Tôhoku Math. Journ.
26 (1974), 333-339.

# ON THE EMBEDDING AS A DOUBLE COMMUTATOR IN A TYPE 1 $AW^*$-ALGEBRA II

KAZUYUKI SAITÔ*)

(Received July 5, 1973)

One of the interesting problems in operator algebras is to find the essential condition for a $C^*$-algebra to have a faithful von Neumann algebra representation. In 1951, I. Kaplansky [4] introduced a class of $C^*$-algebras called $AW^*$-algebras to separate the discussion of the internal structure of a von Neumann algebra from the action of its elements on a Hilbert space and showed that much of the "non spatial theory" of von Neumann algebras can be extended to $AW^*$-algebras; in particular, the lattice structure of the set of projections, type classifications for algebras, etc. can be carried over to $AW^*$-algebras. However, as J. Dixmier [2] showed, this class of $AW^*$-algebras is exactly broader than that of von Neumann algebras.

In 1956, J. Feldman [3] showed that a finite $AW^*$-algebra with a separating set of states which are completely additive on projections can be represented faithfully as a von Neumann algebra and conjectured that the theorem is true without the assumption of finiteness. In 1970, the author [8] proved the same theorem for semi-finite case using the Segal's non-commutative integration theory, although its proof is rather long. Recently, G. K. Pedersen showed that any $C^*$-algebra with weakly closed maximal abelian *-subalgebra is necessarily a von Neumann algebra and as a consequence, he solved the Feldman's conjecture for the general case in the affirmative.

In this paper, the author will prove the following theorem which is the module setting of the Pedersen's idea on one side and is the complete solution of problem of embedding as a double commutator in a type 1 $AW^*$-algebra ([9, 10]) on the other side.

THEOREM 1. *Let $B$ be an $AW^*$-algebra of type 1 with center $Z$ and let $A$ be an $AW^*$-subalgebra (the definition below) of $B$ which contains $Z$, then the double commutator $A''$ in $B$ (that is $A'' = \{x, x \in B, xy = yx$ for all $y \in A\}$) is, exactly, $A$.*

Main tools in this proof are H. Widom's double commutation theorem

*) The author is partially supported by The Sakkokai Foundation.

(Theorem A) and a Stone-algebra-valued measure (the measure taking values in an abelian $AW^*$-algebra) studied by J. D. M. Wright ([11], [12], Theorem B).

In the course of the proof of Theorem 1, we need the following definition and lemmas.

DEFINITION. A $*$-subalgebra $A$ of an $AW^*$-algebra $B$ is an $AW^*$-subalgebra if $A$ is itself an $AW^*$-algebra and for each *orthogonal* set of projections in $A$, the supremum $\sum_\alpha e_\alpha$ of $\{e_\alpha\}$ *calculated in* $B$ is contained in $A$.

Widom's "double commutation theorem" is the following:

THEOREM A ([10]). *Let $\mathfrak{B}(\mathfrak{M})$ be the type 1 $AW^*$-algebra of all bounded module endomorphisms of a faithful $AW^*$-module $\mathfrak{M}$ over an abelian $AW^*$-algebra $Z$ ([6], Theorem 7) and let $A$ be an $AW^*$-subalgebra of $\mathfrak{B}(\mathfrak{M})$ which contains $Z$ (note that we can recognize $Z$ as the center of $\mathfrak{B}(\mathfrak{M})$). Let $a \in A''$, $a = a^*$, then there is a bounded net $\{a_\lambda\}$ of self-adjoint elements of $A$ such that $a_\lambda \to a$ strongly (that is, $(a_\lambda\xi - a\xi, a_\lambda\xi - a\xi) \to 0(O)$) (order convergence in $Z$ ([10]) where $(\ ,\ )$ is the $Z$-valued inner product in $\mathfrak{M}$).*

A Stone-algebra-valued measure $m$ on the Borel sets of a compact space $X$ is *quasi-regular* if $m(K) = \inf\{m(v); v$ is open, $K \subset v\}$ for each closed subset $K$ of $X$ ([11]).

J. D. M. Wright proved the following spectral theorem ([11, 12]).

THEOREM B ([12]). *Let $\mathfrak{M}$ be a faithful $AW^*$-module over an abelian $AW^*$-algebra $Z$, and let $\mathfrak{B}(\mathfrak{M})$ be the $AW^*$-algebra of all bounded module endomorphisms of $\mathfrak{M}$.*

(i) *Then for any normal element $t$ of $\mathfrak{B}(\mathfrak{M})$ $(t^*t = tt^*)$, we can find a unique projection-valued $(\in \mathfrak{B}(\mathfrak{M}))$ quasi-regular measure $m$ on the Borel sets of the spectrum $\sigma(t)$ of $t$ such that $\int_{\sigma(t)} \lambda dm = t$.*

(ii) *For any $\xi$ in $\mathfrak{M}$, define $\mu$ on the Borel sets $E$ of $\sigma(t)$ by $\mu(E) = (m(E)\xi, \xi)$, then $\mu$ is a positive $Z$-valued quasi-regular Borel measure and*

$$\left(\left(\int_{\sigma(t)} f dm\right)\xi, \xi\right) = \int_{\sigma(t)} f d\mu$$

*for any $f \in B^\infty(\sigma(t))$ where $B^\infty(\sigma(t))$ is the space of bounded real-valued Borel functions on $\sigma(t)$.*

(iii) *The mapping $f \in B^\infty(\sigma(t)) \to \int f dm$ is a $*$-homomorphism of $B^\infty(\sigma(t))$ into $\{t, 1\}''$ (double commutator algebra generated by $t$ and 1 in $\mathfrak{B}(\mathfrak{M})$).*

Now we start with the follwing lemma which plays an essential role in the theory of $AW^*$-subalgebra.

LEMMA 1. *Let $N$ be an $AW^*$-algebra and $M$ an $AW^*$-subalgebra of $N$, then $M$ is a Baer*-subring of $N$ in the sense that for any increasing net $\{q_\alpha\}$ of projections in $M$ (not necessarily commuting each other), the supremum $q$ of $\{q_\alpha\}$ (as computed in $N$) is contained in $M$.*

PROOF. Let $q$ be the supremum projection of $\{q_\alpha\}$ calculated in $N$, and $q_0$ the one computed in $M$, then $q \leqq q_0$. Suppose $q_0 - q \neq 0$. Let $\{e_\beta\}$ be the maximal family of orthogonal projections in $M$ such that $(q_0 - q)e_\beta = 0$ for each $\beta$, and let $e_0 = \sum e_\beta$ (in $N$) then by the above definition it follows that $e_0 \in M$ and by the continuity property ([4, Lemma 2.2]), $(q_0 - q)e_0 = 0$. If $(q_0 - q)e = 0$ for some projection $e$ in $M$, then noting that $e \vee e_0 = LP(e + e_0)$ (the left projection in $N$ ([4])) is contained in $M$ by [5, Lemma 2], it follows that $(q_0 - q)(e \vee e_0) = 0$. Hence the maximality of $\{e_\beta\}$ implies that $e \vee e_0 - e_0 = 0$, that is, $e \leqq e_0$. Thus $(q_0 - q)q_\alpha = 0$ for each $\alpha$ shows that $q_\alpha \leqq e_0$ for each $\alpha$ and hence $q_0 \leqq e_0$, which implies that $q_0 - q = (q_0 - q)q_0 \leqq (q_0 - q)e_0(q_0 - q) = 0$. This is a contradiction and $q = q_0$. This completes the proof.

REMARK. By a private letter, Pedersen communicated to the author that he also proved the same result.

LEMMA 2. *Let $\mathfrak{M}$ be a faithful $AW^*$-module over an abelian $AW^*$-algebra $Z$ and let $\mathfrak{B}(\mathfrak{M})$ be the $AW^*$-algebra of all bounded module endomorphisms of $\mathfrak{M}$. Suppose $M$ is an $AW^*$-subalgebra of $\mathfrak{B}(\mathfrak{M})$ whose center coincides with that of $\mathfrak{B}(\mathfrak{M})$ (which is isomorphic with $Z$). Then for any non-negative element $a$ in $M$, $(a + (1/n)1)^{-1}a \uparrow LP(a)$ weakly, that is, $((a + 1(1/n)1)^{-1}a\xi, \eta) \rightarrow (LP(a)\xi, \eta)(O)$ for each $\xi$, $\eta \in \mathfrak{M}$ where $LP(a)$ is the left projection of $a$ in $M$.*

PROOF. Let $A$ be a maximal abelian $AW^*$-subalgebra of $M$ which contains $a$. Observe that $(a + (1/n)1)^{-1}a$ is increasing for $n$ and $(a + (1/n)1)^{-1}a \leqq 1$ for each $n$, by the completeness property ([10, Lemma 1.4]), there exists $f$ in $A''$ (double commutator of $A$ in $\mathfrak{B}(\mathfrak{M})$) such that $((a + (1/n)1)^{-1}a\xi, \xi) \uparrow (f\xi, \xi)(O)$. Since by [10, Theorem 4.1] (the type 1 case of Theorem 1), $A'' = A$, we have $f \in A \subset M$. Next, we shall show that $f = LP(a)$. In fact, by Theorem B and the monotone convergence theorem ([11, Proposition 3.3]),

$$((a + (1/n)1)^{-1}a\xi, \eta) \rightarrow \int_{\sigma(a)} \chi_{\sigma(a)-\{0\}}(\lambda)d(e_\lambda\xi, \eta)(O)$$

where $a = \int_{\sigma(a)} \lambda de_\lambda$ is the spectral decomposition for $a$ in $M$ and $\chi_{\sigma(a)-\{0\}}$

is the characteristic function of the set $\sigma(a) - \{0\}$, which implies that $f = \int_{\sigma(a)} \chi_{\sigma(a)-\{0\}}(\lambda) de_\lambda$. By Theorem B, $f$ is a projection and $af = a$. Hence $f \geqq LP(a)$. On the other hand, $(a + (1/n)1)^{-1} aLP(a) = (a + (1/n)1)^{-1} a \to f$ (strongly) and $(a + (1/n)1)^{-1} aLP(a) \to fLP(a)$ strongly, which implies that $f = fLP(a)$, that is, $f \leqq LP(a)$ and $f = LP(a)$ follows. This completes the proof.

In order to prove our Theorem 1, we need the following key lemma. Although it is a slight modification of [7, Lemma p. 172], main idea is to use Stone-algebra-valued measures (taking values in an abelian $AW^*$-algebra) instead of numerical measures.

LEMMA 3. *Let $\mathfrak{M}$ be a faithful $AW^*$-module over an abelian $AW^*$-algebra $Z$ and let $\mathfrak{B}(\mathfrak{M})$ be the $AW^*$-algebra of all bounded module endomorphisms of $\mathfrak{M}$. Suppose $N$ is an $AW^*$-subalgebra of $\mathfrak{B}(\mathfrak{M})$, then for any self-adjoint element $x \in N''$ (double commutator in $\mathfrak{B}(\mathfrak{M})$), any pair $\xi, \eta$ in $\mathfrak{M}$ with $\|\xi\| \leqq 1$, $\|\eta\| \leqq 1$ and any positive number $\varepsilon$, there are a sequence $\{x_n\}$ of self-adjoint elements in the unit sphere of $N$ and a sequence $\{e_n\}$ of projections in $N$ such that for each $n$,*

(1) $(e_n(x_n - x)^2 e_n \xi, \xi) < 2^{-1} \cdot 4^{-n} \cdot \varepsilon \cdot 1$, $((x_n - x)^2 \xi, \xi) + ((x_n - x)^2 \eta, \eta) < (2/n) \cdot 1$ *in $Z$,*

(2) $e_n(x_n - x)^2 e_n \leqq 2^{-n+1} \cdot 1$,

(3) $e_n \leqq e_{n-1}$, $((e_{n-1} - e_n)\xi, \xi) \leqq 2^{-n+1} \cdot \varepsilon \cdot 1$ *in $Z$.*

PROOF. We will prove by mathematical induction. Put $e_0 = e_1 = 1$. By Theorem A, there is a net $\{x_\alpha\}$ of self-adjoint elements of $N$ with $\|x_\alpha\| \leqq 1$ such that $\|(x_\alpha - x)\xi\| \to 0$ for each $\zeta \in \mathfrak{M}$. Thus we can take an $x_1$ in $\{x_\alpha\}$ with $((x_1 - x)^2 \xi, \xi) < 2^{-1} \cdot 4^{-1} \cdot \varepsilon \cdot 1$ and $((x_1 - x)^2 \eta, \eta) + ((x_1 - x)^2 \xi, \xi) < 1$ in $Z$. Next we will choose a self-adjoint element $x_2$ and a projection $e_2$ in $N$. Let $B$ be the $C^*$-subalgebra of $N''$ generated by $(x_1 - x)^2$ and 1 (identily of $\mathfrak{B}(\mathfrak{M})$), then the restriction of $\omega_\xi$ (where $\omega_\xi(x) = (x\xi, \xi)$ for $x \in N''$) can be represented by a bounded quasi-regular $Z$-valued measure $\mu_\xi$ on the spectrum $\sigma((x_1 - x)^2)$ of $(x_1 - x)^2$ by Theorem B. Again by Theorem B and the continuity of $\mu_\xi$ ([11, Lemma 3.1]), there are a sequence $\{I_n\}$ of open intervals contained in $[2^{-2}, 2^{-1}]$ and a sequence $\{g_n\}$ of continuous functions on $[0, 4]$ with $g_n(t) = 1$ for each $t \in I_n$ such that $\omega_\xi(g_n((x_1 - x)^2)) \to 0(O)$ as $n \to \infty$. Thus by ([10, Lemma 1.1]), for any given non-zero projection $e$ in $Z$, there exists a non-zero projection $e'(\leqq e)$ in $Z$ and an integer $n_0$ such that for all $n \geqq n_0$,

$$(*) \qquad e'\omega_\xi(g_n((x_1 - x)^2)) < 16^{-1} \cdot 2^{-1} \cdot 4^{-2} \cdot \varepsilon \cdot 1 \quad \text{in} \quad Z .$$

For any fixed element $t \in I_{n_0}$, let $f(s) = \chi_t(s)(s \in [0, 4])$ (where $\chi_t$ is the characteristic function of $[0, t]$), then we can take a Borel function $h(0 \leqq h \leqq 1)$ on $[0, 4]$ whose support is contained in $I_{n_0}$ such that $k = h + f$ is continuous on $[0, 4]$. Put $y_\alpha = (x_\alpha - x_1)^2$ and $z_\alpha = (x_\alpha - x)^2$, then by a standard calculation, we have $f(y_\alpha)z_\alpha f(y_\alpha) = 2k(y_\alpha)z_\alpha k(y_\alpha) + 8g(y_\alpha)(g = g_{n_0})$. Since $\|k(y_\alpha)\| \leqq 4$ for each $\alpha$, and $g$ and $k$ are continuous by [10, Lemma 1.5], $\omega_\xi(k(y_\alpha)) \to \omega_\xi(k((x_1 - x)^2))$ and $\omega_\xi(g(y_\alpha)) \to \omega_\xi(g((x_1 - x)^2))$ uniformly in $Z$. This implies that by the equation $(*)$, there exists an $\alpha_1(= \alpha_1(e))$ such that

$$\|e'\omega_\xi(f(y_\alpha)z_\alpha f(y_\alpha))\| < 2^{-1}\cdot 4^{-2}\cdot\varepsilon \quad \text{for each} \quad \alpha \geqq \alpha_1 .$$

Observe that $\omega_\zeta((x_1 - x_\alpha)^2) \to \omega_\zeta((x_1 - x)^2)$ uniformly in $Z$ for each $\zeta \in \mathfrak{M}$, we can choose an $\alpha_2(\geqq \alpha_1)$ such that $\|\omega_\xi((x_1 - x_{\alpha_2}))^2\| < 2^{-1}\cdot 4^{-1}\cdot\varepsilon$ and $\|\omega_\xi((x_{\alpha_2} - x)^2) + \omega_\eta((x_{\alpha_2} - x)^2)\| < 2^{-1}$. Now put $x_2(e) = x_{\alpha_2}e'$ and $e_2(e) = f((x_{\alpha_2} - x_1)^2)e'$, then by [10, Theorem 4.1] (where type 1 case was proved) and Lemma 1, $e_2(e)$ is a projection in $N$ and

(1)′ $e_2(e)(x_2(e) - x_1)^2 e_2(e) \leqq 2^{-1}\cdot 1$,

(2)′ $e'\omega_\xi((x_1 - x_2(e))^2) < 2^{-1}\cdot 4^{-1}\cdot\varepsilon\cdot 1$ in $Z$,

(3)′ $e'\{\omega_\xi((x_2(e) - x)^2) + \omega_\eta((x_2(e) - x)^2)\} < 2^{-1}\cdot 1$ in $Z$,

(4)′ $e'\{\omega_\xi(e_2(e)(x_2(e) - x)^2 e_2(e))\} < 2^{-1}\cdot 4^{-2}\cdot\varepsilon\cdot 1$ in $Z$.

Moreover, let $\mu_\xi$ be the $Z$-valued measure on $\sigma(x_{\alpha_2} - x_1)^2)$ induced by $\omega_\xi$, then the non-negativity of integral ([11, Proposition 3.2]) implies that $e'\omega_\xi(e_1 - e_2) \leqq 2^2 e'\omega_\xi((x_{\alpha_2} - x_1)^2) < 2^{-1}\cdot\varepsilon\cdot e'$ in $Z$ by (2)′.

Thus by the above arguments, there are an orthogonal family $\{e_\beta\}$ of projections in $Z$ with $\sum e_\beta = 1$, a family $\{e_2(\beta)\}$ of non-zero projections in $N$ with $e_2(\beta) = e_2(\beta)e_\beta$ and a family $\{x_2(\beta)\}$ of self-adjoint elements of $N$ with $x_2(\beta) = x_2(\beta)e_\beta$ and $\|x_2(\beta)\| \leqq 1$ for each $\beta$ such that

(1)″ $e_2(\beta)(x_2(\beta) - x_1)^2 e_2(\beta) \leqq 2^{-1}\cdot e_\beta$,

(2)″ $e_\beta\omega_\xi((x_1 - x_2(\beta))^2) < 2^{-1}\cdot 4^{-1}\cdot\varepsilon\cdot e_\beta$ in $Z$,

(3)″ $e_\beta\{(\omega_\xi + \omega_\eta)((x - x_2(\beta))^2)\} < 2^{-1}\cdot e_\beta$ in $Z$,

(4)″ $e_\beta\{\omega_\xi(e_2(\beta)(x_2(\beta) - x)^2 e_2(\beta)) < 2^{-1}\cdot 4^{-2}\cdot\varepsilon\cdot e_\beta$ in $Z$

for each $\beta$. Therefore by the $C^*$-summation process ([4, Lemma 2.5]), we can choose a self-adjoint element $x_2$ in $N$ with $\|x_2\| \leqq 1$, $x_2 e_\beta = x_2(\beta)e_\beta$. Put $e_2 = \sum_\beta e_2(\beta)(\in N)$, and by [4, Lemma 2.5] and a continuity property in $AW^*$-modules ([6, Lemma 2]), it follows that

(a) $e_2(x_2 - x_1)^2 e_2 \leqq 2^{-1}\cdot 1$,

(b) $\omega_\xi((x_1 - x_2)^2) < 2^{-1}\cdot 4^{-1}\cdot\varepsilon\cdot 1$ in $Z$,

(c) $(\omega_\xi + \omega_\eta)((x_2 - x)^2) < 2^{-1}\cdot 1$ in $Z$,

(d) $\omega_\xi(e_2(x_2 - x)^2 e_2) < 2^{-1}\cdot 4^{-2}\cdot\varepsilon\cdot 1$ in $Z$.

By the same argument (but for $[2^{-n-1}, 2^{-n}]$) and the mathematical induction, we can construct $\{e_n\}$, $\{x_n\}$ satisfying (1) – (4) in Lemma 3. This completes the proof.

Before going into the proof of Theorem 1, we shall show the following:

THEOREM 2. *Let $M$ be an $AW^*$-algebra with center $Z_M$. Suppose $M$ has a separating set of bounded $Z_M$-valued non-negative module homomorphisms which are completely additive on projections, then $M$ has a faithful representation as a double commutator in a type* 1 *$AW^*$-algebra whose center is isomorphic to $Z_M$.*

Using Lemma 1, 2, and 3, the proof is a straight forward modification of one given by Pedersen [7] for $Z = \boldsymbol{C}\cdot 1$ (scalar case), however, due to our complicated situation we sketch them.

PROOF. By [10, Theorem 3.1] (representation theorem), we may assume that $M$ is an $AW^*$-subalgebra of the type 1 $AW^*$-algebra $\mathfrak{B}(\mathfrak{M})$ of all bounded module endomorphisms of some faithful $AW^*$-module $\mathfrak{M}$ over $Z_M$. It suffices to prove that for any non-zero projection $p$ in $M''$, there is a non-zero projection $q$ in $M$ with $q \leqq p$. We may assume $p \neq 1$. Then there are non-zero elements $\xi$ and $\eta$ in $\mathfrak{M}$ such that $p\xi = 0$, $p\eta = \eta$ with $\|\xi\| \leqq 1$, $\|\eta\| \leqq 1$. By Lemma 3, for any $\varepsilon > 0$, there are a sequence $\{x_n\}$ of self-adjoint elements in $M$ and a sequence $\{e_n\}$ of projections in $M$ which satisfy (1) – (4) in Lemma 3 (but for $x = 1 - p$). Put $e = \bigwedge_{n=1}^{\infty} e_n$ in $M''$, then $e \in M$ and for each $n$,

(1) $\|\{x_n - (1 - p)\}\xi\| < 1/n$,

(2) $\|\{x_n - (1 - p)\}\eta\| < 1/n$,

(3) $e(x_n - x_{n-1})^2 e \leqq 2^{-n+1}\cdot 1$,

(4) $((1 - e)\xi, \xi) < \varepsilon\cdot 1$ in $Z$.

Since, by (3), $\{x_n e x_n\}$ is a uniformly Cauchy sequence in $M$, there is $y$ in $M$ with $0 \leqq y \leqq 1$, such that $\|x_n e x_n - y\| \to 0$ $(n \to \infty)$. Thus by (1), (2), and (4), $(y\xi, \xi) > (\xi, \xi) - \varepsilon\cdot 1$ in $Z$ and $(y\eta, \eta) = 0$. Let $q(\xi, \varepsilon)$ be the left projection of $y$, then $(y^{1/2} + (1/n)1)^{-1}y^{1/2} \geqq (n/n + 1)y$ (for each integer $n$) implies by Lemma 2, $(q(\xi, \varepsilon)\xi, \xi) \geqq (\xi, \xi) - \varepsilon\cdot 1$ and $q(\xi, \varepsilon)\eta = 0$ in $Z$. For fixed $\eta$, let $q(\xi_1, \varepsilon_1) \vee q(\xi_2, \varepsilon_2)$ be the supremum of $q(\xi_1, \varepsilon_1)(p\xi_1 = 0, \varepsilon_1 > 0)$ and $q(\xi_2, \varepsilon_2)(p\xi_2 = 0, \varepsilon_2 > 0)$ in $M''$, then by Lemma 2, $q(\xi_1, \varepsilon_1) \vee q(\xi_2, \varepsilon_2) \in M$ and $\{q(\xi_1, \varepsilon_1) \vee q(\xi_2, \varepsilon_2)\}\eta = 0$. Let $q_0$ be the supremum of $q(\xi, \varepsilon)$ with $p\xi = 0$ and $\varepsilon > 0$ in $M''$, then by Lemma 1, $q_0 \in M$ and by [10, Lemma 1.4] (the continuity of $\omega_\eta$), $q_0\eta = 0$. On the other hand, $(q_0\xi, \xi) \geqq (\xi, \xi) - \varepsilon\cdot 1$ for all $\xi$ with $p\xi = 0$ and $q_0\eta = 0 (\eta \neq 0)$, which implies $q \geqq 1 - q_0 \neq 0$. This completes the proof.

PROOF OF THEOREM 1. By Theorem 2 and [10, Lemma 4.4], the proof proceeds in entire analogy. So we omit them.

## REFERENCES

[1] S. K. BERBERIAN, Baer*-Rings, Springer-Verlag, Berlin, 1972.
[2] J. DIXMIER, Sur certains espaces considérés par M. H. Stone, Summa Brasil Math., 2 (1951), 151-182.
[3] J. FELDMAN, Embedding of $AW^*$-algebras, Duke Math. J., 23 (1956), 303-307.
[4] I. KAPLANSKY, Projections in Banach algebras, Ann. of Math., 53 (1951), 235-249.
[5] I. KAPLANSKY, Algebras of type 1, Ann. of Math., 56 (1952), 460-472.
[6] I. KAPLANSKY, Modules over operator algebras, Amer. J. Math., 45 (1953), 839-858.
[7] G. K. PEDERSEN, Operator algebras with weakly closed abelian subalgebras, Bull. London Math. Soc., 4 (1972), 171-175.
[8] K. SAITÔ, A non-commutative theory of integration for a semi-finite $AW^*$-algebra and a problem of Feldman, Tôhoku Math. J., 22 (1970), 420-461.
[9] K. SAITÔ, On the embedding as a double commutator in a type 1 $AW^*$-algebra, Tôhoku Math. J., 23 (1971), 541-557.
[10] H. WIDOM, Embedding in algebras of type 1, Duke Math. J., 23 (1956), 309-324.
[11] J. D. MAITLAND WRIGHT, Stone-algebra-valued measures and integrals, Proc. London Math. Soc., 19 (1969), 107-122.
[12] J. D. MAITLAND WRIGHT, A spectral theorem for normal operators on a Kaplansky-Hilbert module, Proc. London Math. Soc., 19 (1969), 258-268.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN