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INFINITESIMAL AFFINE TRANSFORMATIONS OF THE TANGENT BUNDLES WITH SASAKI METRIC

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Introduction. Let (M, g) be a Riemannian manifold with positive definite metric tensor g. Let $T(M)$ be its tangent bundle with the natural projection $\pi: T(M) \rightarrow M$. $T(M)$ admits a natural Riemannian metric g^s called the Sasaki metric. S. Sasaki proved in [2] that the extension $\bar{X}$ (or complete lift X^c) of an infinitesimal isometry X on (M, g) is an infinitesimal isometry on $(T(M), g^s)$ and the vertical lift Y^v of a parallel vector field Y on (M, g) is an infinitesimal isometry on $(T(M), g^s)$. In [5] S. Tanno determined the forms of all infinitesimal isometries on $(T(M), g^s)$. In this paper we determine the forms of all infinitesimal affine transformations on $(T(M), g^s)$. The author wishes to express his sincere gratitude to Professor Tanno who suggested this topic and helpful advices.

1. Notations and basic formulas. Let (M, g) be a Riemannian manifold with positive definite metric tensor g. Let $T(M)$ be the tangent bundle of M with the natural projection $\pi: T(M) \rightarrow M$. For a local coordinate neighborhood $U(x^i)$ in M, let $(\pi^{-1}U)(x^i, y^i)$ be the natural coordinate neighborhood, where indices i, j, k etc. run from 1 to $m = \dim M$.

Let $X = (X^i)$ be a vector field on M. Then the complete lift X^c (in Yano-Kobayashi [3], the extension $\bar{X}$ in Sasaki [2]) and the vertical lift X^v are defined by

$$X^c = (X^i, y^r \partial_r X^i) , \tag{1.1}$$

$$X^v = (0, X^i) , \tag{1.2}$$

respectively, where $\partial_r X^i$ denotes $\partial X^i / \partial x^r$.

For (1, 1)-tensor field $C = (C^i_j)$ on M, a vector field ιC on $T(M)$ is defined by

$$\iota C = (0, C^i_r y^r) . \tag{1.3}$$

For a (1,3)-tensor field $T = (T_{ijk}{}^h)$ on M, a (1, 2)-tensor field $\iota T = ((\iota T)^\alpha_{\beta\gamma})$ on $T(M)$ is defined by

$$(1.4)\quad \begin{cases}(\iota T)_{\bar{j}\bar{k}}^{\ i} = (\iota T)_{\bar{j}\bar{k}}^{\ \bar{i}} = (\iota T)_{j\bar{k}}^{\ \bar{i}} = (\iota T)_{\bar{j}k}^{\ \bar{i}} = (\iota T)_{\bar{j}k}^{\ i} = (\iota T)_{j\bar{k}}^{\ i} = (\iota T)_{jk}^{\ i} = 0\ , \\ (\iota T)_{jk}^{\ \bar{i}} = (T_{rjk}{}^{i} - T_{jrk}{}^{i} + T_{jkr}{}^{i})y^r\ ,\end{cases}$$

where the unbarred indices refer to $x^1, \cdots, x^m$ and the barred indices refer to $y^1, \cdots, y^m$.

By ∇ and $R = (R_{ijk}{}^h)$ we denote the Riemannian connection and the Riemannian curvature tensor of g. By $\Gamma_{jk}^{\ i}$ we denote the coefficients of the connection ∇ of M.

If we put

$$\begin{aligned}&\bar{R}_{\bar{j}\bar{k}}^{\ \alpha} = 0,\ \bar{R}_{\bar{j}k}^{\ i} = \bar{R}_{k\bar{j}}^{\ i} = R_{rjk}{}^{i}y^r,\ \bar{R}_{\bar{j}k}^{\ \bar{i}} = \bar{R}_{k\bar{j}}^{\ \bar{i}} = -\Gamma_{sh}^{\ i}R_{rjk}{}^{h}y^ry^s\ ,\\ &\bar{R}_{jk}^{\ i} = (R_{shk}{}^{i}\Gamma_{rj}^{\ h} + R_{shj}{}^{i}\Gamma_{rk}^{\ h})y^ry^s\ ,\\ &\bar{R}_{jk}^{\ \bar{i}} = \Gamma_{th}^{\ i}(R_{lsk}{}^{h}\Gamma_{rj}^{\ l} + R_{lsj}{}^{h}\Gamma_{rk}^{\ l})y^ry^sy^t\ ,\end{aligned}$$

then the $\bar{R} = (\bar{R}_{\beta\gamma}^{\ \alpha})$ is a (1, 2)-tensor field on $T(M)$.

By ∇^c and ∇^s we denote the Riemannian connection defined by the complete metric g^c and the Sasaki metric g^s on $T(M)$ respectively. If we denote by $\bar{\Gamma}_{\beta\gamma}^{\ \alpha}$ the coefficients of the connection ∇^c (see [3], p. 205), then coefficients $\tilde{\Gamma}_{\beta\gamma}^{\ \alpha}$ of the connection ∇^s (see [2], p. 352) are given by

$$(1.5)\qquad \tilde{\Gamma}_{\beta\gamma}^{\ \alpha} = \bar{\Gamma}_{\beta\gamma}^{\ \alpha} - (1/2)(\iota R)_{\beta\gamma}^{\ \alpha} + (1/2)\bar{R}_{\beta\gamma}^{\ \alpha}\ .$$

If we denote by L_X the Lie derivation by X, then we have the following lemmas.

LEMMA 1.1. *Let X be a vector field on M. Then*

$$\begin{aligned}&L_{X^c}\bar{\Gamma}_{\bar{j}\bar{k}}^{\ \alpha} = 0,\ L_{X^c}\bar{\Gamma}_{\bar{j}k}^{\ i} = 0,\ L_{X^c}\bar{\Gamma}_{\bar{j}k}^{\ \bar{i}} = L_X\Gamma_{jk}^{\ i}\ ,\\ &L_{X^c}\bar{\Gamma}_{jk}^{\ i} = L_X\Gamma_{jk}^{\ i},\ L_{X^c}\bar{\Gamma}_{jk}^{\ \bar{i}} = y^r\partial_r(L_X\Gamma_{jk}^{\ i})\ .\end{aligned}$$

LEMMA 1.2. (Yano and Kobayashi [3])

$$L_{X^c}(\iota R) = \iota(L_XR)\ .$$

LEMMA 1.3. *For $\bar{R} = (\bar{R}_{\beta\gamma}^{\ \alpha})$ we have*

$$\begin{aligned}L_{X^c}\bar{R}_{\bar{j}\bar{k}}^{\ \alpha} &= 0,\ L_{X^c}\bar{R}_{\bar{j}k}^{\ i} = y^rL_XR_{rjk}{}^{i}\ ,\\ L_{X^c}\bar{R}_{\bar{j}k}^{\ \bar{i}} &= -(\Gamma_{sh}^{\ i}L_{X^c}\bar{R}_{\bar{j}k}^{\ h} + \bar{R}_{\bar{j}k}^{\ i}L_X\Gamma_{sh}^{\ i})y^s\ ,\\ L_{X^c}\bar{R}_{jk}^{\ i} &= (\Gamma_{sj}^{\ h}L_{X^c}\bar{R}_{\bar{h}k}^{\ i} + \bar{R}_{\bar{h}k}^{\ i}L_X\Gamma_{sj}^{\ h}\\ &\quad + \Gamma_{sk}^{\ h}L_{X^c}\bar{R}_{\bar{h}j}^{\ i} + \bar{R}_{\bar{h}j}^{\ i}L_X\Gamma_{sk}^{\ h})y^s\ ,\\ L_{X^c}\bar{R}_{jk}^{\ \bar{i}} &= -(\Gamma_{th}^{\ i}L_{X^c}\bar{R}_{jk}^{\ h} + \bar{R}_{jk}^{\ h}L_X\Gamma_{th}^{\ i})y^t\ .\end{aligned}$$

A vector field $Z = (Z^i, Z^{\bar{i}}) = (Z^\alpha)$ on $T(M)$ with affine connection ∇^s is an infinitesimal affine transformation if and only if it satisfies

$$(1.6)\qquad L_Z\tilde{\Gamma}_{\beta\gamma}^{\alpha} = \partial_\beta\partial_\gamma Z^\alpha + Z^\lambda\partial_\lambda\tilde{\Gamma}_{\beta\gamma}^{\alpha} + \tilde{\Gamma}_{\lambda\gamma}^{\alpha}\partial_\beta Z^\lambda + \tilde{\Gamma}_{\beta\lambda}^{\alpha}\partial_\gamma Z^\lambda - \tilde{\Gamma}_{\beta\gamma}^{\lambda}\partial_\lambda Z^\alpha = 0 .$$

By (1.5) we have

$$(1.7)\qquad L_Z\tilde{\Gamma}_{\beta\gamma}^{\alpha} = L_Z\bar{\Gamma}_{\beta\gamma}^{\alpha} - (1/2)L_Z(\iota R)_{\beta\gamma}^{\alpha} + (1/2)L_Z\bar{R}_{\beta\gamma}^{\alpha} .$$

Thus we have the following lemma.

LEMMA 1.4. *Let X be a vector field on M. Then the X^c is an infinitesimal affine transformation of $(T(M), g^s)$ if and only if X itself is an infinitesimal affine transformation of (M, g).*

Next we shall determine the infinitesimal affine transformation Z of $T(M)$ which are of the form (1.3). By a straightforward calculation we get the following lemmas.

LEMMA 1.5. *Let $C = (C_j^i)$ be a* (1, 1)*-tensor field on M. Then we have*

$$L_{\iota C}\bar{\Gamma}_{\bar{j}\bar{k}}^{\alpha} = 0,\ L_{\iota C}\bar{\Gamma}_{\bar{j}k}^{i} = 0,\ L_{\iota C}\Gamma_{\bar{j}k}^{\bar{i}} = \nabla_k C_j^i,\ L_{\iota C}\bar{\Gamma}_{jk}^{i} = 0 ,$$
$$L_{\iota C}\bar{\Gamma}_{jk}^{\bar{i}} = (R_{hjk}{}^{i}C_r^h + R_{jrk}{}^{h}C_h^i + \nabla_j\nabla_k C_r^i + \Gamma_{jr}^{h}\nabla_k C_h^i + \Gamma_{kr}^{h}\nabla_j C_h^i)y^r .$$

LEMMA 1.6.

$$L_{\iota C}(\iota R)_{\bar{j}\bar{k}}^{\alpha} = L_{\iota C}(\iota R)_{\bar{j}k}^{i} = L_{\iota C}(\iota R)_{\bar{j}k}^{\bar{i}} = L_{\iota C}(\iota R)_{jk}^{i} = 0 ,$$
$$L_{\iota C}(\iota R)_{jk}^{\bar{i}} = (R_{hjk}{}^{i}C_r^h + R_{jrk}{}^{h}C_h^i + R_{hkj}{}^{i}C_r^h + R_{krj}{}^{h}C_h^i)y^r .$$

LEMMA 1.7.

$$L_{\iota C}\bar{R}_{\bar{j}\bar{k}}^{\alpha} = 0,\ L_{\iota C}\bar{R}_{\bar{j}k}^{i} = (R_{hjk}{}^{i}C_r^h - R_{hrk}{}^{i}C_j^h)y^r ,$$
$$L_{\iota C}\bar{R}_{\bar{j}k}^{\bar{i}} = -(\bar{R}_{\bar{j}k}^{h}\nabla_h C_s^i + \Gamma_{hs}^{i}L_{\iota C}\bar{R}_{\bar{j}k}^{h})y^s ,$$
$$L_{\iota C}\bar{R}_{jk}^{i} = (\bar{R}_{\bar{h}k}^{i}\nabla_j C_s^h + \Gamma_{js}^{h}L_{\iota C}\bar{R}_{\bar{h}k}^{i} + \bar{R}_{\bar{h}j}^{i}\nabla_k C_s^h + \Gamma_{ks}^{h}L_{\iota C}\bar{R}_{\bar{h}j}^{i})y^s ,$$
$$L_{\iota C}\bar{R}_{jk}^{\bar{i}} = -(\bar{R}_{jk}^{h}\nabla_h C_t^i + \Gamma_{ht}^{i}L_{\iota C}\bar{R}_{jk}^{h})y^t .$$

Thus we have the following lemma.

LEMMA 1.8. *Let $C = (C_j^i)$ be a* (1, 1)*-tensor field on M. Then the vector field ιC on $T(M)$ is an infinitesimal affine transformation of $(T(M), g^s)$ if and only if it satisfies*

(i) $\nabla_k C_j^i = 0$, *and*

(ii) $R_{hjk}{}^{i}C_r^h - R_{hrk}{}^{i}C_j^h = 0$.

PROOF. Suppose that ιC be an infinitesimal affine transformation on $(T(M), g^s)$. Putting $\alpha = \bar{i}$, $\beta = \bar{j}$, $\gamma = k$, and $Z = \iota C$ in (1.7), we have

$$\nabla_k C_j^i + (1/2)L_{\iota C}\bar{R}_{\bar{j}k}^{\bar{i}} = 0 .$$

Hence we get (i). Putting $\alpha = i$, $\beta = \bar{j}$, $\gamma = k$, and $Z = \iota C$ in (1.7), we have

$$(R_{hjk}{}^{i}C^h_r + R_{rhk}{}^{i}C^h_j)y^r = 0 .$$

Hence we have (ii).

Conversely, suppose that C satisfy (i) and (ii). Then we have

$$L_{\iota C}\tilde{\Gamma}_{jk}^{\bar{i}} = (1/2)(R_{kjh}{}^{i}C^h_r - R_{kjr}{}^{h}C^i_h)y^r .$$

By (i) and the Ricci's identity we obtain

$$R_{kjh}{}^{i}C^h_r - R_{kjr}{}^{h}C^i_h = 0 .$$

Thus we see that $L_{\iota C}\tilde{\Gamma}_{\beta\gamma}^{\alpha} = 0$. q.e.d.

2. General infinitesimal affine transformation of $T(M)$. Let $K = (K^i_j)$ be a (1, 1)-tensor field on M. Then the vector field $*K$ on $T(M)$ (S. Tanno [4]) is defined by

$$*K = (K^i_r y^r, \; -\Gamma_{hr}^{\;i}K^h_s y^r y^s) . \tag{2.1}$$

For a vector field $Y = (Y^i)$ on M and a (1, 1)-tensor field K on M we put

$$\bar{X}(Y, K) = Y^v + *K . \tag{2.2}$$

First we shall determine the infinitesimal affine transformations Z of $T(M)$ which are of the form (2.2).

By a straightforward calculation, we have the following lemmas.

LEMMA 2.1. (Yano and Kobayashi [3])

$$L_{Y^v}\bar{\Gamma}_{\bar{j}\bar{k}}^{\alpha} = L_{Y^v}\bar{\Gamma}_{\bar{j}k}^{i} = L_{Y^v}\bar{\Gamma}_{\bar{j}k}^{\bar{i}} = L_{Y^v}\bar{\Gamma}_{jk}^{i} = 0 ,$$
$$L_{Y^v}\bar{\Gamma}_{jk}^{\bar{i}} = L_{Y}\Gamma_{jk}^{i} .$$

LEMMA 2.2. (Yano and Kobayashi [3])

$$L_{Y^v}(\iota R)_{\bar{j}\bar{k}}^{\alpha} = L_{Y^v}(\iota R)_{\bar{j}k}^{i} = L_{Y^v}(\iota R)_{\bar{j}k}^{\bar{i}} = L_{Y^v}(\iota R)_{jk}^{i} = 0 ,$$
$$L_{Y^v}(\iota R)_{jk}^{\bar{i}} = (R_{rjk}{}^{i} + R_{rkj}{}^{i})Y^r .$$

LEMMA 2.3.

$$L_{Y^v}\bar{R}_{\bar{j}\bar{k}}^{\alpha} = 0, \; L_{Y^v}\bar{R}_{\bar{j}k}^{i} = R_{hjk}{}^{i}Y^h ,$$
$$L_{Y^v}\bar{R}_{\bar{j}k}^{\bar{i}} = -(\bar{R}_{\bar{j}k}^{h}\nabla_h Y^i + y^r\Gamma_{rh}^{\;i}L_{Y^v}\bar{R}_{\bar{j}k}^{h}) ,$$
$$L_{Y^v}\bar{R}_{jk}^{i} = \bar{R}_{\bar{h}k}^{i}\nabla_j Y^h + \bar{R}_{\bar{h}j}^{i}\nabla_k Y^h + (\Gamma_{rj}^{\;h}L_{Y^v}\bar{R}_{\bar{h}k}^{i} + \Gamma_{rk}^{\;h}L_{Y^v}\bar{R}_{\bar{h}j}^{i})y^r ,$$
$$L_{Y^v}\bar{R}_{jk}^{\bar{i}} = -(\bar{R}_{jk}^{h}\nabla_h Y^i + y^t\Gamma_{th}^{\;i}L_{Y^v}\bar{R}_{jk}^{h}) .$$

LEMMA 2.4.

$$
\begin{aligned}
&L_{*K}\bar\Gamma_{\bar j\bar k}^{\alpha} = 0,\ L_{*K}\bar\Gamma_{\bar j k}^{i} = \nabla_k K_j^i ,\\
&L_{*K}\bar\Gamma_{\bar j k}^{\bar i} = -(R_{khj}{}^{i}K_r^h + R_{krh}{}^{i}K_j^h + \Gamma_{hr}^{i}\nabla_k K_j^h)y^r ,\\
&L_{*K}\bar\Gamma_{jk}^{i} = (R_{hjk}{}^{i}K_r^h + R_{jrk}{}^{h}K_h^i + \nabla_j\nabla_k K_r^i\\
&\qquad + \Gamma_{kr}^{h}\nabla_j K_h^i + \Gamma_{jr}^{h}\nabla_k K_h^i)y^r ,\\
&L_{*K}\bar\Gamma_{jk}^{\bar i} = (R_{rkh}{}^{i}\nabla_j K_s^h + R_{rjh}{}^{i}\nabla_k K_s^h)y^r y^s - y^s\Gamma_{hs}^{i}L_{*K}\bar\Gamma_{jk}^{h}\\
&\qquad + (R_{rkh}{}^{i}K_p^h + R_{hkp}{}^{i}K_r^h)\Gamma_{js}^{p}y^r y^s\\
&\qquad + (R_{rjh}{}^{i}K_p^h + R_{hjp}{}^{i}K_r^h)\Gamma_{ks}^{p}y^r y^s\\
&\qquad + (\nabla_j R_{hkr}{}^{i} + \nabla_h R_{rjk}{}^{i})K_s^h y^r y^s .
\end{aligned}
$$

LEMMA 2.5.

$$
\begin{aligned}
&L_{*K}(\iota R)_{\bar j\bar k}^{\alpha} = 0,\ L_{*K}(\iota R)_{\bar j k}^{i} = 0 ,\\
&L_{*K}(\iota R)_{\bar j k}^{\bar i} = (\iota R)_{hk}^{\bar i}K_j^h,\ L_{*K}(\iota R)_{jk}^{i} = -(\iota R)_{jk}^{\bar h}K_h^i ,\\
&L_{*K}(\iota R)_{jk}^{\bar i} = (\nabla_h R_{sjk}{}^{i} + \nabla_h R_{skj}{}^{i})K_r^h y^r y^s\\
&\qquad + [((\iota R)_{hk}^{\bar i}\Gamma_{jr}^{p} + (\iota R)_{hj}^{\bar i}\Gamma_{kr}^{p} + (\iota R)_{jk}^{\bar p}\Gamma_{hr}^{i})K_p^h\\
&\qquad + (\iota R)_{hk}^{\bar i}\nabla_j K_r^h + (\iota R)_{hj}^{\bar i}\nabla_k K_r^h]y^r .
\end{aligned}
$$

LEMMA 2.6.

$$
\begin{aligned}
&L_{*K}\bar R_{\bar j k}^{i} = (R_{rkh}{}^{i}K_j^h + R_{rjh}{}^{i}K_k^h)y^r,\ L_{*K}\bar R_{\bar j k}^{\bar i} = -y^s\Gamma_{sp}^{i}L_{*K}\bar R_{\bar j k}^{p} ,\\
&L_{*K}\bar R_{\bar j k}^{i} = (K_s^h\nabla_h R_{rjk}{}^{i} - R_{rjk}{}^{h}\nabla_h K_s^i + R_{rjh}{}^{i}\nabla_k K_s^h)y^r y^s\\
&\qquad + y^s\Gamma_{sk}^{h}L_{*K}\bar R_{\bar h j}^{i} ,\\
&L_{*K}\bar R_{\bar j k}^{\bar i} = R_{rjk}{}^{p}R_{phs}{}^{i}K_t^h y^r y^s y^t - y^t\Gamma_{th}^{i}L_{*K}\bar R_{\bar j k}^{h} ,\\
&L_{*K}\bar R_{jk}^{i} = [(\Gamma_{rk}^{p}\nabla_j K_t^h + \Gamma_{rj}^{p}\nabla_k K_t^h + \Gamma_{rj}^{p}\Gamma_{kt}^{l}K_l^h\\
&\qquad + \Gamma_{rj}^{l}\Gamma_{kt}^{p}K_l^h)R_{sph}{}^{i} + K_t^h(R_{spk}{}^{i}R_{hjr}{}^{p} + R_{spj}{}^{i}R_{hkr}{}^{p}\\
&\qquad + \Gamma_{rj}^{p}\nabla_h R_{spk}{}^{i} + \Gamma_{rk}^{p}\nabla_h R_{spj}{}^{i}) - (R_{spk}{}^{h}\Gamma_{rj}^{p}\\
&\qquad + R_{spj}{}^{h}\Gamma_{rk}^{p})\nabla_h K_t^i]y^r y^s y^t ,\\
&L_{*K}\bar R_{jk}^{\bar i} = (\Gamma_{tl}^{i}\bar R_{jk}^{h}\nabla_h K_u^l - R_{hpt}{}^{i}\bar R_{jk}^{p}K_u^h)y^t y^u - y^u\Gamma_{uh}^{i}L_{*K}\bar R_{jk}^{h} .
\end{aligned}
$$

Thus we have the following lemma.

LEMMA 2.7. *Let $\bar X(Y, K)$ be an infinitesimal affine transformation of $(T(M), g^s)$. Then*

(i) $L_Y\Gamma_{jk}^{i} + (1/2)(R_{khj}{}^{i} + R_{jhk}{}^{i})Y^h = 0,$

(ii) $\nabla_k K_j^i + (1/2)R_{hjk}{}^{i}Y^h = 0,$

(iii) $R_{rjk}{}^{h}\nabla_h Y^i + R_{hkr}{}^{i}K_j^h = 0,$

(iv) $R_{rhk}{}^{i}\nabla_j Y^h + R_{rhj}{}^{i}\nabla_k Y^h + 2\nabla_j\nabla_k K_r^i + 2R_{hjk}{}^{i}K_r^h - R_{kjr}{}^{h}K_h^i = 0,$

(v) $K_s^h\nabla_h R_{rjk}{}^{i} + R_{rjh}{}^{i}\nabla_k K_s^h - R_{rjk}{}^{h}\nabla_h K_s^i$ *is skew-symmetric in r and s.*

Conversely, if Y and K satisfy (i) $\sim$ (v), *then the vector field $\bar X(Y, K)$ defined by* (2.2) *is an infinitesimal affine transformation of $(T(M), g^s)$.*

PROOF. Putting $\alpha = i, \beta = \bar{j}, \gamma = \bar{k}$ and $Z = \bar{X}(Y, K)$ in (1.7) we have

$$R_{rkh}{}^{i}K_{j}^{h} + R_{rjh}{}^{i}K_{k}^{h} = 0 \,. \tag{2.3}$$

Putting $\alpha = i, \beta = \bar{j}, \gamma = k$ and $Z = \bar{X}(Y, K)$ in (1.7), we have

$$\begin{aligned}(\nabla_k K_j^i + (1/2)R_{hjk}{}^{i}Y^h) + (1/2)(K_s^h \nabla_h R_{rjk}{}^{i} \\ + R_{rjh}{}^{i}\nabla_k K_s^h - R_{rjk}{}^{h}\nabla_h K_s^i)y^r y^s = 0 \,,\end{aligned} \tag{2.4}$$

where we have used (2.3). By (2.4) we have (ii) and (v).

Putting $\alpha = \bar{i}, \beta = \bar{j}, \gamma = k$, and $Z = \bar{X}(Y, K)$ in (1.7), we have

$$\begin{aligned}&-(R_{khj}{}^{i}K_r^h + R_{krh}{}^{i}K_j^h + \Gamma_{h\,r}^{\,i}\nabla_k K_j^h)y^r + (1/2)L_{*K}\bar{R}_{\bar{j}k}^{\bar{i}} \\ &-(1/2)(R_{rjk}{}^{h}\nabla_h Y^i + \Gamma_{r\,p}^{\,i}R_{hjk}{}^{p}Y^h)y^r \\ &-(1/2)(R_{rhk}{}^{i} + R_{rkh}{}^{i})K_j^h y^r = 0 \,.\end{aligned}$$

Using (2.3) and (ii) this is written as

$$-(R_{rjk}{}^{h}\nabla_h Y^i + R_{hkr}{}^{i}K_j^h)y^r + L_{*K}\bar{R}_{\bar{j}k}^{\bar{i}} = 0 \,. \tag{2.5}$$

Then we obtain (iii). Putting $\alpha = i, \beta = j, \gamma = k$ and $Z = \bar{X}(Y, K)$ in (1.7) and using (ii), we have

$$\begin{aligned}(R_{rhk}{}^{i}\nabla_j Y^h + R_{rhj}{}^{i}\nabla_k Y^h + 2\nabla_j\nabla_k K_r^i \\ + 2R_{hjk}{}^{i}K_r^h - R_{kjr}{}^{h}K_h^i)y^r + L_{*K}\bar{R}_{jk}^{i} = 0 \,.\end{aligned}$$

Hence we have (iv). Putting $\alpha = \bar{i}, \beta = j, \gamma = k$ and $Z = \bar{X}(Y, K)$ in (1.7), we have

$$\begin{aligned}L_Y\Gamma_{jk}^{i} - (1/2)(R_{hjk}{}^{i} + R_{hkj}{}^{i})Y^h + (1/2)L_{*K}\bar{R}_{jk}^{\bar{i}} \\ + L_{*K}\bar{\Gamma}_{jk}^{\bar{i}} - (1/2)L_{*K}(\iota R)_{jk}^{\bar{i}} + (1/2)L_{Y^v}\bar{R}_{jk}^{\bar{i}} = 0 \,.\end{aligned}$$

Using (2.3), (ii), (iii) and (iv) this is written as

$$L_Y\Gamma_{jk}^{i} - (1/2)(R_{hjk}{}^{i} + R_{hkj}{}^{i})Y^h + (1/2)L_{*K}\bar{R}_{jk}^{\bar{i}} = 0 \,.$$

Hence we get (i).

Conversely, suppose that Y and K satisfy (i) $\sim$ (v). From (iii) we obtain (2.3). By the preceding argument we see that $L_{\bar{X}(Y,K)}\tilde{\Gamma}_{\beta\gamma}^{\alpha} = 0$. Thus we have completed the proof.

Next we shall determine the forms of all infinitesimal affine transformations on $(T(M), g^s)$.

Let $Z = (Z^h, Z^{\bar{h}}) = (Z^\alpha)$ be an infinitesimal affine transformation on $(T(M), g^s)$.

By the Taylor's theorem we have

(2.6) $$Z^h(x^u, y^v) = Z^h(x^u, 0) + \partial_{\bar{r}} Z^h(x^u, 0)y^r + (1/2)\partial_{\bar{r}}\partial_{\bar{s}} Z^h(x^u, 0)y^r y^s + (1/6)\partial_{\bar{r}}\partial_{\bar{s}}\partial_{\bar{t}} Z^h(x^u, 0)y^r y^s y^t + (*)^h ,$$

(2.7) $$Z^h(x^u, y^v) = Z^{\bar{h}}(x^u, 0) + \partial_{\bar{r}} Z^{\bar{h}}(x^u, 0)y^r + (1/2)\partial_{\bar{r}}\partial_{\bar{s}} Z^{\bar{h}}(x^u, 0)y^r y^s + (1/6)\partial_{\bar{r}}\partial_{\bar{s}}\partial_{\bar{t}} Z^{\bar{h}}(x^u, 0)y^r y^s y^t + (*)^{\bar{h}} ,$$

for $(x, y) = (x^u, y^v)$ in the neighborhood of the 0 section, where $(*)^\lambda$ is of the form

$$(*)^\lambda = (1/24)\partial_{\bar{r}}\partial_{\bar{s}}\partial_{\bar{t}}\partial_{\bar{p}} Z^\lambda(x^u, \theta(x, y)y^v)y^r y^s y^t y^p .$$

Then we have the following lemma.

LEMMA 2.8. (S. Tanno [5])

$$X = (X^h) = (Z^h(x, 0)),\ Y = (Y^h) = (Z^{\bar{h}}(x, 0)) ,$$
$$K = (K_r^h) = (\partial_{\bar{r}} Z^h(x, 0)),\ E = (E_{rs}^{\ h}) = (\partial_{\bar{r}}\partial_{\bar{s}} Z^h(x, 0)) ,$$
$$F = (F_{rst}{}^h) = (\partial_{\bar{r}}\partial_{\bar{s}}\partial_{\bar{t}} Z^h(x, 0))$$

are tensor fields on M. Furthermore, if $Z^h(x, 0) = 0$, *then*

$$P = (P_r^h) = (\partial_{\bar{r}} Z^{\bar{h}}(x, 0))$$

is a tensor field on M.

Putting $\alpha = i$, $\beta = j$ and $\gamma = k$ in (1.6), we have

(2.8) $$\partial_j\partial_k Z^i + Z^\lambda\partial_\lambda\tilde{\Gamma}_{jk}^{\ i} + \tilde{\Gamma}_{\lambda k}^{\ i}\partial_j Z^\lambda + \tilde{\Gamma}_{j\lambda}^{\ i}\partial_k Z^\lambda - \tilde{\Gamma}_{jk}^{\ \lambda}\partial_\lambda Z^i = 0 .$$

Substituting (2.6) and (2.7) into (2.8) and taking the part which does not contain y^r, we have

$$\partial_j\partial_k X^i + X^h\partial_h\Gamma_{jk}^{\ i} + \Gamma_{hk}^{\ i}\partial_j X^h + \Gamma_{jh}^{\ i}\partial_k X^h - \Gamma_{jk}^{\ h}\partial_h X^i = 0 .$$

Hence $X = (X^i)$ is an infinitesimal affine transformation on (M, g). By Lemma 1.4, $Z - X^c$ is also infinitesimal affine transformation on $(T(M), g^s)$. Hence we may assume that

(2.9) $$Z^h = K_x^h y^r + (1/2)E_{rs}^{\ h} y^r y^s + (1/6)F_{rst}{}^h y^r y^s y^t + (*)^h ,$$

(2.10) $$Z^{\bar{h}} = Y^h + P_r^h y^r + (1/2)Q_{rs}^{\ h} y^r y^s + (1/6)S_{rst}{}^h y^r y^s y^t + (*)^{\bar{h}} ,$$

where $P = (P_r^h)$ is a tensor field on M and

$$Q_{rs}^{\ h} = \partial_{\bar{r}}\partial_{\bar{s}} Z^{\bar{h}}(x, 0),\ S_{rst}{}^h = \partial_{\bar{r}}\partial_{\bar{s}}\partial_{\bar{t}} Z^{\bar{h}}(x, 0) .$$

Puttting $\alpha = i$, $\beta = \bar{j}$ and $\gamma = \bar{k}$ in (1.6) we have

(2.11) $$E_{jk}^{\ i} + (1/2)(R_{rkh}{}^i K_j^h + R_{rjh}{}^i K_k^h + 2F_{rjk}{}^i)y^r + (---)y^r y^s = 0 .$$

Hence we see that

(2.12) $$E_{jk}^{\ i} = 0 .$$

Putting $\alpha = i$, $\beta = \bar{j}$, and $\gamma = k$ in (1.6) and using (2.12), we have

$$\nabla_k K_j^i + (1/2)R_{hjk}{}^i Y^h + (1/2)(R_{rhk}{}^i P_j^h + R_{hjk}{}^i P_r^h)y^r + (---)y^r y^s = 0 . \tag{2.13}$$

Hence we have

$$\nabla_k K_j^i + (1/2)R_{hjk}{}^i Y^h = 0 , \tag{2.14}$$

$$R_{rhk}{}^i P_j^h + R_{hjk}{}^i P_r^h = 0 . \tag{2.15}$$

Putting $\alpha = \bar{i}$, $\beta = \bar{j}$ and $\gamma = k$ in (1.6) and taking the part which does not contain y^r, we have

$$\nabla_k P_j^i = 0 . \tag{2.16}$$

Hence ιP is an infinitesimal affine transformation of $(T(M), g^s)$ by virtue of the Lemma 1.8. By denoting $Z - \iota P$ again by Z, we may assume that

$$Z^{\bar{h}} = Y^h + (1/2)Q_{rs}^h y^r y^s + (1/6)S_{rst}{}^h y^r y^s y^t + (*)^{\bar{h}} .$$

Putting $\alpha = \bar{i}$, $\beta = \bar{j}$ and $\gamma = \bar{k}$ in (1.6) we have

$$Q_{jk}^i + \Gamma_{hk}^i K_j^h + \Gamma_{hj}^i K_k^h + S_{rjk}{}^i y^r + (---)y^r y^s = 0 ,$$

where we have used (2.12). Hence we have

$$Z^{\bar{h}} = Y^h - \Gamma_{lr}^h K_s^l y^r y^s + (*)^{\bar{h}} . \tag{2.17}$$

Putting $\alpha = \bar{i}$, $\beta = \bar{j}$, and $\gamma = k$ in (1.6) and using (2.14), we have

$$-(1/2)(R_{rjk}{}^h \nabla_h Y^i + 2R_{khj}{}^i K_r^h + R_{khr}{}^i K_j^h)y^r + (---)y^r y^s = 0 .$$

Hence we get

$$R_{rjk}{}^h \nabla_h Y^i + 2R_{khj}{}^i K_r^h + R_{khr}{}^i K_j^h = 0 , \tag{2.18}$$

$$R_{jrk}{}^h \nabla_h Y^i + 2R_{khr}{}^i K_j^h + R_{khj}{}^i K_r^h = 0 . \tag{2.18$'$}$$

Forming (2.18) + (2.18′) we have

$$R_{khr}{}^i K_j^h + R_{khj}{}^i K_r^h = 0 . \tag{2.19}$$

This is equivalent to (2.3). By (2.18) and (2.19) we have

$$R_{rjk}{}^h \nabla_h Y^i + R_{hkr}{}^i K_j^h = 0 . \tag{2.20}$$

By (2.11) and (2.19) we have

$$Z^h = K_r^h y^r + (*)^h . \tag{2.21}$$

Studying the case ($\alpha = \bar{i}$, $\beta = j$, $\gamma = k$), ($\alpha = i$, $\beta = j$, $\gamma = k$) and ($\alpha = i$, $\beta = \bar{j}$, $\gamma = k$) we have (i), (iv) and (v) of Lemma 2.7. Therefore, $\bar{X}(Y,$

K) is an infinitesimal affine transformation on $(T(M), g^s)$.

If we put $\bar{Z} = Z - \bar{X}(Y, K) = (\bar{Z}^\alpha)$, then we have

$$(\bar{Z}^\alpha)_p = (\partial_\beta \bar{Z}^\alpha)_p = 0$$

for $\alpha = i, \bar{i}$ and $\beta = j, \bar{j}$ at a point $p = (x_0, 0)$. Since an infinitesimal affine transformation is determined by the value of its components and their first partial derivatives at a point (cf. Kobayashi and Nomizu [1], p. 232), we have $\bar{Z} = 0$ on $T(M)$.

Thus we have the following theorem.

THEOREM. *Let* $(T(M), g^s)$ *be the tangent bundle with the Sasaki metric of a Riemannian manifold* (M, g). *Let*

(a) $X = (X^i)$ *be an infinitesimal affine transformation of* (M, g),

(b) $C = (C^i_j)$ *be a* (1, 1)*-tensor field on* M *satisfying* (i), (ii) *of Lemma* 1.8,

(c) $Y = (Y^i)$ *and* $K = (K^i_j)$ *be tensor fields on* M *satisfying* (i) $\sim$ (v) *of Lemma* 2.7.

Then the vector field Z *on* $T(M)$ *defined by*

$$Z = X^c + \iota C + \bar{X}(Y, K) \tag{2.22}$$

is an infinitesimal affine transformation on $(T(M), g^s)$.

Conversely, every infinitesimal affine transformation Z *on* $(T(M), g^s)$ *is of the form* (2.22).

REFERENCES

[1] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Interscience Tracts No. 15, 1963.

[2] S. SASAKI, On the geometry of tangent bundles of Riemannian manifolds, Tōhoku Math. J., 10 (1958), 338-354.

[3] K. YANO AND S. KOBAYASHI, Prolongation of tensor fields and connections to tangent bundles I, J. Math. Soc. Japan, 18 (1966), 194-210.

[4] S. TANNO, Infinitesimal isometries on the tangent bundles with complete lift metric, to appear.

[5] ———, Infinitesimal isometries on the tangent bundles with Sasaki metric, to appear.

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