+k}) \int_0^1 R^2(i, k; x)dx$$

$$\leqq C \sum_{i \in \{i_l\}} \{L_s(i-1)\}^{-1} \leqq C \sum_{l=1}^{\infty} l^{-2} < \infty .$$

This concludes the proof of the theorem.

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