

ON THE TOPOLOGY OF QUATERNION KÄHLER MANIFOLDS

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0. Introduction. A $4m$ -dimensional Riemannian manifold is called a quaternion Kähler manifold if the holonomy group is contained in $Sp(m) \cdot Sp(1)$ ($= Sp(m) \times Sp(1)/\{\pm 1\}$). Recently, Ishihara [6] has given a definition equivalent to that above for quaternion Kähler manifolds and obtained many interesting results (cf. [1], [3] and [7]). We shall adopt the definition given by Ishihara [6] and study quaternionic analogue to Kählerian pinching, which will be called *quaternionic pinching*. Kraines [13], using some general results of Klingenberg [8], showed that a compact quaternion Kähler manifold of dimension $4m$ ($m \geq 2$) with quaternionic pinching greater than $9/16$ has the same integral cohomology ring as the quaternionic projective space. On the other hand, Kobayashi [10], using sphere theorem of Berger and Klingenberg, constructed a principal circle bundle over a complete Kähler manifold with Kählerian pinching greater than $4/7$ such that the universal covering space of the bundle space is homeomorphic to a sphere and showed that the Kähler manifold has the same homotopy type as the complex projective space. We shall apply the method developed by Kobayashi to quaternion Kähler manifolds.

In § 1, we give the definition of a quaternion Kähler manifold and construct principal $Sp(1)$ -bundle over it under certain topological conditions which will be naturally satisfied if the quaternionic pinching number is greater than $9/16$. In § 2, we define a Riemannian metric in the principal $Sp(1)$ -bundle constructed above by a similar method as that given in [10] and calculate its Riemannian curvature tensor. In § 3, using the structure equation obtained in § 2, we determine the quaternionic pinching number such that the bundle space of the $Sp(1)$ -bundle has Riemannian pinching greater than $1/4$ and prove

THEOREM. *Let M be a complete quaternion Kähler manifold of dimension $4m$ ($m \geq 2$) with quaternionic pinching greater than $10/13$. Then, $\pi_q(M) = \pi_q(HP^m)$ for all q .*

1. Definitions and construction of principal $Sp(1)$ -bundle. Let H^m be the m -dimensional right module over quaternions H and $\{1, e_1, e_2, e_3\}$

be the usual base of H over R . The symplectic group $Sp(m)$ is defined as the set of all endomorphisms of H^m which preserve the symplectic product $(p, q) = \sum_{i=1}^m p_i \bar{q}_i$ where $p = (p_1, \dots, p_m)$ and $q = (q_1, \dots, q_m) \in H^m$. In particular, $Sp(1)$ is the set of unit quaternions. Hence it is diffeomorphic to a 3-dimensional sphere S^3 and its Lie algebra $\mathfrak{sp}(1)$ is the set of pure quaternions. It is well-known that there exists a homomorphism f of $Sp(1)$ onto $SO(3)$ whose kernel is $\{\pm 1\}$ and the induced Lie algebra isomorphism is given by

$$(1.1) \quad f(\mu) = 2 \begin{pmatrix} 0 & -\mu_3 & \mu_2 \\ \mu_3 & 0 & -\mu_1 \\ -\mu_2 & \mu_1 & 0 \end{pmatrix}$$

for $\mu = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 \in \mathfrak{sp}(1)$.

Next, we shall define a quaternion Kähler manifold. Let F_0, G_0 and H_0 be linear transformations represented by the right actions on $H^m = R^{4m}$ by e_1, e_2 and e_3 respectively and V_0 be the linear subspace of linear transformations of R^{4m} spanned by F_0, G_0 and H_0 . Then $SO(3)$ acts effectively on V_0 in such a way that

$$(1.2) \quad s(\mu_1 F_0 + \mu_2 G_0 + \mu_3 H_0) = \dot{\mu}_1 F_0 + \dot{\mu}_2 G_0 + \dot{\mu}_3 H_0; \quad \dot{\mu}_\alpha = \sum_{\beta=1}^3 s_{\alpha\beta} \mu_\beta$$

where $s = (s_{\alpha\beta}) \in SO(3)$. Let M be a connected $4m$ -dimensional Riemannian manifold with metric g . M is called a quaternion Kähler manifold if there is a subbundle V of the tensor bundle of type $(1, 1)$ over M with standard fiber V_0 and structure group $SO(3)$ such that the following conditions (a) and (b) are satisfied (see [6]):

(a) In any coordinate neighborhood U of M , there is a local base $\{F, G, H\}$ of the bundle V , where F, G and H are tensor fields of type $(1, 1)$ in U such that each of F, G and H forms an almost Hermitian structure together with g and they satisfy

$$(1.3) \quad \begin{aligned} F^2 = G^2 = H^2 = -I, \quad FG = -GF = H, \\ GH = -HG = F, \quad HF = -FH = G, \end{aligned}$$

I being the identity tensor field of type $(1, 1)$ in M .

(b) If ϕ is a local cross-section of the bundle V , then $\nabla_X \phi$ is also a local cross-section of V for any vector field X in M , where ∇ denotes the Riemannian connection of M .

Let A and A_0 be the tensor fields of type $(2, 2)$ in M and R^{4m} defined by

$$A = F \otimes F + G \otimes G + H \otimes H, \quad A_0 = F_0 \otimes F_0 + G_0 \otimes G_0 + H_0 \otimes H_0$$

respectively. Then the condition (a) implies that A is globally defined on M and (b) is equivalent to $\nabla A = 0$ (cf. [6]).

The holonomy group of a $4m$ -dimensional quaternion Kähler manifold M is contained in $Sp(m) \cdot Sp(1)$ and hence the structure group $O(4m)$ of the orthogonal frame bundle over M can be reduced to $Sp(m) \cdot Sp(1)$ (cf. [3] and [6]). The bundle space of the reduced bundle \mathfrak{F} with structure group $Sp(m) \cdot Sp(1)$ consists of orthogonal frames with respect to which the components of A coincide with that of A_0 with respect to the natural basis of R^{4m} . It is easily verified that $\mathfrak{F}/Sp(m)$ is a principal bundle over M with structure group $SO(3)$ and that the vector bundle V stated in the definition of quaternion Kähler manifold is the associated vector bundle of $\mathfrak{F}/Sp(m)$ with standard fiber V_0 .

Let M be a quaternion Kähler manifold of dimension $4m$ and $P = \mathfrak{F}/Sp(m)$ be the associated principal bundle of V with structure group $SO(3)$. By means of the condition (b), the Riemannian connection ∇ leaves the bundle V invariant. So, ∇ induces naturally a connection in V and hence induces a connection Γ in P . We now prove (cf. [5]).

PROPOSITION 1.1. *If Z_2 -cohomology groups $H^1(M, Z_2)$ and $H^2(M, Z_2)$ vanish, then there exists a principal bundle \tilde{M} over M with structure group $Sp(1)$ such that $P = \tilde{M}/\{\pm 1\}$.*

PROOF. The exact sequence of groups

$$1 \rightarrow Z_2 \rightarrow Sp(1) \xrightarrow{f} SO(3) \rightarrow 1$$

induces an exact sequence of the cohomology sets of M with coefficients in the corresponding sheaves of germs of differentiable mappings

$$H^1(M, Z_2) \rightarrow H^1(M, Sp(1)) \xrightarrow{f} H^1(M, SO(3)) \rightarrow H^2(M, Z_2).$$

By our assumption, we have $H^1(M, Sp(1)) \approx H^1(M, SO(3))$. On the other hand, $H^1(M, Sp(1))$ and $H^1(M, SO(3))$ can be considered as sets of principal bundles over M with structure groups $Sp(1)$ and $SO(3)$ respectively. Thus there is a principal bundle \tilde{M} over M with structure group $Sp(1)$ such that $P = f(\tilde{M}) = \tilde{M}/\{\pm 1\}$. q.e.d.

If φ denotes the connection form of Γ in P , then we obtain

PROPOSITION 1.2. *Let \tilde{M} be the principal $Sp(1)$ -bundle such that $\tilde{M}/\{\pm 1\} = P$. Then there exists a connection $\tilde{\Gamma}$ in \tilde{M} such that the connection form ω of $\tilde{\Gamma}$ is given by $f^*\varphi = f \cdot \omega$, where f in the left hand side is the bundle map $f: \tilde{M} \rightarrow P$ and f in the right hand side is the*

Lie algebra isomorphism $f: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(3)$.

PROOF. Define ω by $\omega = f^{-1}(f^*\varphi)$. Noting that f is an isomorphism of $\mathfrak{sp}(1)$ onto $\mathfrak{so}(3)$ and $f^{-1} \text{ad}(f(\lambda^{-1})) = \text{ad}(\lambda^{-1})f^{-1}$ for any $\lambda \in Sp(1)$, we see that ω is a connection form in \tilde{M} such that $f^*\varphi = f \cdot \omega$ (for detail, see [11] vol I, p. 82). q.e.d.

Let $S(\sigma)$ be the sectional curvature of a quaternion Kähler manifold M corresponding to a plane section σ . Taking a local base $\{F, G, H\}$ of V , we can set

$$(1.4) \quad \cos^2 \alpha(\sigma) = g(FX, Y)^2 + g(GX, Y)^2 + g(HX, Y)^2, \quad 0 \leq \alpha(\sigma) \leq \frac{\pi}{2}$$

for each plane section σ , where X and Y are orthonormal vectors spanning σ . We can easily show that $\alpha(\sigma)$ is independent of the choice of orthonormal vectors X and Y spanning σ and the choice of a local base $\{F, G, H\}$ of V . Thus we say that the *quaternionic pinching* of M is greater than δ ($\delta > 0$) if there is a positive number K such that

$$(1.5) \quad \delta K < 4S(\sigma)/(1 + 3 \cos^2 \alpha(\sigma)) \leq K$$

for any plane section σ . By normalizing metric, we may set $K = 1$ in (1.5). Here we note that $S(\sigma) = (1 + 3 \cos^2 \alpha(\sigma))/4$ for any σ if M is of constant Q-sectional curvature 1 (see [6]). If the quaternionic pinching of M is greater than δ , then the Riemannian pinching of M is greater than $\delta/4$.

2. Structure equations of the fibering $\pi: \tilde{M} \rightarrow M$. In this section, we shall make use of the following convention on the range of indices:

$$1 \leq \alpha, \beta, \gamma, \varepsilon \leq 3, \quad 4 \leq i, j, k, l \leq 4m + 3.$$

Let M be a $4m$ -dimensional quaternion Kähler manifold and suppose that there exists the principal $Sp(1)$ -bundle \tilde{M} over M with projection π considered in the preceding section. Since the connection form ω and the curvature form Ω of the connection \tilde{F} in \tilde{M} take values in the Lie algebra $\mathfrak{sp}(1)$, they can be written as

$$(2.1) \quad \omega = \sum \omega_\alpha e_\alpha,$$

$$(2.2) \quad \Omega = \sum \Omega_\alpha e_\alpha,$$

where ω_α and Ω_α are 1-forms and 2-forms on \tilde{M} respectively. The structure equation of the connection \tilde{F} is given by

$$(2.3) \quad \Omega_1 = d\omega_1 + 2\omega_2 \wedge \omega_3, \quad \Omega_2 = d\omega_2 + 2\omega_3 \wedge \omega_1, \quad \Omega_3 = d\omega_3 + 2\omega_1 \wedge \omega_2.$$

Now, we define a Riemannian metric \tilde{g} on \tilde{M} by

$$(2.4) \quad \tilde{g}(\tilde{X}, \tilde{Y}) = g(\pi\tilde{X}, \pi\tilde{Y}) + a^2b^2 \sum \omega_\alpha(\tilde{X})\omega_\alpha(\tilde{Y})$$

for any tangent vectors \tilde{X} and \tilde{Y} of \tilde{M} , where a and b are non-zero real numbers which will be fixed in a moment. Then \tilde{g} is a Riemannian metric tensor on \tilde{M} . We shall denote by $\tilde{\nabla}$ the Riemannian connection defined by the metric tensor \tilde{g} and denote by $\langle \tilde{X}, \tilde{Y} \rangle$ (resp. $\langle X, Y \rangle$) the inner product $\tilde{g}(\tilde{X}, \tilde{Y})$ (resp. $g(X, Y)$) of vectors \tilde{X} and \tilde{Y} of \tilde{M} (resp. X and Y of M).

Well, we shall give some properties of fundamental vector fields and basic vector fields. The basic vector field \tilde{X} corresponding to a vector field X in the base manifold M is the unique horizontal vector field such that $\pi\tilde{X} = X$. Let h and v denote the projections of the tangent spaces of \tilde{M} onto the horizontal and vertical subspaces respectively. The following lemma is easily verified (see [14]).

LEMMA 2.1. *If \tilde{X} (resp. \tilde{Y}) is the basic vector field corresponding to a vector field X (resp. Y) of M and μ^* (resp. ν^*) is the fundamental vector field corresponding to an element μ (resp. ν) of $\mathfrak{sp}(1)$, then the following properties hold:*

- (1) $\langle \tilde{X}, \tilde{Y} \rangle = \langle X, Y \rangle \cdot \pi,$
- (2) $h[\tilde{X}, \tilde{Y}]$ is the basic vector field corresponding to $[X, Y],$
- (3) $h\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ is the basic vector field corresponding to $\nabla_X Y,$
- (4) $[\tilde{X}, \mu^*] = 0,$
- (5) $[\mu^*, \nu^*]$ is the fundamental vector field corresponding to $[\mu, \nu].$

To calculate the Riemannian curvature tensor \tilde{R} of \tilde{g} at any fixed point $\tilde{x} \in \tilde{M}$, we shall take a special orthonormal frame field on a neighborhood of \tilde{x} . Let X_α^* be the fundamental vector field corresponding to e_α/ab for each α . Clearly, they are orthonormal vector fields satisfying

$$(2.5) \quad [X_\alpha^*, X_\beta^*] = \sum C_{\alpha\beta\gamma} X_\gamma^*,$$

where $C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma}$ and

$$(2.6) \quad C_{123} = C_{231} = C_{312} = 2/ab, \quad C_{\alpha\beta\gamma} = 0 \text{ otherwise.}$$

Let X_4, \dots, X_{4m+3} be orthonormal vector fields in a neighborhood of $x = \pi(\tilde{x})$ such that $\nabla_{X_i} X_j = 0$ at x for any i and j . The basic vector fields \tilde{X}_i corresponding to X_i are orthonormal vector fields such that $\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j$ is vertical on the fiber passing through \tilde{x} for any i and j . Therefore, we have local orthonormal frame field $\{X_1^*, X_2^*, X_3^*, \tilde{X}_4, \dots, \tilde{X}_{4m+3}\}$ around \tilde{x} .

First, we shall study the Riemannian connection $\tilde{\nu}$.

LEMMA 2.2. *The components of the connection $\tilde{\nu}$ with respect to the orthonormal frame field taken above are given by*

- (1) $\langle \tilde{\nu}_{X_\alpha^*} X_\beta^*, X_\gamma^* \rangle = 1/2 \langle [X_\alpha^*, X_\beta^*], X_\gamma^* \rangle, \langle \tilde{\nu}_{X_\alpha^*} X_\beta^*, \tilde{X}_i \rangle = 0,$
- (2) $\langle \tilde{\nu}_{X_\alpha^*} \tilde{X}_i, X_\beta^* \rangle = 0, \langle \tilde{\nu}_{X_\alpha^*} \tilde{X}_i, \tilde{X}_j \rangle = -1/2 \langle X_\alpha^*, [\tilde{X}_i, \tilde{X}_j] \rangle,$
- (3) $\langle \tilde{\nu}_{\tilde{X}_i} X_\alpha^*, X_\beta^* \rangle = 0, \langle \tilde{\nu}_{\tilde{X}_i} X_\alpha^*, \tilde{X}_j \rangle = -1/2 \langle X_\alpha^*, [\tilde{X}_i, \tilde{X}_j] \rangle,$
- (4) $\langle \tilde{\nu}_{\tilde{X}_i} \tilde{X}_j, X_\alpha^* \rangle = 1/2 \langle X_\alpha^*, [\tilde{X}_i, \tilde{X}_j] \rangle, \langle \tilde{\nu}_{\tilde{X}_i} \tilde{X}_j, \tilde{X}_k \rangle = \langle \nabla_{X_i} X_j, X_k \rangle \cdot \pi$

in a neighborhood of \tilde{x} .

PROOF. Using the standard formula

$$2\langle \tilde{\nu}_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle = \tilde{X} \langle \tilde{Y}, \tilde{Z} \rangle + \tilde{Y} \langle \tilde{Z}, \tilde{X} \rangle - \tilde{Z} \langle \tilde{X}, \tilde{Y} \rangle - \langle \tilde{X}, [\tilde{Y}, \tilde{Z}] \rangle + \langle \tilde{Y}, [\tilde{Z}, \tilde{X}] \rangle + \langle \tilde{Z}, [\tilde{X}, \tilde{Y}] \rangle$$

for any vector fields \tilde{X}, \tilde{Y} and \tilde{Z} , we shall prove this lemma. If we note the Definition (2.4) of \tilde{g} and (5) given in Lemma 2.1, then we can verify the first assertion by using the above formula for $\tilde{X} = X_\alpha^*, \tilde{Y} = X_\beta^*$ and $\tilde{Z} = X_\gamma^*$. The assertions (2), (3) and (4) follow similarly from the above standard formula and Lemma 2.1. q.e.d.

REMARK. By (1) given in Lemma 2.2, we see that each fiber is totally geodesic in the bundle space \tilde{M} .

LEMMA 2.3. *If we set $\Omega_{\alpha ij} = \Omega_\alpha(\tilde{X}_i, \tilde{X}_j)$, then we have*

- (1) $\tilde{\nu}_{X_\alpha^*} X_\beta^* = 1/2 [X_\alpha^*, X_\beta^*],$
- (2) $\tilde{\nu}_{X_\alpha^*} \tilde{X}_i = \tilde{\nu}_{\tilde{X}_i} X_\alpha^* = ab \sum \Omega_{\alpha ij} \tilde{X}_j,$
- (3) $v\tilde{\nu}_{\tilde{X}_i} \tilde{X}_j = 1/2 v[\tilde{X}_i, \tilde{X}_j] = -ab \sum \Omega_{\alpha ij} X_\alpha^*,$

in a neighborhood of \tilde{x} .

PROOF. The first assertion follows immediately from (1) of Lemma 2.2. To prove (2) and (3), we use the structure equation (2.3). By (2), (3) and (4) of Lemma 2.2, it suffices to show

$$ab\Omega_{\alpha ij} = -\frac{1}{2} \langle X_\alpha^*, [\tilde{X}_i, \tilde{X}_j] \rangle,$$

which follows from

$$\Omega_{\alpha ij} = d\omega_\alpha(\tilde{X}_i, \tilde{X}_j) = -\frac{1}{2} \omega_\alpha([\tilde{X}_i, \tilde{X}_j]) = -\frac{1}{2ab} \langle X_\alpha^*, [\tilde{X}_i, \tilde{X}_j] \rangle.$$

q.e.d.

Next, we shall obtain the covariant derivative of Ω_α .

LEMMA 2.4. *If we set $\tilde{\nu}_\alpha \Omega_{\beta ij} = (\tilde{\nu}_{X_\alpha^*} \Omega_\beta)(\tilde{X}_i, \tilde{X}_j)$, then we obtain*

$$(2.7) \quad \tilde{\nu}_\alpha \Omega_{\beta ij} = -ab \sum (\Omega_{\alpha il} \Omega_{\beta lj} - \Omega_{\beta il} \Omega_{\alpha lj}) + \sum C_{\alpha\beta\gamma} \Omega_{\gamma ij}.$$

PROOF. Exterior differentiating (2.3), we have

$$\begin{aligned} \frac{1}{2}d\Omega_1 &= d\omega_2 \wedge \omega_3 - \omega_2 \wedge d\omega_3, & \frac{1}{2}d\Omega_2 &= d\omega_3 \wedge \omega_1 - \omega_3 \wedge d\omega_1, \\ \frac{1}{2}d\Omega_3 &= d\omega_1 \wedge \omega_2 - \omega_1 \wedge d\omega_2, \end{aligned}$$

and using (2.3) once more, we obtain

$$\begin{aligned} \frac{1}{2}d\Omega_1 &= \Omega_2 \wedge \omega_3 - \omega_2 \wedge \Omega_3, & \frac{1}{2}d\Omega_2 &= \Omega_3 \wedge \omega_1 - \omega_3 \wedge \Omega_1, \\ \frac{1}{2}d\Omega_3 &= \Omega_1 \wedge \omega_2 - \omega_1 \wedge \Omega_2. \end{aligned}$$

We shall compare the values of the left hand side with those of the right hand side of equations above for vectors X_α^* , \tilde{X}_i and \tilde{X}_j . For example, the left hand side of the first equation is given by

$$\begin{aligned} \frac{1}{2}d\Omega_1(X_\alpha^*, \tilde{X}_i, \tilde{X}_j) &= \frac{1}{6}X_\alpha^*\Omega_1(\tilde{X}_i, \tilde{X}_j) \\ &= \frac{1}{6}\{(\tilde{\nu}_{X_\alpha^*}\Omega_1)(\tilde{X}_i, \tilde{X}_j) + \Omega_1(\tilde{\nu}_{X_\alpha^*}\tilde{X}_i, \tilde{X}_j) + \Omega_1(\tilde{X}_i, \tilde{\nu}_{X_\alpha^*}\tilde{X}_j)\} \\ &= \frac{1}{6}\{\tilde{\nu}_\alpha\Omega_{1ij} + ab \sum (\Omega_{\alpha il}\Omega_{1lj} - \Omega_{1il}\Omega_{\alpha lj})\}, \end{aligned}$$

where we have used (2) of Lemma 2.3. On the other hand, the right hand side of the first equation is given by

$$\begin{aligned} &(\Omega_2 \wedge \omega_3 - \omega_2 \wedge \Omega_3)(X_\alpha^*, \tilde{X}_i, \tilde{X}_j) \\ &= \frac{1}{3}\{\omega_3(X_\alpha^*)\Omega_2(\tilde{X}_i, \tilde{X}_j) - \omega_2(X_\alpha^*)\Omega_3(\tilde{X}_i, \tilde{X}_j)\} = \frac{1}{6} \sum C_{\alpha l}{}^r \Omega_{r ij}. \end{aligned}$$

Thus we have

$$\tilde{\nu}_\alpha\Omega_{1ij} = -ab \sum (\Omega_{\alpha il}\Omega_{1lj} - \Omega_{1il}\Omega_{\alpha lj}) + \sum C_{\alpha l}{}^r \Omega_{r ij}.$$

q.e.d.

We shall express the components of the curvature tensor \tilde{R} with respect to the orthonormal frame $\{X_1^*, X_2^*, X_3^*, \tilde{X}_4, \dots, \tilde{X}_{4m+3}\}$ in terms of Ω_α and the curvature tensor R of the base manifold M .

PROPOSITION 2.1. *The components of the curvature tensor \tilde{R} with respect to $\{X_1^*, X_2^*, X_3^*, \tilde{X}_4, \dots, \tilde{X}_{4m+3}\}$ are given at \tilde{x} by*

- (1) $\tilde{R}_{\alpha\beta\gamma\epsilon} = (1/a^2b^2)(\delta_{\alpha\epsilon}\delta_{\beta\gamma} - \delta_{\beta\epsilon}\delta_{\alpha\gamma}),$
- (2) $\tilde{R}_{\alpha\beta\gamma j} = 0,$
- (3) $\tilde{R}_{\alpha\beta ij} = -a^2b^2 \sum (\Omega_{\alpha il}\Omega_{\beta lj} - \Omega_{\beta il}\Omega_{\alpha lj}) + ab \sum C_{\alpha\beta}{}^r \Omega_{r ij},$

$$(4) \quad \tilde{R}_{\alpha\beta j} = a^2b^2 \sum \Omega_{\alpha j l} \Omega_{\beta l i} + (ab/2) \sum C_{\alpha\beta}{}^{\gamma} \Omega_{\gamma i j},$$

$$(5) \quad \tilde{R}_{\alpha i j k} = -ab \tilde{\nu}_i \Omega_{\alpha j k},$$

$$(6) \quad \tilde{R}_{i j k l} = R_{i j k l} - a^2b^2 \sum (\Omega_{\alpha i l} \Omega_{\alpha j k} - \Omega_{\alpha i k} \Omega_{\alpha j l} - 2\Omega_{\alpha i j} \Omega_{\alpha k l}),$$

where $\tilde{R}_{\alpha\beta\gamma\epsilon} = \langle \tilde{R}(X_\alpha^*, X_\beta^*) X_\gamma^*, X_\epsilon^* \rangle, \dots, R_{i j k l} = \langle R(X_i, X_j) X_k, X_l \rangle$ and $\tilde{\nu}_i \Omega_{\alpha j k} = (\tilde{\nu}_{\tilde{X}_i} \Omega_\alpha)(\tilde{X}_j, \tilde{X}_k)$. Formulas (1), \dots , (6) determine all components of \tilde{R} .

PROOF. By (1) of Lemma 2.3, we have

$$\begin{aligned} & \tilde{R}(X_\alpha^*, X_\beta^*) X_\gamma^* \\ &= \tilde{\nu}_{X_\alpha^*} \tilde{\nu}_{X_\beta^*} X_\gamma^* - \tilde{\nu}_{X_\beta^*} \tilde{\nu}_{X_\alpha^*} X_\gamma^* - \tilde{\nu}_{[X_\alpha^*, X_\beta^*]} X_\gamma^* \\ &= \frac{1}{4} [X_\alpha^*, [X_\beta^*, X_\gamma^*]] - \frac{1}{4} [X_\beta^*, [X_\alpha^*, X_\gamma^*]] - \frac{1}{2} [[X_\alpha^*, X_\beta^*], X_\gamma^*]. \end{aligned}$$

Using Jacobi's identity, we obtain

$$\tilde{R}(X_\alpha^*, X_\beta^*) X_\gamma^* = -\frac{1}{4} [[X_\alpha^*, X_\beta^*], X_\gamma^*],$$

from which we have

$$\tilde{R}_{\alpha\beta\gamma\epsilon} = \frac{1}{4} \langle [X_\alpha^*, X_\beta^*], [X_\gamma^*, X_\epsilon^*] \rangle = \frac{1}{a^2b^2} (\delta_{\alpha\epsilon} \delta_{\beta\gamma} - \delta_{\beta\epsilon} \delta_{\alpha\gamma}),$$

i.e., the formula (1). Similarly, we obtain (2) by using the above result.

We have, from (2) of Lemma 2.3,

$$\tilde{\nu}_{X_\alpha^*} \tilde{\nu}_{X_\beta^*} \tilde{X}_i = ab \sum (\tilde{\nu}_\alpha \Omega_{\beta i l}) \tilde{X}_l + a^2b^2 \sum \Omega_{\alpha i k} \Omega_{\beta k l} \tilde{X}_l$$

and, from (2.5),

$$\tilde{\nu}_{[X_\alpha^*, X_\beta^*]} \tilde{X}_i = ab \sum C_{\alpha\beta}{}^\gamma \Omega_{\gamma i l} \tilde{X}_l.$$

Therefore, $\tilde{R}_{\alpha\beta i j}$ is given by

$$\tilde{R}_{\alpha\beta i j} = ab(\tilde{\nu}_\alpha \Omega_{\beta i j} - \tilde{\nu}_\beta \Omega_{\alpha i j}) + a^2b^2 \sum (\Omega_{\alpha i l} \Omega_{\beta l j} - \Omega_{\beta i l} \Omega_{\alpha l j}) - ab \sum C_{\alpha\beta}{}^\gamma \Omega_{\gamma i j}.$$

Substituting (2.7) in this equation, we obtain (3). The equation (4) follows similarly from (2) of Lemma 2.3 and the fact that $[X_\alpha^*, \tilde{X}_i] = 0$.

$\tilde{R}_{\alpha i j k}$ is given by

$$\tilde{R}_{\alpha i j k} = \langle \tilde{\nu}_{X_\alpha^*} \tilde{\nu}_{\tilde{X}_i} \tilde{X}_j - \tilde{\nu}_{\tilde{X}_i} \tilde{\nu}_{X_\alpha^*} \tilde{X}_j, \tilde{X}_k \rangle.$$

We consider the right hand side at \tilde{x} where $\tilde{\nu}_{\tilde{X}_i} \tilde{X}_j$ is vertical. Since $\langle \tilde{\nu}_{\tilde{X}_i} \tilde{X}_j, \tilde{X}_k \rangle$ is constant on the fiber passing through \tilde{x} , we have

$$\langle \tilde{\nu}_{X_\alpha^*} \tilde{\nu}_{\tilde{X}_i} \tilde{X}_j, \tilde{X}_k \rangle = X_\alpha^* \langle \tilde{\nu}_{\tilde{X}_i} \tilde{X}_j, \tilde{X}_k \rangle - \langle \tilde{\nu}_{\tilde{X}_i} \tilde{X}_j, \tilde{\nu}_{X_\alpha^*} \tilde{X}_k \rangle = 0.$$

Thus we obtain

$$\tilde{R}_{\alpha i j k} = -\langle \tilde{\nu}_{\tilde{X}_i} \tilde{\nu}_{X_\alpha^*} \tilde{X}_j, \tilde{X}_k \rangle = -ab \tilde{\nu}_i \Omega_{\alpha j k} \text{ at } \tilde{x},$$

which proves the equation (5).

Using (3) of Lemma 2.1 and (3) of Lemma 2.3, we have

$$\langle \tilde{F}_{\tilde{X}_i} \tilde{F}_{\tilde{X}_j} \tilde{X}_k, \tilde{X}_l \rangle = \langle \nabla_{X_i} \nabla_{X_j} X_k, X_l \rangle \cdot \pi - a^2 b^2 \sum \Omega_{ajk} \Omega_{ail},$$

and taking account of (1), (2) and (3) of Lemma 2.1 and (3) of Lemma 2.3, we obtain

$$\langle \tilde{F}_{[\tilde{X}_i, \tilde{X}_j]} \tilde{X}_k, \tilde{X}_l \rangle = -2a^2 b^2 \sum \Omega_{aij} \Omega_{akl} + \langle \nabla_{[X_i, X_j]} X_k, X_l \rangle \cdot \pi.$$

These equations imply (6). q.e.d.

Finally, we shall rewrite the equations obtained in Proposition 2.1 in terms of a local base of V , what are called the structure equations of the fibering $\pi: \tilde{M} \rightarrow M$. Let τ be a local cross-section of \tilde{M} defined on a neighborhood U of x such that $\tau(x) = \tilde{x}$ and the differential map of τ maps the tangent space of M at x onto the horizontal space at \tilde{x} . For each point y of U , $(f \circ \tau)(y) \in P$ can be considered as a linear map from V_0 to the fiber over y of V where f is the bundle map $f: \tilde{M} \rightarrow P$. If we set

$$\begin{aligned} J_1(y) &= ((f \circ \tau)(y))(F_0), & J_2(y) &= ((f \circ \tau)(y))(G_0), \\ J_3(y) &= ((f \circ \tau)(y))(H_0), \end{aligned}$$

then $\{J_1, J_2, J_3\}$ is a local base of V defined on U satisfying the condition (a) stated in §1. Taking account of (1.1) and (1.2), we see that the covariant derivatives of J_α are given by, for any X ,

$$(2.8) \quad \begin{aligned} \nabla_X J_1 &= 2\theta_3(X)J_2 - 2\theta_2(X)J_3 \\ \nabla_X J_2 &= -2\theta_3(X)J_1 + 2\theta_1(X)J_3 \\ \nabla_X J_3 &= 2\theta_2(X)J_1 - 2\theta_1(X)J_2, \end{aligned}$$

where $\theta_\alpha = \tau^* \omega_\alpha$ ($\alpha = 1, 2, 3$). If we set $\Theta_\alpha = \tau^* \Omega_\alpha$ for $\alpha = 1, 2, 3$, then we have from (2.3)

$$(2.9) \quad \Theta_1 = d\theta_1 + 2\theta_2 \wedge \theta_3, \quad \Theta_2 = d\theta_2 + 2\theta_3 \wedge \theta_1, \quad \Theta_3 = d\theta_3 + 2\theta_1 \wedge \theta_2$$

(in detail, see [12]). By Berger [2], we know that a quaternion Kähler manifold is an Eistein manifold. Thus we have

$$(2.10) \quad 4\Theta_\alpha(X, Y) = \frac{-r}{4m(m+2)} \langle J_\alpha X, Y \rangle \quad \text{for each } \alpha,$$

where r is the scalar curvature of M (see [6]).

LEMMA 2.5. *If we set $1/b = r/(16m(m+2))$ and $J_{\alpha jk} = \langle J_\alpha X_j, X_k \rangle$, then we have*

$$(2.11) \quad \Omega_{ajk} = \frac{-1}{b} J_{ajk} \quad \text{on } U$$

and $\tilde{F}_i \Omega_{ajk} = 0$ at \tilde{x} .

PROOF. Noting that $h\tau_*X_j = \tilde{X}_j$ on U for each j , we obtain

$$\begin{aligned} \Omega_\alpha(\tilde{X}_j, \tilde{X}_k)_{\tau(y)} &= \Omega_\alpha(\tau_*X_j, \tau_*X_k)_{\tau(y)} = (\tau^*\Omega_\alpha)(X_j, X_k)_y \\ &= \theta_\alpha(X_j, X_k)_y = -\frac{1}{b}\langle J_\alpha X_j, X_k \rangle_y \end{aligned}$$

for any point y of U . Thus we have

$$\begin{aligned} (\tilde{\nabla}_{\tilde{X}_i}\Omega_\alpha)(\tilde{X}_j, \tilde{X}_k) &= \tilde{X}_i \cdot \Omega_\alpha(\tilde{X}_j, \tilde{X}_k) = -\frac{1}{b}X_i \cdot \langle J_\alpha X_j, X_k \rangle \\ &= -\frac{1}{b}\langle (\nabla_{X_i}J_\alpha)X_j, X_k \rangle \text{ at } x, \end{aligned}$$

because $\nabla_{X_i}X_j = 0$ at x . Since we have, for example,

$$\begin{aligned} \nabla_{X_i}J_1 &= 2\theta_3(X_i)J_2 - 2\theta_2(X_i)J_3 = 2(\tau^*\omega_3)(X_i)J_2 - 2(\tau^*\omega_2)(X_i)J_3 \\ &= 2\omega_3(\tilde{X}_i)J_2 - 2\omega_2(\tilde{X}_i)J_3 = 0 \text{ at } x, \end{aligned}$$

we obtain $\tilde{\nabla}_{\tilde{X}_i}\Omega_{\alpha jk} = 0$ at \tilde{x} .

q.e.d.

In the sequel, we shall set $1/b = r/(16m(m + 2))$. By Lemma 2.5, we can rewrite the equations in Proposition 2.1 in terms of $J_{\alpha ij}$.

PROPOSITION 2.2. *The components of the curvature tensor \tilde{R} are expressed by R_{ijkl} and $J_{\alpha ij}$ at \tilde{x} as follows:*

- (1) $\tilde{R}_{\alpha\beta\gamma\epsilon} = (1/a^2b^2)(\delta_{\alpha\epsilon}\delta_{\beta\gamma} - \delta_{\beta\epsilon}\delta_{\alpha\gamma})$,
- (2) $\tilde{R}_{\alpha\beta\gamma j} = \tilde{R}_{\alpha ijk} = 0$,
- (3) $\tilde{R}_{\alpha\beta ij} = -a^2 \sum (J_{\alpha il}J_{\beta lj} - J_{\beta il}J_{\alpha lj}) - a \sum C_{\alpha\beta}{}^l J_{\gamma ij}$,
- (4) $\tilde{R}_{\alpha i\beta j} = a^2 \sum J_{\alpha jl}J_{\beta li} - a/2 \sum C_{\alpha\beta}{}^l J_{\gamma ij}$,
- (5) $\tilde{R}_{ijkl} = R_{ijkl} - a^2 \sum (J_{\alpha il}J_{\alpha jk} - J_{\alpha ik}J_{\alpha jl} - 2J_{\alpha ij}J_{\alpha kl})$.

3. **Main theorem.** As in the preceding section, we now assume that M is a quaternion Kähler manifold of dimension $4m$ ($m \geq 2$) and that there exists the principal $Sp(1)$ -bundle \tilde{M} explained in §1. By using Proposition 2.2, we shall study the Riemannian pinching of the bundle space \tilde{M} with metric \tilde{g} .

For arbitrary fixed point \tilde{x} of \tilde{M} , by taking local orthonormal frame field $\{X_1^*, X_2^*, X_3^*, \tilde{X}_4, \dots, \tilde{X}_{4m+3}\}$ around \tilde{x} as in §2, we can identify the tangent space of \tilde{M} at \tilde{x} with the Euclidean space R^{4m+3} with usual inner product \langle, \rangle . Let R_{ijkl} be a set of real numbers satisfying the same algebraic conditions as the Riemannian curvature tensor. We assume that indices A, B, C and D run over the range $\{1, \dots, 4m + 3\}$ and let \tilde{R}_{ABCD} be a set of real numbers subject to the same algebraic conditions as the Riemannian curvature tensor and satisfy (1) ~ (5) of Proposition 2.2 where we put $J_1 = F, J_2 = G$ and $J_3 = H$. For each 2-dimen-

sional subspace $\tilde{\sigma}$ of R^{4m+3} , we define $S(\tilde{\sigma})$ by $S(\tilde{\sigma}) = \sum \tilde{R}_{ABCD} X^A Y^B Y^C X^D$ where $\tilde{X} = (X^A)$ and $\tilde{Y} = (Y^A)$ form an orthonormal basis of $\tilde{\sigma}$. Clearly, $S(\tilde{\sigma})$ is independent of the choice of \tilde{X} and \tilde{Y} . Then $S(\tilde{\sigma})$ is given by

$$\begin{aligned} S(\tilde{\sigma}) = & \sum \tilde{R}_{\alpha\beta\gamma\epsilon} X^\alpha Y^\beta Y^\gamma X^\epsilon + \sum \tilde{R}_{\alpha\beta kl} X^\alpha Y^\beta Y^k X^l \\ & + \sum \tilde{R}_{\alpha j\gamma l} X^\alpha Y^j Y^\gamma X^l + \sum \tilde{R}_{\alpha j k\epsilon} X^\alpha Y^j Y^k X^\epsilon \\ & + \sum \tilde{R}_{i\beta\gamma l} X^i Y^\beta Y^\gamma X^l + \sum \tilde{R}_{i\beta k\epsilon} X^i Y^\beta Y^k X^\epsilon \\ & + \sum \tilde{R}_{i j\gamma\epsilon} X^i Y^j Y^\gamma X^\epsilon + \sum \tilde{R}_{i j k l} X^i Y^j Y^k X^l. \end{aligned}$$

If we set $X = (X^i)$ and $Y = (Y^i)$ (resp. $\xi = (X^\alpha)$ and $\eta = (Y^\alpha)$), then they are elements of R^{4m} (resp. R^3) with usual inner product which will be also denoted by \langle, \rangle . We now have

$$\begin{aligned} \sum \tilde{R}_{\alpha\beta\gamma\epsilon} X^\alpha Y^\beta Y^\gamma X^\epsilon &= \frac{1}{a^2 b^2} (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2), \\ \sum \tilde{R}_{\alpha\beta kl} X^\alpha Y^\beta Y^k X^l &= 2\left(\frac{1}{b} - a^2\right)L, \\ \sum \tilde{R}_{\alpha j\gamma l} X^\alpha Y^j Y^\gamma X^l &= -a^2 \langle \xi, \eta \rangle \langle X, Y \rangle + \left(\frac{1}{b} - a^2\right)L, \\ \sum \tilde{R}_{\alpha j k\epsilon} X^\alpha Y^j Y^k X^\epsilon &= a^2 |\xi|^2 |Y|^2, \\ \sum \tilde{R}_{i\beta\gamma l} X^i Y^\beta Y^\gamma X^l &= a^2 |\eta|^2 |X|^2, \\ \sum \tilde{R}_{i\beta k\epsilon} X^i Y^\beta Y^k X^\epsilon &= -a^2 \langle \xi, \eta \rangle \langle X, Y \rangle + \left(\frac{1}{b} - a^2\right)L, \\ \sum \tilde{R}_{i j\gamma\epsilon} X^i Y^j Y^\gamma X^\epsilon &= 2\left(\frac{1}{b} - a^2\right)L, \\ \sum \tilde{R}_{i j k l} X^i Y^j Y^k X^l &= \sum R_{ijkl} X^i Y^j Y^k X^l \\ &\quad - 3a^2 \{ \langle FX, Y \rangle^2 + \langle GX, Y \rangle^2 + \langle HX, Y \rangle^2 \}, \end{aligned}$$

where $|X| = \langle X, X \rangle^{1/2}$, $|Y| = \langle Y, Y \rangle^{1/2}$, $|\xi| = \langle \xi, \xi \rangle^{1/2}$, $|\eta| = \langle \eta, \eta \rangle^{1/2}$ and $L = (\xi^i \eta^2 - \xi^2 \eta^1) \langle HX, Y \rangle + (\xi^2 \eta^3 - \xi^3 \eta^2) \langle FX, Y \rangle + (\xi^3 \eta^1 - \xi^1 \eta^3) \langle GX, Y \rangle$. Therefore we have

PROPOSITION 3.1. *If \tilde{X} and \tilde{Y} are orthonormal vectors which span a 2-dimensional subspace $\tilde{\sigma}$ of R^{4m+3} and X and Y (ξ and η) are $R^{4m}(R^3)$ -components of \tilde{X} and \tilde{Y} , then $S(\tilde{\sigma})$ is given by*

$$\begin{aligned} (3.1) \quad S(\tilde{\sigma}) = & \frac{1}{a^2 b^2} (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) + a^2 (|\xi|^2 |Y|^2 + |\eta|^2 |X|^2 \\ & - 2 \langle \xi, \eta \rangle \langle X, Y \rangle) + 6\left(\frac{1}{b} - a^2\right)L + \sum R_{ijkl} X^i Y^j Y^k X^l \\ & - 3a^2 (\langle FX, Y \rangle^2 + \langle GX, Y \rangle^2 + \langle HX, Y \rangle^2). \end{aligned}$$

Let σ be a 2-dimensional subspace of R^{4m} and let $Z = (Z^i)$ and $W = (W^i)$ form an orthonormal basis for σ . Define $S(\sigma)$ and $\alpha(\sigma)$ by

$$S(\sigma) = \sum R_{ijkl} Z^i W^j W^k Z^l,$$

$$\cos^2 \alpha(\sigma) = \langle FZ, W \rangle^2 + \langle GZ, W \rangle^2 + \langle HZ, W \rangle^2, \quad 0 \leq \alpha(\sigma) \leq \frac{\pi}{2}.$$

Then both of $S(\sigma)$ and $\alpha(\sigma)$ are independent of the choice of Z and W .

LEMMA 3.1. *Let X and Y be elements of R^{4m} . If X and Y are linearly independent and span a 2-dimensional subspace σ of R^{4m} , then we have*

$$(3.2) \quad \langle FX, Y \rangle^2 + \langle GX, Y \rangle^2 + \langle HX, Y \rangle^2 = (|X|^2 |Y|^2 - \langle X, Y \rangle^2) \cos^2 \alpha(\sigma).$$

PROOF. Set

$$Z = X/|X|, \quad W = (|X|^2 Y - \langle X, Y \rangle X) / (|X| (|X|^2 |Y|^2 - \langle X, Y \rangle^2)^{1/2}).$$

Then Z and W form an orthonormal basis for σ and we have

$$\begin{aligned} \langle FX, Y \rangle^2 &= \langle FZ, W \rangle^2 (|X|^2 |Y|^2 - \langle X, Y \rangle^2), \\ \langle GX, Y \rangle^2 &= \langle GZ, W \rangle^2 (|X|^2 |Y|^2 - \langle X, Y \rangle^2), \\ \langle HX, Y \rangle^2 &= \langle HZ, W \rangle^2 (|X|^2 |Y|^2 - \langle X, Y \rangle^2). \end{aligned}$$

Therefore we obtain (3.2).

q.e.d.

Let \tilde{X} and \tilde{Y} be orthonormal vectors which span $\tilde{\sigma}$ and let X and Y (resp. ξ and η) be R^{4m} (resp. R^3)-parts of \tilde{X} and \tilde{Y} respectively. Then we have

$$(3.3) \quad |X|^2 + |\xi|^2 = 1, \quad |Y|^2 + |\eta|^2 = 1, \quad \langle X, Y \rangle + \langle \xi, \eta \rangle = 0,$$

from which,

$$(3.4) \quad |X|^2 |Y|^2 - \langle X, Y \rangle^2 = 1 - |\xi|^2 - |\eta|^2 + |\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2.$$

The following proposition follows immediately from (3.1), (3.2), (3.3) and (3.4):

PROPOSITION 3.2. (1) *If X and Y are linearly dependent, then*

$$(3.5) \quad S(\tilde{\sigma}) = \left(\frac{1}{a^2 b^2} - a^2 \right) (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) + a^2.$$

(2) *If X and Y are linearly independent and span a 2-dimensional subspace σ of R^{4m} , then*

$$(3.6) \quad S(\bar{\sigma}) = \left(\frac{1}{a^2 b^2} - 2a^2\right)(|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) + a^2(|\xi|^2 + |\eta|^2) \\ + 6\left(\frac{1}{b} - a^2\right)L + (|X|^2 |Y|^2 - \langle X, Y \rangle^2)(S(\sigma) \\ - 3a^2 \cos^2 \alpha(\sigma)).$$

Let a be a positive number not greater than $1/2$ and assume that $S(\sigma)$ satisfies the inequality

$$(3.7) \quad 4a^2 \leq 4S(\sigma)/(1 + 3 \cos^2 \alpha(\sigma)) \leq 1$$

for any 2-dimensional subspace σ of R^{4m} .

LEMMA 3.2. *The bound of $1/b$ is given by*

$$(3.8) \quad a^2 \leq \frac{1}{b} \leq \frac{1}{4}$$

where $1/b = r/(16m(m + 2))$ and $r = \sum_{i,j} R_{ijji}$.

PROOF. Let $\{Z_i\}$ be an orthonormal basis of R^{4m} such that $Z_{m+q} = FZ_q$, $Z_{2m+q} = GZ_q$ and $Z_{3m+q} = HZ_q$ for $q = 1, \dots, m$. Then r is given by $r = \sum_{i \neq j} S(\sigma_{ij})$ where σ_{ij} are 2-dimensional subspaces spanned by Z_i and Z_j ($i \neq j$). A straightforward computation shows (3.8). q.e.d.

Lemma 3.3. *The bound of L is given by*

$$(3.9) \quad -|\xi| |\eta| |X| |Y| \leq L \leq |\xi| |\eta| |X| |Y|.$$

PROOF. By Cauchy-Schwarz's inequality, we have

$$L^2 \leq \{(\xi^1 \eta^2 - \xi^2 \eta^1)^2 + (\xi^2 \eta^3 - \xi^3 \eta^2)^2 + (\xi^3 \eta^1 - \xi^1 \eta^3)\} \{\langle FX, Y \rangle^2 \\ + \langle GX, Y \rangle^2 + \langle HX, Y \rangle^2\} = (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) \{\langle FX, Y \rangle^2 \\ + \langle GX, Y \rangle^2 + \langle HX, Y \rangle^2\}.$$

Using (3.2), we obtain

$$L^2 \leq (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2)(|X|^2 |Y|^2 - \langle X, Y \rangle^2) \cos^2 \alpha(\sigma) \\ \leq |\xi|^2 |\eta|^2 |X|^2 |Y|^2. \quad \text{q.e.d.}$$

Using Proposition 3.2, Lemmas 3.2 and 3.3, we can show

PROPOSITION 3.3. *Let a be any real number such that $1/12 \leq a^2 \leq 1/4$ and suppose that (3.7) holds for any 2-dimensional subspace of R^{4m} . Let X and Y be R^{4m} -parts of orthonormal vectors \tilde{X} and \tilde{Y} which span a 2-dimensional subspace $\bar{\sigma}$ of R^{4m+3} . If X and Y are linearly dependent, then*

$$a^2 \leq S(\bar{\sigma}) \leq \frac{1}{16a^2}.$$

If X and Y span a 2-dimensional subspace σ of R^{4m} , then

$$\frac{5}{2}a^2 - \frac{3}{8} \leq S(\bar{\sigma}) \leq 1 - 3a^2,$$

where the following inequalities hold:

$$1 - 3a^2 \geq \frac{1}{16a^2}, \quad a^2 \geq \frac{5}{2}a^2 - \frac{3}{8}.$$

PROOF. We use the following inequalities:

$$\begin{aligned} 1 &\geq |\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 \geq 0, & 1 &\geq |X|^2 |Y|^2 - \langle X, Y \rangle^2 \geq 0, \\ \frac{1}{a^2 b^2} - a^2 &\geq 0 \text{ (Lemma 3.2)}, & |\xi| |\eta| |X| |Y| &\leq \frac{1}{4}. \end{aligned}$$

If X and Y are linearly dependent, then, from (3.5) and Lemma 3.2, we have

$$a^2 \leq S(\bar{\sigma}) = \left(\frac{1}{a^2 b^2} - a^2 \right) (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) + a^2 \leq \frac{1}{a^2 b^2} \leq \frac{1}{16a^2}.$$

If X and Y span a 2-dimensional subspace σ of R^{4m} , then we have

$$\begin{aligned} S(\sigma) - 3a^2 \cos^2 \alpha(\sigma) &\geq a^2, \\ S(\sigma) - 3a^2 \cos^2 \alpha(\sigma) &\leq \frac{1}{4} \{1 + 3(1 - 4a^2) \cos^2 \alpha(\sigma)\} \leq 1 - 3a^2. \end{aligned}$$

Using (3.6), we shall find a lower bound for $S(\bar{\sigma})$. From Lemmas 3.2 and 3.3, we have

$$\begin{aligned} S(\bar{\sigma}) &= \left(\frac{1}{a^2 b^2} - 2a^2 \right) (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) + a^2 (|\xi|^2 + |\eta|^2) + 6 \left(\frac{1}{b} - a^2 \right) L \\ &\quad + (1 - |\xi|^2 - |\eta|^2 + |\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) (S(\sigma) - 3a^2 \cos^2 \alpha(\sigma)) \\ &\geq \left(\frac{1}{a^2 b^2} - 2a^2 \right) (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) + a^2 (|\xi|^2 + |\eta|^2) - \frac{3}{2} \left(\frac{1}{b} - a^2 \right) \\ &\quad + (1 - |\xi|^2 - |\eta|^2 + |\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) a^2 \\ &= \left(\frac{1}{a^2 b^2} - a^2 \right) (|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2) - \frac{3}{2} \left(\frac{1}{b} - a^2 \right) + a^2 \\ &\geq a^2 - \frac{3}{2} \left(\frac{1}{b} - a^2 \right) \\ &\geq \frac{5}{2} a^2 - \frac{3}{8}. \end{aligned}$$

Next, we shall find an upper bound for $S(\bar{\sigma})$. By Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
 S(\tilde{\sigma}) &= \left(\frac{1}{a^2b^2} - 2a^2\right)(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2) + a^2(|\xi|^2 + |\eta|^2) \\
 &\quad + 6\left(\frac{1}{b} - a^2\right)L + (|X|^2|Y|^2 - \langle X, Y \rangle^2)(S(\sigma) - 3a^2 \cos^2 \alpha(\sigma)) \\
 &\leq \left(\frac{1}{a^2b^2} - 2a^2\right)(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2) + a^2(|\xi|^2 + |\eta|^2) \\
 &\quad + 6\left(\frac{1}{b} - a^2\right)|\xi||\eta||X||Y| + (1 - |\xi|^2 - |\eta|^2 + |\xi|^2|\eta|^2 \\
 &\quad - \langle \xi, \eta \rangle^2)(1 - 3a^2) \\
 &= \left(\frac{1}{a^2b^2} - a^2\right)(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2) - (1 - 4a^2)(|\xi|^2 + |\eta|^2 \\
 &\quad - |\xi|^2|\eta|^2 + \langle \xi, \eta \rangle^2) + 6\left(\frac{1}{b} - a^2\right)|\xi||\eta||X||Y| + 1 - 3a^2 \\
 &\leq \left(\frac{1}{a^2b^2} - a^2\right)|\xi|^2|\eta|^2 - (1 - 4a^2)(|\xi|^2 + |\eta|^2 - |\xi|^2|\eta|^2) \\
 &\quad + 6\left(\frac{1}{b} - a^2\right)|\xi||\eta|(1 - |\xi|^2)^{1/2}(1 - |\eta|^2)^{1/2} + 1 - 3a^2.
 \end{aligned}$$

If we set

$$\begin{aligned}
 c_1 &= \frac{1}{a^2b^2} - a^2, & c_2 &= 1 - 4a^2, & c_3 &= \frac{1}{b} - a^2, \\
 t &= |\xi|^2, & s &= |\eta|^2,
 \end{aligned}$$

then we have

$$\begin{aligned}
 0 &\leq t \leq 1, & 0 &\leq s \leq 1, & c_1 &\geq 0, & c_2 &\geq 0, & c_3 &\geq 0, \\
 c_1 - c_2 &\leq 0.
 \end{aligned}$$

The last inequality follows from $1 - 3a^2 \geq 1/16a^2$. By simple calculus, we see that the function

$$h(t, s) = c_1ts - c_2(t + s - ts) + 6c_3\{ts(1 - t)(1 - s)\}^{1/2}$$

attains the maximum value 0 at (0, 0) in the square $\{(t, s) \in \mathbb{R}^2; 0 \leq t, s \leq 1\}$. Therefore we obtain

$$S(\tilde{\sigma}) \leq 1 - 3a^2. \qquad \text{q.e.d.}$$

We must prove the following theorem to state our main Theorem 3.2.

THEOREM 3.1. *Let M be a complete quaternion Kähler manifold of dimension $4m$ ($m \geq 2$) with quaternionic pinching greater than δ . If*

$\delta \geq 9/16$, then there exists a principal $\mathrm{Sp}(1)$ -bundle \tilde{M} over M and a Riemannian metric on \tilde{M} with Riemannian pinching greater than $(5\delta - 3)/(8 - 6\delta)$.

PROOF. By Kraines [13], we see that M has the same integral cohomology ring as the quaternion projective space HP^m . Since $H^1(HP^m, \mathbb{Z}) = 0$, $H^2(HP^m, \mathbb{Z}) = 0$ and $H^3(HP^m, \mathbb{Z}) = 0$, the exact cohomology sequence

$$H^1(M, \mathbb{Z}) \rightarrow H^1(M, \mathbb{Z}_2) \rightarrow H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2) \rightarrow H^3(M, \mathbb{Z})$$

implies $H^1(M, \mathbb{Z}_2) = H^2(M, \mathbb{Z}_2) = 0$. Thus, by Proposition 1.1, we see that there exists a principal $\mathrm{Sp}(1)$ -bundle \tilde{M} over M such that $\tilde{M}/\{\pm 1\} = P$. Setting $4a^2 = \delta$ in Proposition 3.3, we have

$$\frac{5}{8}\delta - \frac{3}{8} < S(\tilde{\sigma}) \leq 1 - \frac{3}{4}\delta. \quad \text{q.e.d.}$$

THEOREM 3.2. Let M be a complete quaternion Kähler manifold of dimension $4m$ ($m \geq 2$) with quaternionic pinching greater than $10/13$. Then $\pi_q(M) = \pi_q(HP^m)$ for all q .

PROOF. Theorem 3.1 implies that there exists a principal $\mathrm{Sp}(1)$ -bundle \tilde{M} over M and a Riemannian metric on \tilde{M} with Riemannian pinching greater than $1/4$. M is simply connected by a theorem of Synge [16]. Using the exact homotopy sequence of the fibering $S^3 \rightarrow \tilde{M} \rightarrow M$, we see that \tilde{M} is also simply connected. Thus, by sphere theorem of Berger and Klingenberg (cf. [4]), \tilde{M} is homeomorphic with a sphere S^{4m+3} . Since the fiber $\pi^{-1}(x) \approx S^3$ is contractible in \tilde{M} to the point $\tilde{x} \in \pi^{-1}(x)$ leaving \tilde{x} fixed, we have (cf. [15])

$$\pi_q(M) \approx \pi_{q-1}(S^3) + \pi_q(S^{4m+3}) \quad (q \geq 2).$$

We have also, from Hopf fibering $S^3 \rightarrow S^{4m+3} \rightarrow HP^m$,

$$\pi_q(HP^m) \approx \pi_{q-1}(S^3) + \pi_q(S^{4m+3}) \quad (q \geq 1).$$

These complete the proof.

q.e.d.

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