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NOTE ON TOEPLITZ OPERATORS

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1. For $p = 2, \infty$, let L^p be the usual class of Lebesgue measurable functions on the unit circle of the complex plane and let H^p be the closed subspace of L^p of functions whose Fourier coefficients vanish on the negative integers. We say a function ϕ in L^∞ is inner if $\phi \in H^\infty$ and $|\phi| = 1$ a.e.. For $\phi \in L^\infty$, we denote by L_ϕ the Laurent operator on L^2 defined by $L_\phi f = \phi \cdot f$ for every f in L^2 and by T_ϕ the Toeplitz operator on H^2 defined by $T_\phi = PL_\phi P$ where P is the orthogonal projection of L^2 onto H^2. We say a Toeplitz operator T_ϕ is analytic if $\phi \in H^\infty$.

In order to study Toeplitz operators T_ϕ, it is natural to examine the behavior of ϕ as a function in L^∞ and, in fact, almost known results are given by using the functional method. However, the following characterization of Toeplitz operators makes it possible to deal with them as operators with some relation to a simple unilateral shift and it seems that such a treatment is useful to study Toeplitz operators, systematically.

The purposes of this note are to study some of properties on Toeplitz operators from this point of view and to show the relations between several results given by A. Brown and G. Douglas, A. Brown and P. R. Halmos, R. Goor, P. Hartman and A. Wintner or T. Itô and T. K. Wong, independently and ours.

At first, we prove several results on invariant subspaces of L_ϕ and next, using these, we give a sufficient condition that a function ϕ of T_ϕ is inner in terms of subspace of H^2 and lastly, we prove a result on the spectrum of T_ϕ.

We state here a characterization of Toeplitz operators and also analytic Toeplitz operators given by A. Brown and P. R. Halmos.

THEOREM 1. ([3]) *A necessary and sufficient condition that an operator A on H^2 be a Toeplitz operator (or an analytic Toeplitz operator) is that $T_z^* A T_z = A$ (or $T_z A = A T_z$). Since a simple unilateral shift V on a Hilbert space K is unitarily equivalent to T_z on H^2, a necessary and sufficient condition that an operator B on K be unitarily equivalent to a Toeplitz operator (or an analytic Toeplitz operator) is that $V^*BV = B$ (or $VB =$*

BV).

For our purpose, we need the following result given by A. Beurling.

THEOREM 2. ([1]) *Let $\mathscr{M}$ be a non-zero closed subspace of H^2. Then $\mathscr{M}$ is invariant under T_z if and only if $\mathscr{M} = T_\psi H^2$, where ψ is an inner function.*

By Theorems 1 and 2, we have easily the following

COROLLARY 1. *Every invariant subspace of a simple unilateral shift is hyper-invariant, that is, invariant under every bounded linear operator which commutes with the simple unilateral shift.*

2. THEOREM 3. *If T_ϕ is non-analytic, then the only invariant subspace of L_ϕ which includes H^2 is L^2 itself.*

PROOF. Let $\mathscr{M} = \bigvee \{L_\phi^n f: f \in H^2, n \geqq 0\}$ (which denotes the smallest closed subspace of L^2 containing $L_\phi^n f$, $f \in H^2, n \geqq 0$), then $\mathscr{M}$ is the smallest invariant subspace of L_ϕ which includes H^2. Hence we have only to prove $\mathscr{M} = L^2$. Since L_z commutes with L_ϕ and since H^2 is invariant under L_z, $\mathscr{M}$ is invariant under L_z. If $\mathscr{M}$ reduces L_z, then $\mathscr{M} = L^2$ because L^2 is the minimal unitary extension space of T_z, that is, L^2 is the smallest subspace which includes H^2 and reduces L_z. If $\mathscr{M}$ is a non-reducing invariant subspace of L_z, then $L_z | \mathscr{M}$ is a simple unilateral shift on $\mathscr{M}$ because L_z is a simple bilateral shift on L^2. Since L_z commutes with L_ϕ, $L_z | \mathscr{M}$ commutes with $L_\phi | \mathscr{M}$. By Corollary 1, H^2 is invariant under $L_\phi | \mathscr{M}$ and hence invariant under L_ϕ. Therefore, $\phi \in H^\infty$. This contradicts with the hypothesis that T_ϕ is non-analytic.

COROLLARY 2. *If ϕ is a non-constant function in L^∞, then the only subspace of L^2 that includes H^2 and reduces L_ϕ is L^2 itself* ([3]). *Since, if T_ϕ is analytic, then L_ϕ is a normal extension of T_ϕ, in the case where ϕ is a non-constant function in H^∞, L_ϕ is the minimal normal extension of T_ϕ.*

PROOF. Let $\mathscr{M}$ be a subspace of L^2 that includes H^2 and reduces L_ϕ and let $\mathscr{M} \neq L^2$, then, by Theorem 3, T_ϕ and T_ϕ^* are analytic and hence $\phi \in H^\infty \cap \overline{H^\infty} = \{\lambda 1\}$ where $\overline{H^\infty}$ denotes the complex conjugate of H^∞. This contradicts with the hypothesis that ϕ is non-constant.

Since $L^2 \ominus H^2$ is a non-reducing invariant subspace of L_z^*, $L_z^* | L^2 \ominus H^2$ is a simple unilateral shift on $L^2 \ominus H^2$ with the minimal unitary extension L_z^*. Hence, by Theorem 1, we may think that $(I - P)L_\phi^*(I - P)$ is a Toeplitz operator on $L^2 \ominus H^2$ with the corresponding Laurent operator L_ϕ^*. If T_ϕ is non-analytic, then $\phi \notin H^\infty$ and hence $\bar{\phi} \notin L^\infty \cap (L^2 \ominus H^2)$

where $\bar{\phi}$ denotes the complex conjugate of ϕ. Therefore, we may think $(I - P)L_\phi^*(I - P)$ is a non-analytic Toeplitz operator on $L^2 \ominus H^2$. Hence, by Theorem 3, the only invariant subspace of L_ϕ^* which includes $L^2 \ominus H^2$ is L^2 itself. And hence we have the following

THEOREM 4. *If T_ϕ is non-analytic, then the only invariant subspace of L_ϕ contained in H^2 is $\{0\}$ itself.*

As an application of Theorem 4, we have

THEOREM 5. *For a T_ϕ such as $\| T_\phi \| \leqq 1$, if $\{f \in H^2: \| T_\phi^n f \| = \| f \|, n \geqq 0\} \neq \{0\}$, then ϕ is inner.*

PROOF. Let $\mathscr{M} = \{f \in H^2: \| T_\phi^n f \| = \| f \|, n \geqq 0\}$ and let

$$\mathscr{N} = \bigvee \{L_\phi^n f: f \in \mathscr{M}, n \geqq 0\} ,$$

then, for $f \in \mathscr{M}$, we have $\|f\| = \| T_\phi f\| = \| PL_\phi f\| \leqq \| L_\phi f\| \leqq \|f\|$ because it is known that $\| L_\phi \| = \| T_\phi \|$. This implies that $T_\phi f = L_\phi f$ and $L_\phi^* L_\phi f = f$. Hence, we have $\| f \| = \| T_\phi^2 f \| = \| T_\phi L_\phi f \| = \| PL_\phi^2 f \| \leqq \| L_\phi^2 f \| \leqq \| f \|$ and $T_\phi^2 f = L_\phi^2 f$. Similarly, we have $T_\phi^n f = L_\phi^n f$ for all $f \in \mathscr{M}$ and $n \geqq 0$. Clearly, $\mathscr{M}$ is invariant under T_ϕ and it is a closed subspace of H^2 because $\mathscr{M} = \bigcap_{n=0}^\infty \{f \in H^2: T_\phi^{*n} T_\phi^n f = f\}$. Therefore $\mathscr{N} = \mathscr{M}$. Since $\mathscr{M} \neq \{0\}$ and since $\mathscr{M} = \mathscr{N}$ is a invariant subspace of L_ϕ contained in H^2, by Theorem 4, T_ϕ is analytic, i.e., $\phi \in H^\infty$. And hence we have $T_\phi^* T_\phi = PL_\phi^* PL_\phi P = PL_\phi^* L_\phi P = PL_{|\phi|^2} P$. This implies that $T_\phi^* T_\phi$ is also a Toeplitz operator on H^2. Since, for $f \in \mathscr{M}$, $L_\phi^* L_\phi f = f$, we have $L_{|\phi|^2}\mathscr{M} = \mathscr{M}$ and hence, by Theorem 4, $T_\phi^* T_\phi$ is also analytic, i.e., $|\phi|^2 \in H^\infty$. Thus, $|\phi| = \lambda 1$ a.e.. For $f \in \mathscr{M}$, $f = L_{|\phi|^2} f = |\phi|^2 f = \lambda^2 f$ and we have $|\phi| = 1$ a.e.. Therefore, ϕ is inner.

COROLLARY 3. *If T_ϕ is norm-achieved (i.e., for some non-zero vector f in H^2, $\| T_\phi f \| = \| T_\phi \| \| f \|$), hyponormal (i.e., $T_\phi^* T_\phi \geqq T_\phi T_\phi^*$), then ϕ is a scalar multiple of an inner function. And, as a special case, a necessary and sufficient condition that T_ϕ be isometric is that ϕ be inner* ([3]).

PROOF. We may assume that $\| T_\phi \| = 1$. Let $\mathscr{M} = \{f \in H^2: \| T_\phi f \| = \| f \|\}$, then, by the hypothesis, $\mathscr{M} \neq \{0\}$ and, by the hyponormality of T_ϕ, $T_\phi \mathscr{M} \subset \mathscr{M}$. Hence, we have $\mathscr{M} = \{f \in H^2: \| T_\phi^n f \| = \| f \|, n \geqq 0\} \neq \{0\}$. Therefore, by Theorem 5, ϕ is inner.

REMARK. In above Corollary 3, we need the hyponormality of T_ϕ only to show $\mathscr{M}$ is invariant under T_ϕ. It is known that there are more wide classes of operators which have $\mathscr{M}$ as an invariant subspace of them. For example, the following operator is that: An operator T on a Hilbert space H such that $\| T^k x \| \geqq \| Tx \|^k$ for all unit vector $x \in H$

where $k \geqq 2$ is a fixed integer.

COROLLARY 4. *For a non-constant function ϕ in L^∞, if T_ϕ is a contraction (i.e., $\|T_\phi\| \leqq 1$), then it is completely non-unitary, that is, T_ϕ has no non-zero reducing subspaces restricted to which T_ϕ is unitary* ([4]). *And, as a complementary case, the only unitary Toeplitz operators are the scalars of modulus* 1 ([3]).

PROOF. By the decomposition theorem of contractions, the unitary part of T_ϕ is the restriction of T_ϕ on $H^{(u)} = \{f \in H^2: \|T_\phi^n f\| = \|T_\phi^{*n} f\| = \|f\|, n \geqq 0\}$. Hence we have only to prove $H^{(u)} = \{0\}$. If $H^{(u)} \neq \{0\}$, then, by Theorem 5, $\phi \in H^\infty \cap \overline{H^\infty} = \{\lambda 1\}$. This contradicts with the hypothesis.

From above corollary, it is natural the following question arises: Is every non-normal Toeplitz operator completely non-normal? As a partial answer of this question, we have

THEOREM 6. *If ϕ is a non-constant function in H^∞, then T_ϕ is completely non-normal, that is, T_ϕ has no non-zero reducing subspaces restricted to which T_ϕ is normal.*

PROOF. Let $\mathscr{M}$ be a reducing subspace of T_ϕ such that $T_\phi | \mathscr{M}$ is normal, then, for $f \in \mathscr{M}$, $\|L_\phi^* f\| \geqq \|PL_\phi^* f\| = \|T_\phi^* f\| = \|T_\phi f\| = \|L_\phi f\| = \|L_\phi^* f\|$ because $\phi \in H^\infty$. This implies that $L_\phi^* f = PL_\phi^* f = T_\phi^* f \in \mathscr{M}$. Hence $\mathscr{M}$ is invariant under L_ϕ^*. Since ϕ is a non-constant function in H^∞, T_ϕ^* is non-analytic and hence, by Theorem 4, $\mathscr{M} = \{0\}$.

3. Next, we prove the following result on the spectrum of Toeplitz operators.

THEOREM 7. *If ϕ is a non-constant function in L^∞, then $\sigma_p(T_\phi) \cap \overline{\sigma_p(T_\phi^*)} = \emptyset$, where $\sigma_p(T_\phi)$ denotes the point spectrum of T_ϕ and the bar denotes the complex conjugate.*

PROOF. If $\sigma_p(T_\phi) \cap \overline{\sigma_p(T_\phi^*)} \neq \emptyset$, then we may assume $0 \in \sigma_p(T_\phi) \cap \overline{\sigma_p(T_\phi^*)}$ because $T_\phi - \lambda I$ is also a Toeplitz operator. Then there is a non-zero vector f in H^2 such that $T_\phi f = 0$. Since, for $n \geqq 0$, $T_z^{*n} T_\phi T_z^n f = T_\phi f = 0$, we have

$$T_\phi T_z^n f = \lambda_1 T_z^{n-1} e_0 + \lambda_2 T_z^{n-2} e_0 + \cdots + \lambda_{n-1} T_z e_0 + \lambda_n e_0$$

where $e_0(z) = 1$. Let $\mathscr{M} = \bigvee \{T_z^n f : n \geqq 0\}$ and let $\mathscr{N} = \{g \in H^2: T_\phi g = 0\}$, then, if $\{T_z^n f : n \geqq 0\}$ is not included in $\mathscr{N}$, then there is a positive integer n_0 such that $T_\phi T_z^{n_0} f = \lambda e_0$, $\lambda \neq 0$. Since $H^2 = \bigvee \{T_z^n e_0 : n \geqq 0\}$, $T_\phi \mathscr{M}$ is dense in H^2 and hence $T_\phi H^2$ is dense in H^2. This implies that

$\{g \in H^2 : T_\phi^* g = 0\} = \{0\}$ and contradicts with $0 \in \sigma_p(T_\phi^*)$. Therefore, $\mathscr{M} \subset \mathscr{N}$ because $\mathscr{N}$ is clearly a closed subspace. Since $\mathscr{M} \neq \{0\}$ and since $\mathscr{M}$ is invariant under T_z, by Theorem 2, $\mathscr{M} = T_\psi H^2$ where ψ is some inner function. And hence $\mathscr{M} \subset \mathscr{N}$ implies $T_\phi T_\psi = 0$ and we have $T_\phi = 0$ because ψ is inner. Therefore, by Theorem 1, $\phi \in H^\infty \cap \overline{H^\infty} = \{\lambda 1\}$. This contradicts with the hypothesis.

COROLLARY 5. ([3]) *The only idempotent Toeplitz operators are* 0 *and* I.

PROOF. Since $\sigma_p(T_\phi) \cap \overline{\sigma_p(T_\phi^*)} \supset \{0, 1\}$, by Theorem 7, $T_\phi = \lambda I$ such as $\lambda^2 = \lambda$. Therefore $T_\phi = 0$ or $T_\phi = I$.

COROLLARY 6. *For a non-constant function ϕ in L^∞, if T_ϕ is hyponormal, then $\sigma_p(T_\phi) = \varnothing$. And, as a special case, for a non-constant function ϕ in L^∞, if T_ϕ is self-adjoint, then $\sigma_p(T_\phi) = \varnothing$* ([5]).

PROOF. If T_ϕ is hyponormal, then $T_\phi - \lambda I$ is also hyponormal and hence $T_\phi f = \lambda f$ implies that $T_\phi^* f = \bar{\lambda} f$. Therefore, by Theorem 7, $\sigma_p(T_\phi) = \varnothing$.

COROLLARY 7. ([2]) *If T_ϕ is non-zero, partially isometric, then ϕ or $\bar{\phi}$ is inner.*

PROOF. In the case where ϕ is constant, $T_\phi = \lambda I$, $|\lambda| = 1$ and, by Theorem 5, ϕ is inner. In the case where ϕ is non-constant, by Theorem 7, if $0 \in \sigma_p(T_\phi)$, then $0 \notin \sigma_p(T_\phi^*)$ and hence T_ϕ^* is isometric. Therefore, by Corollary 3, $\bar{\phi}$ is inner. Similarly, if $0 \in \sigma_p(T_\phi^*)$, then ϕ is inner.

In [7], we proved the following result: Let A be a nearly normal operator on a Hilbert space K (i.e., A commutes with A^*A) with the polar decomposition $A = V|A|$ and let W be an isometry on K. Then, if W commutes with A, then W commutes with V and $|A|$ also. From this and by Theorem 1 and Corollary 7, we have

COROLLARY 8. ([6]) *If T_ϕ is nearly normal, analytic, then ϕ is a scalar multiple of an inner function.*

PROOF. Let $T_\phi = V|T_\phi|$ be the polar decomposition of T_ϕ, then, by above result and by Theorem 1, T_z commutes with V and $|T_\phi|$ and hence V and $|T_\phi|$ are analytic Toeplitz operators on H^2. Since $|T_\phi|$ is self-adjoint, $|T_\phi| = \lambda I$ and $T_\phi = \lambda V$. Therefore, by Corollary 7, ϕ or $\bar{\phi}$ is a scalar multiple of an inner function because V is a partial isometry. Since T_ϕ is analytic, ϕ is a scalar multiple of an inner function.

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Misonou kindly communicated to the author the following result concerning our Theorem 7 (essentially, same as Theorem 7) was proved in the proof of Theorem (4.1) of L. A. Coburn [8], using the functional method: If, for a non-zero essentially bounded function ϕ, there exist functions f and g in H^2 such that $T_\phi f = 0$ and $T_{\bar{\phi}} g = 0$, then either $f = 0$ a.e. or $g = 0$ a.e..

REFERENCES

[1] A. BEURLING, On two problems concerning linear transformations in Hilbert space, Acta Math., 81 (1949), 239-255.

[2] A. BROWN AND R. G. DOUGLAS, Partially isometric Toeplitz operators, Proc. Amer. Math. Soc., 16-4 (1965), 681-682.

[3] A. BROWN AND P. R. HALMOS, Algebraic properties of Toeplitz operators, J. Reine Angew. Math., 213 (1963), 89-102.

[4] R. GOOR, On Toeplitz operators which are contractions, Proc. Amer. Math. Soc., 34-1 (1972), 191-192.

[5] P. HARTMAN AND A. WINTNER, The spectra of Toeplitz's matrices, Amer. J. Math., 76 (1954), 867-882.

[6] T. ITÔ AND T. K. WONG, Subnormality and quasinormality of Toeplitz operators, Proc. Amer. Math. Soc., 34-1 (1972), 157-164.

[7] T. YOSHINO, On the commuting extensions of nearly normal operators, Tôhoku Math. J., 25-3 (1973), 263-272.

[8] L. A. COBURN, Weyl's theorem for non-normal operators, Michigan Math. J., 13 (1966), 285-288.

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