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THE INDEX THEOREM FOR CLOSED GEODESICS

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1. Let $c = c(t)$, $0 \leqq t \leqq \omega$, $|\dot{c}(t)| = 1$, be a closed geodesic on a Riemannian manifold M of dimension $n+1$. By considering c as critical point of the energy integral E on the Hilbert manifold M of closed H^1-curves on M, the index of c is defined as the index of the Hessian $D^2E(c)$ with respect to the Hilbert product on the tangent space $T_c\Lambda M$, cf. [2]. On the other hand, we can associate to the closed geodesic a $2n$-dimensional vector bundle over the circle S_ω of length ω as follows. Let

$$\tau^{2n}\colon T^{2n}T_1M \longrightarrow T_1M$$

be the subbundle of the tangent bundle of T_1M formed by the vectors orthogonal to the geodesic spray. Then we have from the immersion

$$\dot{c}\colon S_\omega \longrightarrow T_1M$$

an induced bundle which we denote by

$$\tau^{2n}\colon V^{2n} \longrightarrow S_\omega .$$

Moreover, the decomposition of τ^{2n} into its horizontal and its vertical subbundles τ_h^n and τ_v^n gives a corresponding decomposition over S_ω:

$$\tau_h^n\colon V_h^n \longrightarrow S_\omega ; \qquad \tau_v^n\colon V_v^n \longrightarrow S_\omega .$$

On τ^{2n} we have a symplectic structure defined by (with $(X_h, X_v) \in V_h^n \oplus V_v^n$)

$$2\alpha((X_h, X_v), (Y_h, Y_v)) := \langle X_h, Y_v\rangle - \langle Y_h, X_v\rangle$$

and we have the geodesic flow ϕ_t which carries the fibre $V^{2n}(t_0)$ over $t_0 \in S_\omega$ into the fibre $V^{2n}(t_0+t)$ over $t_0+t \bmod \omega \in S_\omega$ by associating to an element

$$(A, B) \in V_h^n(t_0) \oplus V_v^n(t_0)$$

the value at $t+t_0$ of the Jacobi field $Y(t)$ and its covariant derivative $\nabla Y(t)$ determined by

$$\tilde{Y}(t_0) := (Y(t_0), \nabla Y(t_0)) = (A, B) .$$

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The geodesic flow respects the symplectic structure on τ^{2n}. In particular we have the map

$$P:= d\phi_\omega: V^{2n}(0) \to V^{2n}(0)$$

of the fibre over 0 onto the fibre over $\omega = 0 \in S_\omega$, the so-called (linear) Poincaré map. The closed geodesic c is non-degenerate if and only if P has no eigenvalue $=1$.

In this paper we will show how the index of c defined above can also be expressed by using the bundle τ^{2n} and its structure. For the special case that P has no eigenvalue ρ with $|\rho| = 1$ this was done already in the paper [4]. We thus get a natural analogue of the Morse index theorem for geodesic segments, cf. M. Morse [5]. The general case where we allow the eigenvalue $\rho = 1$ for P will be discussed in a later paper; there we will also establish the relation of our theory with Bott's paper [1] on the iteration of closed geodesics and the Sturm intersection theory; this means that we will have to derive an index theorem for the space of H^1-vector fields ξ along c satisfying the boundary condition $\xi(\omega) = \rho\xi(0)$ for every complex number ρ with $|\rho| = 1$.

The results of this paper were announced in [3].

2. We start with some results of the geometry of a linear symplectic space V^{2n} with symplectic form α:

LEMMA 2.1. *Let $P: V^{2n} \to V^{2n}$ be a linear symplectic map. Then there exists, up to a symplectic isomorphism, a unique decomposition*

$$V^{2n} = V^{2p}_{in} \oplus V^{2q}_{un}$$

into invariant non-degenerate subspaces V^{2p}_{in} and V^{2q}_{un} such that $P|V^{2p}_{in}$ does not belong to a compact subgroup of $Sp(n)$ and has an invariant isotropic subspace V^p_{in}. $P|V^{2q}_{un}$ has only eigenvalues ρ with $|\rho| = 1$ and possesses a decomposition $V^{2q'}_{nc} \oplus V^{2q''}_{co}$ into non-degenerate subspaces with the following properties: $V^{2q''}_{co}$ is spanned by proper eigenvectors and $P|V^{2q''}_{co}$ belongs to a compact subgroup of $Sp(n)$. $V^{2q'}_{nc}$ can be written as a sum of non-degenerate subspaces having a base of the form

$$(P\bar{\rho} - 1)^j X\,, \qquad 0 \leqq j \leqq 2l$$

with $(P\bar{\rho} - 1)^{2l+1}X = 0$, $l > 0$, ρ an eigenvalue with $|\rho| = 1$. $P|V^{2q'}_{nc}$ does not belong to a compact subgroup.

PROOF. We complexify V^{2n} and write again V^{2n}. The symplectic form α is extended to a hermitian symplectic form. With ρ also $\bar{\rho}$, ρ^{-1} and $\bar{\rho}^{-1}$ are eigenvalues. Let $V(\rho)$ be the generalized eigenspace belonging to the eigenvalue ρ of P. For $|\rho| \neq 1$ we put $V(\rho) \oplus V(\bar{\rho}^{-1})$ into V^{2p}_{in} and one of the summands into V^p_{in}. For $|\rho| = 1$ we can decompose $V(\rho)$ into non-degenerate subspaces $V'(\rho) \oplus V''(\rho)$ such that $V''(\rho)$ consists entirely of eigenvectors whereas $V'(\rho)$ can be written as orthogonal sum of non-degenerate space having a base of the form

$$(*) \qquad (P\bar{\rho} - 1)^j X, 0 \leqq j \leqq k, k > 0 .$$

$V''(\rho)$ is taken into $V^{2q''}_{co} \subset V^{2q}_{un}$. If in $(*)$ we have $k = 2l > 0$ then the subspace is taken into $V^{2q'}_{nc} \subset V^{2q}_{un}$. If, however, $k = 2l - 1$ then the subspace is taken into V^{2p}_{in} and its invariant isotropic subspace spanned by the vectors $(P\bar{\rho} - 1)^j X$ with $l \leqq j \leqq 2l - 1$ is taken into V^p_{in}.

LEMMA 2.2. *Let V^q_{un} be an isotropic subspace of V^{2q}_{un}. Assume that 1 is not an eigenvalue of $P|V^{2q}_{un}$. Then*

$$Q(X, Y) := -2\alpha(X, (P-1)^{-1} Y)$$

is a quadratic form on V^q_{un}. Its nullspace consists of $(P-1)(V^q_{un} \cap P^{-1} V^q_{un})$.

PROOF. 1. Put

$$(P-1)^{-1} X = S , \qquad (P-1)^{-1} Y = T .$$

Then

$$\begin{aligned} 0 = 2\alpha((P-1)S, (P-1)T) &= -2\alpha(X, (P-1)^{-1} Y) \\ &+ 2\alpha((P-1)S, PT) = Q(X, Y) + 2\alpha(S, T) \\ &- 2\alpha(S, PT) = Q(X, Y) + 2\alpha((P-1)T, S) \\ &= Q(X, Y) + 2\alpha(Y, (P-1)X) = Q(X, Y) - Q(Y, X) . \end{aligned}$$

2. Assume that $Y \in V^q_{un}$ is such that $Q(X, Y) = 0$ for all $X \in V^q_{un}$. That means, since V^q_{un} is an isotropic subspace of maximal dimension in V^{2q}_{un}, that $(P-1)^{-1} Y = Y' \in V^q_{un}$. Hence, $Y = PY' - Y' \in V^q_{un}$, i.e., $Y' \in V^q_{un} \cap P^{-1} V^q_{un}$. Conversely, if $(P-1)^{-1} Y \in V^q_{un} \cap P^{-1} V^q_{un}$, so $Y = PY' - Y' \in V^q_{un}$ and $Y \in$ nullspace of Q.

LEMMA 2.3. *Let*

$$V^{2n} = V^{2p}_{in} \oplus V^{2q}_{un}$$

be an orthogonal decomposition into non-degenerate subspaces. Let V^p_{in} be an isotropic subspace of V^{2p}_{in} and let V^n_v be an isotropic subspace of V^{2n}.

Claim. (i) *The projection of*

$$V^n_v \cap (V^p_{in} \oplus V^{2q}_{un})$$

into V^{2q}_{un} modulo V^p_{in} gives a q-dimensional isotropic subspace $V^q_{un} \subset V^{2q}_{un}$.

(ii) *The space*

$$V_e^n := V_{in}^p \oplus V_{un}^q$$

is an n-dimensional isotropic subspace of V^{2n}.

(iii) *There exists a q-dimensional subspace* $V_v^q \subset V_v^n$ *such that*

$$V_e^n = V_{in}^p \oplus V_v^q \ .$$

PROOF. Let $V_{in}^k := V_v^n \cap V_{in}^p$ have dimension $k \geqq 0$. Let V_{in}^{*p} be an isotropic complement of V_{in}^p in V_{in}^{2p}. The projection V_{in}^{*l} of V_v^n into V_{in}^{*p} modulo $V_{in}^p \oplus V_{un}^{2q}$ shall have dimension l. Since every $X_{in}^* \in V^{*l}$ can be complemented by an $X' \in V_{in}^p \oplus V_{un}^{2q}$ to give an element $X^* + X' \in V_v^n$ we have, for every $X \in V_{in}^k$,

$$\alpha(X^*, X) = \alpha(X^* + X', X) = 0 \ .$$

Since $\alpha \,|\, V_{in}^p \oplus V_{in}^{*p}$ is non-degenerate it follows that $k + l \leqq p$. Hence, $\dim V_v^n \cap (V_{in}^p \oplus V_{un}^{2q}) = n - l \geqq n + k - p$ and therefore, the projection into V_{un}^{2q} has dimension $\geqq n - p = q$. Note, however, that this projection is isotropic. Indeed, elements X and Y of the projection can be complemented by elements X' and Y' of V_{in}^p so as to give elements $X + X'$ and $Y + Y'$ of the isotropic space V_v^n; i.e.,

$$0 = \alpha(X + X', Y + Y') = \alpha(X, Y) \ ,$$

since $\alpha(V_{in}^p, V_{un}^{2q}) = 0$.
Hence, the projection has the exact dimension q and can be denoted by V_{un}^q. Thus we have proved (i).

(ii) follows immediately from this.

But also (iii) is clear from the definition of V_{un}^q.

3. Let $c = c(t)$, $0 \leqq t \leqq \omega$, $|\dot{c}(t)| = 1$, be a closed geodesic. To the fibre $V^{2n}(0)$ over $0 = \omega \in S_\omega$ of the bundle τ^{2n} we apply the results of (2), with $P := d\phi_\omega$ and $V_v^n := V_v^n(0) =$ fibre over $0 = \omega$ of the vertical bundle $\tau_v^n \colon V_v^n \to S_\omega$. For each $t \in [0, \omega]$, $d\phi_t V_e^n(0)$ will be an isotropic subspace of $V^{2n}(t)$. In general, however, $d\phi_\omega V_e^n(0) = PV_e^n(0)$ will be different from $V_e^n(0)$, i.e., in general we will not obtain in this way a bundle over S_ω. An exception would be the case that $p = n$, e.g., if all eigenvalues ρ of P satisfy $|\rho| \neq 1$, cf. [4]. For each $t \in [0, \omega]$ we define the space

$$W(t) := V_v^n(t) \cap d\phi_t V_e^n(0) \ .$$

Put $\dim W(t) = \iota(t)$. $\iota(t) = 0$, except for a finite number of value, as follows from the

PROPOSITION 3.1. *Let* $\dim W(t_0) = \iota_0 > 0$. *Choose a base* $\tilde{Y}_i(t) := (Y_i(t), \nabla\phi_i(t))$, $1 \leqq i \leqq n$, *of Jacobi fields for* $d\phi_t V_e^n(0)$ *such that the* $\tilde{Y}_i(t_0)$, $1 \leqq i \leqq$

ι_0, *are a base of* $W(t)$.

Claim. (i) *The elements*

$$\nabla Y_i(t_0),\ 1 \leqq i \leqq \iota_0,\ Y_j(t_0),\ \iota_0 + 1 \leqq j \leqq n$$

of $T_{c(t_0)}M$ *form a base for the* n*-dimensional space orthogonal to* $\dot{c}(t_0) \in T_{c(t_0)}M$.

(ii) *For all* $t \neq t_0$, *sufficiently near* t_0, *the* $Y_i(t)$, $1 \leqq i \leqq n$, *are linearly independent.*

PROOF. Since $\tilde{Y}(t_0) = (Y(t_0), \nabla Y(t_0)) \in W(t_0)$ means that $Y(t_0) = 0$ it follows that the $Y_j(t_0)$, $j > \iota_0$, are linearly independent. Clearly, also the $\nabla Y_i(t_0), 1 \leqq i \leqq \iota_0$, are linearly independent. From $\alpha \mid d\phi_{t_0} V_e^n(0) \equiv 0$ it follows that $\langle \nabla Y_i(t_0), Y_j(t_0) \rangle = 0$ for $1 \leqq i \leqq \iota_0$, hence (i) does hold. (ii) simply follows from the observation that for $\tilde{Y}(t) \neq 0$, $Y(t_0) = 0$ implies $\nabla Y(t_0) \neq 0$.

We can now formulate the main result of this paper, i.e., the index theorem for closed geodesics.

THEOREM 3.2. *Let* $c = c(t)$, $0 \leqq t \leqq \omega$, *be a non-degenerate closed geodesic. Using the previous notations we then have*

$$\operatorname{Index} c = \sum_{0 < t \leqq \omega} \iota(t) + \operatorname{Index} Q$$

where $\iota(t) = \dim W(t)$ *and* Q *is the quadratic form defined in* 2.2 *on the space* $V_{un}^q(0)$.

PROOF. 1. We define for each $t_0 \in [0, \omega]$ an injective map

$$\zeta \colon W(t_0) \longrightarrow T_c\Lambda$$

as follows: Write $\tilde{Y}(t_0) \in W(t_0)$ as

$$\tilde{Y}(t_0) = \tilde{Y}_{in}(t_0) + \tilde{Y}_{un}(t_0) \in d\phi_{t_0} V_{in}^p(0) \oplus d\phi_{t_0} V_{un}^q(0) .$$

Put $(P - 1)^{-1} \tilde{Y}_{in}(t) = \tilde{Z}_{in}(t)$, $(P - 1)^{-1} \tilde{Y}_{un}(t) = \tilde{Z}_{un}(t)$. Note that $\tilde{Z}_{in}(t) \in d\phi_t V_{in}^p(0)$ whereas $\tilde{Z}_{un}(t) \in d\phi_t V_{un}^{2q}(0)$. We define an element $\zeta \in T_c\Lambda$, i.e., a continuous vector field along c by

$$\zeta(t) = \begin{cases} Z_{in}(t + \omega) + Z_{un}(t + \omega),\ 0 \leqq t \leqq t_0 \\ Z_{in}(t) + Z_{un}(t),\ t_0 \leqq t \leqq \omega . \end{cases}$$

$\zeta(t)$ is even differentiable except possibly at $t = t_0$ where we have:

$$\nabla \zeta(t_0 -) - \nabla \zeta(t_0 +) = \nabla Y(t_0) .$$

The injectivity of $W(t_0) \longrightarrow T_c\Lambda$ is obvious.

2. For $t_0 \neq t_0'$, the spaces $\zeta W(t_0)$ and $\zeta W(t_0')$ will be linearly independent except possibly if $|t_0 - t_0'| = \omega$, e.g., $t_0 = \omega$, $t_0' = 0$. In this case,

$$\zeta(\tilde{Y}'(\omega)) = \zeta(\tilde{Y}(0))$$

means

$$\tilde{Z}'_{in}(t) + \tilde{Z}'_{un}(t) = \tilde{Z}_{in}(t+\omega) + \tilde{Z}_{un}(t+\omega)$$

i.e.,

$$\tilde{Y}'(0) = \tilde{Y}(\omega) \in W(0) \cap W(\omega) .$$

Let $W^*(0)$ be a complement of $W(0) \cap W(\omega)$ in $W^*(0)$. Then the linear map

$$\zeta\colon W := \bigoplus_{0<t\leqq\omega} W(t) \oplus W^*(0) \to T_c\Lambda$$

is injective. Denote the image in $T_c\Lambda$ by ζW.

$$\dim \zeta W = J_c + \dim W^*(0)$$

where we have put $\sum_{0<t\leqq\omega} \iota(t) = J_c$.

3. We define a linear map

$$\lambda\colon \zeta W \to V^q_{un}(0)$$

by associating to an element $\zeta = \zeta(\tilde{Y}(t_0))$ with

$$\tilde{Y}(t_0) = \tilde{Y}_{in}(t_0) + \tilde{Y}_{un}(t_0) \in d\phi_{t_0} V^p_{in}(0) \oplus d\phi_{t_0} V^q_{un}(0)$$

the element $\tilde{Y}_{un}(0)$. Note that λ is surjective: Indeed, from 2.3 (iii) we know that every $\tilde{Y}_{un}(0) \in V^q_{un}(0)$ occurs in an element $\tilde{Y}(0) = \tilde{Y}_{in}(0) + \tilde{Y}_{un}(0) \in V^n_v(0) \cap V^n_e(0) = W(0)$. Moreover, $W(\omega) \cap W(0) \oplus W^*(0) = W(0)$. It follows that $\dim\ker\lambda = J_c - \dim(W(\omega) \cap W(0))$. We claim: If $\zeta = \zeta(\tilde{Y}(t_0))$, $\zeta' = \zeta(\tilde{Y}'(t'_0))$, then

$$D^2E(c)(\zeta, \zeta') = Q(\lambda\zeta, \lambda\zeta') .$$

For the proof we can assume: $t'_0 \leqq t_0$. We find with the expressions given in 1:

$$\begin{aligned} D^2E(c)(\zeta, \zeta') &= \int_0^\omega \frac{d}{dt}\langle \nabla\zeta, \zeta'\rangle dt \\ &\quad - \int_0^\omega \langle \nabla^2(\zeta) + R(\zeta), \zeta'\rangle dt \\ &= \langle \nabla Y(t_0), Z'_{in}(t_0) + Z'_{un}(t_0)\rangle \\ &= -2\alpha(\tilde{Y}_{in} + \tilde{Y}_{un}, \tilde{Z}'_{in} + \tilde{Z}'_{un}) \\ &= -2\alpha(\tilde{Y}_{un}, \tilde{Z}'_{un}) = Q(\lambda\zeta, \lambda\zeta') , \end{aligned}$$

because $2\alpha(\tilde{Y}_{in}, \tilde{Z}'_{in}) = 0$.

It follows that index $c \geqq \dim\ker\lambda + \text{Index } Q + \text{nullity } Q = J_c + \text{Index } Q$, since nullity $Q = \dim W(\omega) \cap W(0)$, cf. 2.3.

4. To prove that actually equality does hold, we show that any $\xi \in$

$T_c\Lambda$ satisfying

$$D^2E(c)(\zeta, \xi) = 0, \quad \text{for all} \quad \zeta \in \zeta W, \quad \text{and}$$
$$D^2E(c)(\xi, \xi) \leqq 0$$

belongs already to ζW or nullspace of $D^2E(c)$ which is 1-dimensional with the element $\dot{c}(t)$ as generator, because c is supposed to be non-degenerate. Indeed, the first condition means for $\zeta = \zeta(\tilde{Y}(t_0))$:

$$\langle \nabla Y(t_0), \xi(t_0)\rangle = 0 .$$

Using 3.1 we have that $\xi(t)$ can be written in the form

$$\xi(t) = \sum_i w^i(t) Y_i(t)$$

where the $\tilde{Y}_i(t)$, $1 \leqq i \leqq n$, form a basis of Jacobi fields for $d\phi_t V_e^n(0)$. We can assume that $\tilde{Y}_i(0) \in V_v^n(0)$ for $i > p$, i.e., $w^i(0) = 0$ for $i > p$. Since the $\tilde{Y}_i(0)$, $1 \leqq i \leqq p$, and the $\tilde{Y}_i(\omega)$, $1 \leqq i \leqq p$, span the same space $V_{in}^p(0) = V_{in}^p(\omega)$ and since $\xi(\omega) = \xi(0)$ we have: $w^i(\omega) = 0$ for $i > p$. With this we get:

$$\begin{aligned} 0 \geqq D^2E(c)(\xi, \xi) &= -\int_0^\omega \langle \sum_i w^i(\nabla^2 Y_i + R(Y_i)), \sum_j w^j Y_j\rangle dt \\ &\quad + \int_0^\omega \sum_{i,j} \dot{w}^i w^j 2\alpha(\tilde{Y}_i, \tilde{Y}_j) dt + \int_0^\omega |\sum_i \dot{w}^i Y_i|^2 dt \\ &\quad + \int_0^\omega \frac{d}{dt}\langle \sum_i w^i \nabla Y_i, \sum_j w^j Y_j\rangle dt \\ &\geqq \sum_{j,i} w^j(0) w^i(\omega) 2\alpha(\tilde{Y}_j(0), \tilde{Y}_i(\omega)) = 0 . \end{aligned}$$

Hence, $\dot{w}^i(t) = 0$, the $w^i(t)$ are locally constant. But then one sees easily that $\xi(t) \in \zeta W$ or nullspace of $D^2E(c)$ which is 1-dimensional with the element $\dot{c}(t)$ as generator, because c is supposed to be non-degenerate. Thus our theorem is proved.

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